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Coupled Fixed Point and Best Proximity Point Results Involving Simulation Functions *

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ABSTRACT: The purpose of this paper is to prove coupled fixed point theorems using simulation functions that extend the results of Kojasteh et al [1]. As an application we prove a coupled best proximity points using simulation functions.

Key Words: Coupled fixed points, best proximity points, simulation functions, partially ordered set.

Contents

1. Introduction and Preliminaries

Let (X, d) be a metric space and A be a nonempty subset of X. A mapping $T : A \to X$ has a fixed point in A if the fixed point equation $Tx = x$ has a solution. That is $x \in A$ is a fixed point of T if $d(Tx, x) = 0$. Suppose that the fixed point equation $Tx = x$ does not possess any solution. Then $d(x, Tx) > 0$ for all $x \in A$. In this situation, the goal is to find a point $x \in A$ such that $d(x, Tx)$ is the minimum in some sense.

Definition 1.1. *Let A and B are two nonempty subsets of a metric space (X, d) and consider a mapping $T : A \to B$. We say that $z \in A$ is a best proximity point of T if*

$$d(z, Tz) = d(A, B) := inf\{d(x, y) : x \in A, y \in B\}.$$

Suppose that $d(A, B) = 0$, then a best proximity point of T is a fixed point of T.

The existence and convergence of best proximity point is an interesting field of optimization theory and recently attracted the attention of many researchers [11,12,13,14,15,16]. Also best proximity points in ordered metric spaces are studied by many authors [17,18,19]. Let $A_0 = \{x \in A : d(x, y) = d(A, B), \text{for some } y \in B\}$ and $B_0 = \{y \in B : d(x, y) = d(A, B), \text{for some } x \in A\}$.

Kirk et. al [16] gave the sufficient conditions that guarantee that A_0 and B_0 are nonempty. In 2006 T.Gnana Bhaskar and V.Lakshmikantham [20] introduced the concept of the mixed monotone property and obtained some coupled fixed point theorems for mappings that satisfy the mixed monotone property and gave some applications in the existence and uniqueness of a solution of a periodic boundary value problem. After the result of Bhaskar et. al. [20] there are lots of work done by many authors [21,22,23,24] and reference there in.

The concept of coupled best proximity point theorem is introduced by W.Sintunavarat and P.Kumam [25] and proved coupled best proximity theorem for cyclic contractions. For several improvements and generalizations see in [25,26,27,28].

Recently Kojasteh et al. [1] introduced the class of simulation function as follows

Definition 1.2. *We say that $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is a simulation function if it satisfies the following conditions:*

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1. $\zeta(0,0) = 0$;
2. $\zeta(t,s) < s - t$ for all $t, s \in (0,\infty)$;
3. if (a_n) and (b_n) are two sequences in $(0,\infty)$, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n > 0 \Rightarrow \limsup_{n\to\infty} \zeta(a_n, b_n) < 0.$$

The examples for simulation function are presented in [1,2,3,4,5,6,7,8,9,10]. Class of such functions will be denoted by $\mathscr{Z}$.

Example 1.3. *Let $\phi_i : [0,\infty) \to [0,\infty)$ be continuous functions with $\phi_i(t) = 0$ if, and only if, $t = 0$. For $i = 1,2,3,4,5,6$ we define the mappings $\zeta_i : [0,\infty) \times [0,\infty) \to \mathbb{R}$, as follows*

1. $\zeta_1(t,s) = \phi_1(s) - \phi_2(t)$ for all $t, s \in [0,\infty)$, where $\phi_1(t) < t \le \phi_2(t)$ for all $t > 0$.
2. $\zeta_2(t,s) = s - \dfrac{f(t,s)}{g(t,s)} t$ for all $t, s \in [0,\infty)$, where $f, g : [0,\infty) \times [0,\infty) \to [0,\infty)$ are two continuous functions with respect to each variable such that $f(t,s) > g(t,s)$ for all $t, s > 0$.
3. $\zeta_3(t,s) = s - \phi_3(s) - t$ for all $t, s \in [0,\infty)$.
4. If $\phi : [0,\infty) \to [0,1)$ is function such that $\limsup_{t\to r^+} \phi(t) < 1$ for all $r > 0$, and we define $\zeta_4(t,s) = s\phi(s) - t$ for all $s, t \in [0,\infty)$.
5. If $\eta : [0,\infty) \to [0,\infty)$ is an upper semi-continuous mapping such that $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$, and we define $\zeta_5(t,s) = \eta(s) - t$ for all $s, t \in [0,\infty)$.

It is easy to verify that each function $\zeta_i (i = 1,2,3,4,5)$ forms a simulation function.

Definition 1.4. *[1] Let $T : X \to X$ be a given operator. Where X is a nonempty set equipped with a metric d. We say that T is a $\mathscr{Z}$- contraction with respect to $\zeta \in \mathscr{Z}$ if*

$$\zeta(d(Tx,Ty), d(x,y)) \ge 0, \text{for all } x, y \in X.$$

The following fixed point theorem is proved by authors in [1], which generalizes many previous results from the literature including the Banach fixed point theorem.

Theorem 1.5. *[1] Let $T : X \to X$ be a given map, where X is a nonempty set equipped with a metric d such that (X,d) is complete. Suppose that T is a $\mathscr{Z}$- contraction with respect to $\zeta \in \mathscr{Z}$. Then T has a unique fixed point. Moreover, for any $x \in X$, the sequence $(T^n x)$ converges to this fixed point.*

Further many authors generalizes [1], we refer to [2,3,4,5,6,7,8,9,10]. Now we recall the basic definitions and facts.

Definition 1.6. *Let $(X,\preceq)$ be a partially ordered set and $F : X \times X \to X$. We say that F has the mixed monotone property if $F(x,y)$ is monotone non decreasing in x and is monotone non increasing in y, that is for any $x, y \in X$,*

$$x_1, x_2 \in X,\ x_1 \preceq x_2 \quad \text{implies} \quad F(x_1,y) \preceq F(x_2,y)$$

and

$$y_1, y_2 \in X,\ y_1 \preceq y_2 \quad \text{implies} \quad F(x,y_1) \succeq F(x,y_2).$$

Example 1.7. *Let $X = \{(1,0),(0,1)\} \subset \mathbb{R}^2$ with the order*

$$(x,y) \preceq (u,v) \Leftrightarrow x \le u \text{ and } y \ge v.$$

Then $(X,\preceq)$ be a partially ordered set. Let $F : X \times X \to X$ be defined by $F(x,y) = x$.

Note that, every element in X is comparable to itself. In this case we can easily check that F has the mixed monotone property.

The nontrivial case is the following, for any $x, y \in X$,

$$(0,1) \preceq (1,0) \text{ implies } (0,1) = F((0,1), y) \preceq F((1,0), y) = (1,0)$$

and

$$(0,1) \preceq (1,0) \text{ implies } x = F(x,(0,1)) \succeq F(x,(1,0)) = x.$$

Definition 1.8. *An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.*

The following theorem is the main theorem proved by Bhaskar and Lakshmikantham [20].

Theorem 1.9. *[20] Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X and assume that there exists $k \in [0, 1)$ with*

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}[d(x,u) + d(y,v)],$$

for any $x \succeq u$ and $y \preceq v$. If there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0)$$

and we suppose that either F is continuous or X satisfies the following property:
if (x_n) is a non decreasing sequence with $x_n \to x$ then $x_n \preceq x$ for each $n \in \mathbb{N}$
and if (y_n) is a non increasing sequence with $y_n \to y$ then $y_n \succeq y$ for each $n \in \mathbb{N}$
then F has a coupled fixed point.

Motivated by the results of [1] and [20], in this article we introduce the concept called $\mathscr{Z}$- coupled contraction and prove a coupled fixed point theorem for $\mathscr{Z}$- coupled contraction which is a generalization of Theorem1.9. Also we define a proximally $\mathscr{Z}$- coupled contraction and prove a coupled best proximity point using proximally $\mathscr{Z}$- coupled contraction. Our results generalize and unify the existing results in the literature.

Kumam et. al. [27] introduced the proximal mixed monotone property as follows,

Definition 1.10. *Let $(X, d, \preceq)$ be a partially ordered metric space and A, B are nonempty subsets of X. A mapping $F : A \times A \to B$ is said to have a proximal mixed monotone property if $F(x, y)$ is proximally non decreasing in x and is proximally non increasing in y, that is, for all $x, y \in A$*

$$\left.\begin{array}{l} x_1 \preceq x_2, \\ d(u_1, F(x_1, y)) = d(A,B) \\ d(u_2, F(x_2, y)) = d(A,B) \end{array}\right\} \Rightarrow u_1 \preceq u_2$$

and

$$\left.\begin{array}{l} y_1 \preceq y_2, \\ d(u_3, F(x, y_1)) = d(A,B) \\ d(u_4, F(x, y_2)) = d(A,B) \end{array}\right\} \Rightarrow u_4 \preceq u_3$$

where $x_1, x_2, y_1, y_2, u_1, u_2, u_3, u_4 \in A$.

One can see that, if $A = B$ in the above definition, the notion of the proximal mixed monotone property reduces to that of the mixed monotone property.

Example 1.11. *Let $X = \{(0,1), (1,0), (-1,0), (0,-1)\}$ and consider the usual order $(x, y) \preceq (z, t) \Leftrightarrow x \leq z$ and $y \leq t$.*

Thus $(X, \preceq, d_2)$ be a partially ordered metric space with respect to the Euclidean metric d_2. Let $F : A \times A \to B$ be defined as $F((x_1, x_2), (y_1, y_2)) = (-x_2, -x_1)$ and $d_2(A, B) = \sqrt{2}$. Note that the only comparable points in A are $x \preceq x$ for $x \in A$, then it is easy to verify that F has the proximal mixed monotone property.

Example 1.12. *Let* $X = \{(0,1),(1,0),(-1,0),(0,-1)\}$ *and consider the usual order* $(x,y) \preceq (z,t) \Leftrightarrow x \leq z$ *and* $y \geq t$.

Thus $(X, \preceq, d_2)$ *be a partially ordered metric space with respect to the Euclidean metric* d_2. *Let* $F : A \times A \to B$ *be defined as* $F((x_1,x_2),(y_1,y_2)) = (-x_2,-x_1)$ *and* $d_2(A,B) = \sqrt{2}$.

Note that every element in A is comparable to itself. In this case we can easily verify that F has the mixed monotone property. The nontrivial case is that, for any $x, y \in A$

$$\left.\begin{array}{l}(0,1) \preceq (1,0),\\ d((0,1),F((0,1),y)) = \sqrt{2}\\ d((1,0),F((1,0),y)) = \sqrt{2}\end{array}\right\} \Rightarrow (0,1) \preceq (1,0)$$

and

$$\left.\begin{array}{l}(0,1) \preceq (1,0),\\ d(u,F(x,(0,1))) = \sqrt{2}\\ d(u,F(x,(1,0))) = \sqrt{2}\end{array}\right\} \Rightarrow u \preceq u$$

for some $u \in A$. *Observe that for any* $x \in A$ *there exists a unique* $u \in A$.

Lemma 1.13. *[27] Let* $(X, \preceq, d)$ *be a partially ordered metric space and* A, B *are nonempty subsets of* X. *Assume* A_0 *is nonempty. A mapping* $F : A \times A \to B$ *has the proximal mixed monotone property with* $F(A_0 \times A_0) \subseteq B_0$ *whenever* x_0, x_1, x_2, y_0, y_1 *in* A_0 *such that*

$$\left.\begin{array}{l}x_0 \preceq x_1 \text{ and } y_0 \succeq y_1,\\ d(x_1,F(x_0,y_0)) = d(A,B)\\ d(x_2,F(x_1,y_1)) = d(A,B)\end{array}\right\} \Rightarrow x_1 \preceq x_2.$$

Lemma 1.14. *[27] Let* $(X, \preceq, d)$ *be a partially ordered metric space and* A, B *are nonempty subsets of* X. *Assume* A_0 *is nonempty. A mapping* $F : A \times A \to B$ *has the proximal mixed monotone property with* $F(A_0 \times A_0) \subseteq B_0$ *whenever* x_0, x_1, x_2, y_0, y_1 *in* A_0 *such that*

$$\left.\begin{array}{l}x_0 \preceq x_1 \text{ and } y_0 \succeq y_1,\\ d(y_1,F(y_0,x_0)) = d(A,B)\\ d(y_2,F(y_1,x_1)) = d(A,B)\end{array}\right\} \Rightarrow y_1 \succeq y_2.$$

2. Coupled fixed point theorems

Now we are defining $\mathscr{Z}$- coupled contraction and proximally $\mathscr{Z}$- coupled contraction as follows.

Definition 2.1. *Let* $(X, \preceq, d)$ *be a partially ordered metric space. Let* $F : X \times X \to X$ *be a mapping and* $\zeta \in \mathscr{Z}$. *Then F is called a* $\mathscr{Z}$*-coupled contraction with respect to* ζ *if the following condition is satisfied,*

$$\zeta(max\{d(F(x,y),F(u,v)),d(F(y,x),F(v,u))\}, max\{d(x,u),d(y,v)\}) \geq 0$$

for all $x, y, u, v \in X$ *with* $x \preceq u$ *and* $y \succeq v$.

Definition 2.2. *Let* $(X, \preceq, d)$ *be a partially ordered metric space and* A, B *are nonempty subsets of* X. *A mapping* $F : A \times A \to B$ *is said to be proximally* $\mathscr{Z}$*- coupled contraction on A there exists* $\zeta \in \mathscr{Z}$ *such that*

$$\left.\begin{array}{l}x_1 \preceq x_2 \text{ and } y_1 \succeq y_2,\\ d(u_1,F(x_1,y_1)) = d(A,B)\\ d(u_2,F(x_2,y_2)) = d(A,B)\\ d(v_1,F(y_1,x_1)) = d(A,B)\\ d(v_2,F(y_2,x_2)) = d(A,B)\end{array}\right\}$$

$\Rightarrow \zeta(max\{d(u_1,u_2),d(v_1,v_2)\}, max\{d(x_1,x_2),d(y_1,y_2)\}) \geq 0$

for all $x_1, x_2, y_1, y_2, u_1, u_2, v_1, v_2 \in A$.

Theorem 2.3. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X and F is a $\mathscr{Z}$-coupled contraction with respect to ζ. If there exists $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$, $y_0 \succeq F(y_0, x_0)$ then F has a coupled fixed point.*

Proof. Let $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0) \quad \text{and} \quad y_0 \succeq F(y_0, x_0).$$

Let

$$x_1 = F(x_0, y_0), \quad y_1 = F(y_0, x_0).$$

Then $x_0 \preceq x_1$ and $y_0 \succeq y_1$. Again let $x_2 = F(x_1, y_1)$ and $y_2 = F(y_1, x_1)$, since F has a mixed monotone property we get $x_1 \preceq x_2$ and $y_1 \succeq y_2$. Continuing in this way we construct two sequences (x_n) and (y_n) in X such that

$$x_{n+1} = F(x_n, y_n), \quad y_{n+1} = F(y_n, x_n)$$

and

$$x_0 \preceq x_1 \preceq x_2 \preceq ...x_n \preceq x_{n+1} \preceq ..., \quad y_0 \succeq y_1 \succeq y_2 \succeq ...y_n \succeq y_{n+1} \succeq ... \quad .$$

Using the $\mathscr{Z}$-coupled contraction condition and since $x_{n-1} \preceq x_n$ and $y_{n-1} \succeq y_n$ we get,

$$0 \leq \zeta(max\{d(x_{n+1}, x_n), d(y_{n+1}, y_n)\}, max\{d(x_n, x_{n-1}), d(y_n, y_{n-1})\}). \tag{2.1}$$

Suppose that $max\{d(x_{n+1}, x_n), d(y_{n+1}, y_n)\} = 0$ for some $n \in \mathbb{N}$. Then we have

$$d(x_{n+1}, x_n) = 0 = d(y_{n+1}, y_n),$$

implies that

$$d(F(x_n, y_n), x_n) = 0 = d(F(y_n, x_n), y_n).$$

Therefore $x_n = F(x_n, y_n)$ and $y_n = F(y_n, x_n)$. Hence the claim.

Now we discuss the non trivial case, such that for all $n \in \mathbb{N}$,

$max\{d(x_{n+1}, x_n), d(y_{n+1}, y_n)\} \neq 0$ and $max\{d(x_n, x_{n-1}), d(y_n, y_{n-1})\} \neq 0$, using condition (2) of simulation function, equation 2.1 becomes

$$0 < max\{d(x_n, x_{n-1}), d(y_n, y_{n-1})\} - max\{d(x_{n+1}, x_n), d(y_{n+1}, y_n)\}$$

$$max\{d(x_{n+1}, x_n), d(y_{n+1}, y_n)\} < max\{d(x_n, x_{n-1}), d(y_n, y_{n-1})\}.$$

So, $(max\{d(x_{n+1}, x_n), d(y_{n+1}, y_n)\})$ is a non negative decreasing sequence, which implies that there exists $r \geq 0$ such that

$$\lim_{n\to\infty} max\{d(x_{n+1}, x_n), d(y_{n+1}, y_n)\} = r.$$

Suppose that $r > 0$, using the property (iii) of simulation function and (2.1) becomes,

$$0 \leq \limsup_{n\to\infty} \zeta(max\{d(x_{n+1}, x_n), d(y_{n+1}, y_n)\}, max\{d(x_n, x_{n-1}), d(y_n, y_{n-1})\}) < 0$$

which a is contradiction. As a consequence, we have

$$\lim_{n\to\infty} max\{d(x_{n+1}, x_n), d(y_{n+1}, y_n)\} = 0. \tag{2.2}$$

Now, we claim that (x_n) and (y_n) are Cauchy sequences. Suppose the contrary, there exists $\epsilon > 0$ for which we can find subsequences $(x_{m(k)})$, $(x_{n(k)})$ of (x_n) and $(y_{m(k)})$, $(y_{n(k)})$ of (y_n) with $n(k) > m(k) > k$ such that

$$max\{d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})\} \geq \epsilon. \tag{2.3}$$

Further, we can choose $n(k)$ corresponding to $m(k)$, such that $n(k)$ is the smallest integer with $n(k) > m(k)$ and satisfying (2.3). Then

$$max\{d(x_{n(k)-1}, x_{m(k)}), d(y_{n(k)-1}, y_{m(k)})\} < \epsilon. \tag{2.4}$$

Since, $x_{n(k)-1} \succeq x_{m(k)-1}$ and $y_{n(k)-1} \preceq y_{m(k)-1}$, using the $\mathscr{Z}$-coupled contraction condition we obtain,

$$\begin{aligned} 0 \leq \zeta(&max\{d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})\}, \\ &max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\}) \end{aligned} \tag{2.5}$$

On the other hand, the triangular inequality and (2.4) gives us,

$$\begin{aligned} d(x_{n(k)}, x_{m(k)}) &\leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) \\ &< d(x_{n(k)}, x_{n(k)-1}) + \epsilon \end{aligned} \tag{2.6}$$

$$\begin{aligned} d(y_{n(k)}, y_{m(k)}) &\leq d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)}) \\ &< d(y_{n(k)}, y_{n(k)-1}) + \epsilon \end{aligned} \tag{2.7}$$

From (2.3), (2.6) and (2.7) we get,

$$\begin{aligned} \epsilon &\leq max\{d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})\} \\ &\leq max\{d(x_{n(k)-1}, x_{m(k)}), d(y_{n(k)-1}, y_{m(k)})\} < \epsilon. \end{aligned} \tag{2.8}$$

As $k \to \infty$ in the last inequality we have,

$$\lim_{k\to\infty} max\{d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})\} = \epsilon. \tag{2.9}$$

Again using the triangular inequality and (2.4) gives us,

$$\begin{aligned} d(x_{n(k)-1}, x_{m(k)-1}) &\leq d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)-1}) \\ &< \epsilon + d(x_{m(k)}, x_{m(k)-1}) \end{aligned} \tag{2.10}$$

$$\begin{aligned} d(y_{n(k)-1}, y_{m(k)-1}) &\leq d(y_{n(k)-1}, y_{m(k)}) + d(y_{m(k)}, y_{m(k)-1}) \\ &< \epsilon + d(y_{m(k)}, y_{m(k)-1}). \end{aligned} \tag{2.11}$$

From (2.10) and (2.11) we get,

$$\begin{aligned} &max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\} < \\ &\qquad \{d(x_{m(k)}, x_{m(k)-1}, d(y_{m(k)}, y_{m(k)-1})\} + \epsilon \end{aligned} \tag{2.12}$$

Using the triangular inequality we have,

$$d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)})$$
$$d(y_{n(k)}, y_{m(k)}) \leq d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)})$$

from (2.3) and by the previous two inequalities we have,

$$\begin{aligned} \epsilon &\leq max\{d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})\} \\ &\leq max\{d(x_{n(k)}, x_{n(k)-1}), d(y_{n(k)}, y_{n(k)-1})\} \\ &+ max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\} \\ &+ max\{d(x_{m(k)-1}, x_{m(k)}), d(y_{m(k)-1}, y_{m(k)})\}. \end{aligned} \tag{2.13}$$

From (2.12) and (2.13) we have,

$$\begin{aligned} \epsilon &- max\{d(x_{n(k)}, x_{n(k)-1}), d(y_{n(k)}, y_{n(k)-1})\} \\ &- max\{d(x_{m(k)-1}, x_{m(k)}), d(y_{m(k)-1}, y_{m(k)})\} \\ &\leq max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\} \\ &< \{d(x_{m(k)}, x_{m(k)-1}, d(y_{m(k)}, y_{m(k)-1})\} + \epsilon. \end{aligned}$$

Letting $k \to \infty$ in the previous equation and by (2.2) we have,

$$\lim_{k\to\infty} max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\} = \epsilon \tag{2.14}$$

Using (2.9), (2.14) and the property (iii) of simulation function to (2.5), we get

$$0 \le \zeta(max\{d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})\},$$
$$max\{d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})\}) < 0$$

which is a contradiction, therefore $\epsilon = 0$. Implies that (x_n) and (y_n) are Cauchy sequences. Since (X, d) is a complete metric space there exists $x, y \in X$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Using the continuity of F we get,

$$x = \lim_{n\to\infty} F(x_n, y_n) = F(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = F(x, y)$$
$$y = \lim_{n\to\infty} F(y_n, x_n) = F(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n) = F(y, x)$$

and this proves (x, y) is a coupled fixed point of F. □

Example 2.4. *Let $X = \{(1, 0), (0, 1)\} \subset \mathbb{R}^2$ with the usual order $(x, y) \preceq (u, v) \Leftrightarrow x \le u$ and $y \le v$.*

Thus $(X, \preceq)$ is a partially ordered set, also (X, d_2) is a complete metric space considering d_2 the Euclidean metric. Let $F : X \times X \to X$ be defined by $F(x, y) = x$.

Obviously, F is continuous and has the mixed monotone property. The only comparable pairs of points in X are $x \preceq x$ for $x \in X$. Hence F satisfies the $\mathscr{Z}$-coupled contraction trivially. Moreover,

$$(1, 0) \preceq F((1, 0), (0, 1)) = (1, 0)$$

$$(0, 1) \succeq F((0, 1), (1, 0)) = (0, 1).$$

It can be easily shown that $((1, 0), (0, 1))$ and $((0, 1), (1, 0))$ are coupled fixed points of F.

In what follows, we prove that Theorem 2.3 is still valid for F not necessarily continuous, assuming the following hypothesis in X. X has the property that,

$$\text{if } (x_n) \text{ is a non decreasing sequence with } x_n \to x \text{ then } x_n \preceq x, \text{for all } n \in \mathbb{N} \tag{2.15}$$
$$\text{if } (y_n) \text{ is a non increasing sequence with } y_n \to y \text{ then } y_n \succeq y, \text{for all } n \in \mathbb{N} \tag{2.16}$$

Theorem 2.5. *Assume the conditions (2.14) and (2.15) instead of continuity of F in Theorem 2.3, then the conclusion of Theorem 2.3 holds.*

Proof. Following the proof of Theorem 2.3 we only have to check that (x, y) is a coupled fixed point of F.

Since (x_n) is non decreasing and $x_n \to x$ and (y_n) is non increasing and $y_n \to y$ also using our assumption $x_n \preceq x$ and $y_n \succeq y$ for all $n \in \mathbb{N}$, by the $\mathscr{Z}$-coupled contraction condition we get,

$$0 \le \zeta(max\{d(F(x, y), F(x_n, y_n)), d(F(y, x), F(y_n, x_n))\}, max\{d(x, x_n), d(y, y_n)\}). \tag{2.17}$$

Case I: In (2.17), suppose that

$$max\{d(F(x, y), F(x_n, y_n)), d(F(y, x), F(y_n, x_n))\} = 0$$

for some $n \in \mathbb{N}$. Implies that

$$x_{n+1} = F(x_n, y_n) = F(x, y) \quad \text{and} \quad y_{n+1} = F(y_n, x_n) = F(y, x),$$

from (2.15) and (2.16) we get,

$$x_{n+1} = F(x,y) \preceq x \quad \text{and} \quad y_{n+1} = F(y,x) \succeq y.$$

Using the monotonicity of (x_n) and (y_n) and the mixed monotone property of F, we get

$$x_{n+2} = F(x_{n+1}, y_{n+1}) \preceq F(x, y_{n+1}) \preceq F(x,y)$$

implies that $x_{n+2} = F(x,y)$, similarly we can verify that $y_{n+2} = F(y,x)$, this is true for all $m \geq n$. Since $x_n \to x$ and $y_n \to y$ we get $x = F(x,y)$ and $y = F(y,x)$.

Now suppose that $max\{d(x,x_n), d(y,y_n)\} = 0$, for some $n \in \mathbb{N}$. Then we get $x_n = x$ and $y_n = y$, implies that $x_{n+1} = F(x_n,y_n) = F(x,y)$ and $y_{n+1} = F(y_n,x_n) = F(y,x)$. By (2.15) and (2.16) we get $x = x_n \preceq x_{n+1} = F(x,y) \preceq x$ and $y = y_n \succeq y_{n+1} = F(y,x) \succeq y$. Hence the claim.

Case II: Now consider (2.17) such that for all $n \in \mathbb{N}$,
$max\{d(F(x,y), F(x_n,y_n)), d(F(y,x), F(y_n,x_n))\} \neq 0$ and $max\{d(x,x_n), d(y,y_n)\} \neq 0$. By the property (2) of simulation function, (2.17) becomes

$$0 < max\{d(x,x_n), d(y,y_n)\} - max\{d(F(x,y), F(x_n,y_n)), d(F(y,x), F(y_n,x_n))\}.$$

As $n \to \infty$ the previous inequality becomes

$$\lim_{n\to\infty} max\{d(F(x,y), x_{n+1}), d(F(y,x), y_{n+1})\} \leq 0$$

which implies that, $x_{n+1} \to F(x,y)$ and $y_{n+1} \to F(y,x)$. Since $x_n \to x$ and $y_n \to y$ and by the uniqueness of the limits we have, $F(x,y) = x$ and $F(y,x) = y$ and this finishes the proof. □

The sufficient condition for the uniqueness of the coupled fixed point for Theorems (2.3) and (2.5) is the following,

$$\text{for } (x,y),(z,t) \in X \times X \text{ there exists } (u,v) \in X \times X \text{ which is comparable to } (x,y) \text{ and } (z,t). \tag{2.18}$$

Theorem 2.6. *Adding condition (2.18) to the hypothesis of Theorem 2.3 (resp. Theorem 2.5) we obtain uniqueness of the coupled fixed point of F.*

Proof. From Theorem 2.3(resp. Theorem 2.5) the set of coupled fixed points of F is non empty. Suppose that (x,y) and (z,t) are coupled fixed points of F, that is

$$x = F(x,y), y = F(y,x), z = F(z,t) \text{ and } t = F(t,z).$$

Let (u,v) be an element in $X \times X$ which is comparable to (x,y) and (z,t). Suppose that $(u,v) \preceq (x,y)$. We construct sequences (u_n) and (v_n) defined by

$$u_0 = u, v_0 = v, u_{n+1} = F(u_n,v_n), v_{n+1} = F(v_n,u_n).$$

We claim that $(u_n,v_n) \preceq (x,y)$ for each $n \in \mathbb{N}$. To prove this, we use mathematical induction. For $n = 0$, we have $(u_0,v_0) = (u,v) \preceq (x,y)$, which gives $u_0 \preceq x$ and $v_0 \succeq y$ and suppose that $(u_n,v_n) \preceq (x,y)$, then using the mixed monotone property of F, we get

$$u_{n+1} = F(u_n,v_n) \preceq F(x,v_n) \preceq F(x,y) = x$$
$$v_{n+1} = F(v_n,u_n) \succeq F(y,u_n) \succeq F(y,x) = y$$

this implies that $u_n \preceq x$ and $v_n \succeq y$ for all $n \in \mathbb{N}$, using the $\mathscr{Z}$-coupled contraction condition we get,

$$0 \leq \zeta(max\{d(x,u_n), d(y,v_n)\}, max\{d(x,u_{n-1}), d(y,v_{n-1})\}). \tag{2.19}$$

Suppose that $max\{d(x,u_n),d(y,v_n)\}=0$ for some $n\in\mathbb{N}$, we get $u_{n+1}=F(u_n,v_n)=F(x,y)=x$ and $v_{n+1}=F(v_n,u_n)=F(y,x)=y$, this is true for all $m\geq n$. As $n\to\infty$ $u_n\to x$ and $v_n\to y$. Similarly we can show that for (z,t) such that $u_n\to z$ and $v_n\to t$. By the uniqueness of the limit we get $x=z$ and $y=t$.

Now we consider the case such that $max\{d(x,u_n),d(y,v_n)\}\neq 0$ and $max\{d(x,u_{n-1}),d(y,v_{n-1})\}\neq 0$, for all $n\in\mathbb{N}$. Using the condition (2) of simulation function (2.19) becomes

$$\begin{aligned}0<max\{d(x,u_{n-1}),d(y,v_{n-1})\}-max\{d(x,u_n),d(y,v_n)\}\\ max\{d(x,u_n),d(y,v_n)\}<max\{d(x,u_{n-1}),d(y,v_{n-1})\}\end{aligned}$$

which implies that $max\{d(x,u_n),d(y,v_n)\}$ is a decreasing sequence and bounded below by 0, and for some $r\geq 0$ we have,

$$\lim_{n\to\infty} max\{d(x,u_n),d(y,v_n)\}=r.$$

Suppose that $r>0$, (2.19) becomes,

$$0\leq\limsup_{n\to\infty}\zeta(max\{d(x,u_n),d(y,v_n)\},max\{d(x,u_{n-1}),d(y,v_{n-1})\})<0$$

which is a contradiction. Therefore $r=0$, consequently $\lim_{n\to\infty} max\{d(x,u_n),d(y,v_n)\}=0$ and this gives us $u_n\to x$ and $v_n\to y$.

Using the similar argument for (z,t) we have, $u_n\to z$ and $v_n\to t$ and the uniqueness of the limit gives $x=z$ and $y=t$. This proofs our claim. □

Theorem 2.7. *Under the assumptions of Theorem 2.3(resp.Theorem2.5), suppose that x_0 and y_0 are comparable, then the coupled fixed point $(x,y)\in X\times X$ satisfies $x=y$.*

Proof. Suppose that $x_0\preceq y_0$ (similarly for $y_0\preceq x_0$).

We claim that $x_n\preceq y_n$ for all $n\in\mathbb{N}$, where $x_{n+1}=F(x_n,y_n)$ and $y_{n+1}=F(y_n,x_n)$.

The inequality is true for $n=0$. Assume that $x_n\preceq y_n$, using the mixed monotone property of F, we have

$$x_{n+1}=F(x_n,y_n)\preceq F(y_n,y_n)\preceq F(y_n,x_n)=y_{n+1},$$

this proofs our claim. Suppose that $x_n=y_n$ for some $n\in\mathbb{N}$, implies that $x_{n+1}=y_{n+1}$ and this is true for all $m\geq n$. Since $x_n\to x$ and $y_n\to y$ we get $x=y$. Assume that $x_n\neq y_n$ for all $n\in\mathbb{N}$ and using the $\mathscr{Z}$-coupled contraction condition we get,

$$\begin{aligned}0&\leq\zeta(max\{d(F(x_n,y_n),F(y_n,x_n)),d(F(y_n,x_n),F(x_n,y_n))\},\\ &\quad max\{d(x_n,y_n),d(y_n,x_n)\})\\ 0&\leq\zeta(d(x_{n+1},y_{n+1}),d(x_n,y_n))\\ &<d(x_n,y_n)-d(x_{n+1},y_{n+1})\end{aligned}\tag{2.20}$$

this implies that $(d(x_n,y_n))$ is decreasing, there exists $r\geq 0$ we get $\lim_{n\to\infty} d(x_n,y_n)=r$. Suppose that $r>0$, (2.20) becomes,

$$0\leq\limsup_{n\to\infty}\zeta(max\{d(x_{n+1},y_{n+1}),d(y_{n+1},x_{n+1})\},max\{d(x_n,y_n),d(y_n,x_n)\})<0$$

which is a contradiction, and hence $\lim_{n\to\infty} d(x_n,y_n)=0$.

Since $x_n\to x$ and $y_n\to y$ and we have

$$0=\lim_{n\to\infty} d(x_n,y_n)=d(\lim_{n\to\infty} x_n,\lim_{n\to\infty} y_n)=d(x,y),$$

and we have $x=y$. Hence the claim. □

In the following corollaries we obtain some known coupled fixed point results via the simulation function.

Corollary 2.8. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X such that there exists $k \in [0, 1)$ satisfying*

$$d(F(x, y), F(u, v)) \leq k\ max\{d(x, u), d(y, v)\}$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$.

Suppose that either F is continuous or X satisfies condition (2.15) and (2.16).

If there exists $x_0, y_0 \in X$ with

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0)$$

then F has a coupled fixed point.

Proof. Define $\zeta^* : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by

$$\zeta^*(t, s) = ks - t, \text{ for all } t, s \in [0, \infty) \text{ and } k \in [0, 1).$$

Note that the mapping F is a $\mathscr{Z}$- contraction with respect to $\zeta^* \in \mathscr{Z}$. Applying Theorems 2.3 and 2.5 and taking $\zeta = \zeta^*$, we obtain the corollary. □

Remark 2.1. *Notice that Theorem 1.1 of Bhaskar and Lakshmikantham [20] is a consequence of corollary 2.8. The contractive condition appearing in Theorem 1.1 is that,*

$$d(F(x, y), d(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)] \text{ for any } x \succeq u \text{ and } y \preceq v$$

with $k \in [0, 1)$ implies

$$\begin{aligned} d(F(x, y), F(u, v)) &\leq \frac{k}{2}[d(x, u) + d(y, v)] \\ &\leq \frac{k}{2}.2.max\{d(x, u), d(y, v)\} \\ &= k\ max\{d(x, u), d(y, v)\} \quad \text{for any } x \succeq u \text{ and } y \preceq v \end{aligned}$$

and applying Corollary 2.8 we obtain the desired result.

3. Coupled best proximity point theorems

Theorem 3.1. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let A and B be nonempty closed subsets of the metric space (X, d) such that $A_0 \neq \emptyset$. Let $F : A \times A \to B$ satisfy the following conditions.*

1. *F is continuous proximally $\mathscr{Z}$-coupled contraction on A having the proximal mixed monotone property on A such that $F(A_0 \times A_0) \subseteq B_0$.*

2. *There exists elements (x_0, y_0) and (x_1, y_1) in $A_0 \times A_0$ such that*

$$\begin{aligned} d(x_1, F(x_0, y_0)) &= d(A, B) \qquad \text{with } x_0 \preceq x_1 \text{ and} \\ d(y_1, F(y_0, x_0)) &= d(A, B) \qquad \text{with } y_0 \succeq y_1. \end{aligned}$$

Then there exists $(x, y) \in A \times A$ such that $d(x, F(x, y)) = d(A, B)$ and $d(y, F(y, x)) = d(A, B)$.

Proof. From the hypothesis there exists elements (x_0, y_0) and (x_1, y_1) in $A_0 \times A_0$ such that

$$d(x_1, F(x_0, y_0)) = d(A, B) \qquad \text{with } x_0 \preceq x_1 \text{and}$$
$$d(y_1, F(y_0, x_0)) = d(A, B) \qquad \text{with } y_0 \succeq y_1.$$

Since $F(A_0 \times A_0) \subseteq B_0$, there exists (x_2, y_2) in $A \times A$ such that

$$d(x_2, F(x_1, y_1)) = d(A, B) \quad \text{and}$$
$$d(y_2, F(y_1, x_1) = d(A, B)).$$

Using lemma 1.13 and lemma 1.14, we have $x_1 \preceq x_2$ and $y_1 \succeq y_2$.

Continuing in this way, we construct two sequences (x_n) and (y_n) in A_0 such that

$$d(x_{n+1}, F(x_n, y_n)) = d(A, B), \quad \forall n \in \mathbb{N} \text{ with} \tag{3.1}$$
$$x_0 \preceq x_1 \preceq x_2 \preceq \ldots \preceq x_n \preceq x_{n+1} \preceq \ldots \quad \text{and}$$
$$d(y_{n+1}, F(y_n, x_n)) = d(A, B), \quad \forall n \in \mathbb{N} \text{ with} \tag{3.2}$$
$$y_0 \succeq y_1 \succeq y_2 \succeq \ldots \succeq y_n \succeq y_{n+1} \succeq \ldots .$$

Using the proximally $\mathscr{Z}$-coupled contraction condition on A and since $d(x_n, F(x_{n-1}, y_{n-1})) = d(A, B)$, $d(y_n, F(y_{n-1}, x_{n-1})) = d(A, B)$ and we have $x_{n-1} \preceq x_n$, $y_{n-1} \succeq y_n$ which implies that,

$$0 \leq \zeta(max\{d(x_{n+1}, x_n), d(y_{n+1}, y_n)\}, max\{d(x_n, x_{n-1}), d(y_n, y_{n-1})\}) \tag{3.3}$$

Suppose that $max\{d(x_{n+1}, x_n), d(y_{n+1}, y_n)\} = 0$, for some $n \in \mathbb{N}$. From (3.1) and (3.2) we get $d(x_n, F(x_n, y_n)) = d(A, B)$ and $d(y_n, F(y_n, x_n)) = d(A, B)$. Hence the claim.

Now assume that $max\{d(x_{n+1}, x_n), d(y_{n+1}, y_n)\} \neq 0$ and $max\{d(x_n, x_{n-1}), d(y_n, y_{n-1})\} \neq 0$ for all $n \in \mathbb{N}$. Then (3.3) becomes,

$$0 < max\{d(x_n, x_{n-1}), d(y_n, y_{n-1})\} - max\{d(x_{n+1}, x_n), d(y_{n+1}, y_n)\}$$

$$max\{d(x_{n+1}, x_n), d(y_{n+1}, y_n)\} < max\{d(x_n, x_{n-1}), d(y_n, y_{n-1})\}.$$

So, $(max\{d(x_{n+1}, x_n), d(y_{n+1}, y_n)\})$ is a non negative decreasing sequence. Using Theorem 2.3 we can show that (x_n) and (y_n) are Cauchy sequences.

Since A is a closed subset of a complete metric space X, there exists $x, y \in A$ such that $x_n \to x$ and $y_n \to y$. Therefore $(x_n, y_n) \to (x, y)$ in $A \times A$. Using the continuity of F, we have $F(x_n, y_n) \to F(x, y)$ and $F(y_n, x_n) \to F(y, x)$.

By the continuity of the metric function d implies that $d(x_{n+1}, F(x_n, y_n)) \to d(x, F(x, y))$ and $d(y_{n+1}, F(y_n, x_n)) \to d(y, F(y, x))$.

Since from (3.1) and (3.2) we see that the sequences $d(x_{n+1}, F(x_n, y_n))$ and $d(y_{n+1}, F(y_n, x_n))$ are constant sequences with the value $d(A, B)$. Therefore $d(x, F(x, y)) = d(A, B)$ and $d(y, F(y, x)) = d(A, B)$. Hence the proof of the theorem. □

Example 3.2. *Let* $X = \{(0, 1), (1, 0), (-1, 0), (0, -1)\}$ *and consider the usual order* $(x, y) \preceq (z, t) \Leftrightarrow x \leq z$ *and* $y \leq t$.

Thus $(X, \preceq)$ *is a partially ordered set, also* (X, d_2) *is a complete metric space considering* d_2 *the Euclidean metric. Let* $A = \{(0, 1), (1, 0)\}$ *and* $B = \{(0, -1), (-1, 0)\}$ *be a closed subset of* X. *Then* $d_2(A, B) = \sqrt{2}$, $A = A_0$ *and* $B = B_0$. *Let* $F : A \times A \to B$ *be defined as*

$$F((x_1, x_2), (y_1, y_2)) = (-x_2, -x_1).$$

Then it can be seen that F *is continuous such that* $F(A_0 \times A_0) \subseteq B_0$. *The only comparable pairs of points in* A *are* $x \preceq x$ *for* $x \in A$, *hence the proximal mixed monotone property and the proximally* $\mathscr{Z}$*-coupled contraction condition on* A *are satisfied trivially.*

It can be shown that the other hypothesis of the theorem are also satisfied. However, F *has three coupled best proximity points* $((0, 1), (0, 1))$, $((0, 1), (1, 0))$ *and* $((1, 0), (1, 0))$.

Theorem 3.3. *Assume the conditions (2.15) and (2.16) and A_0 is closed in X instead of continuity of F in Theorem 3.1, then the conclusion of Theorem 3.1 holds.*

Proof. Following the proof of Theorem 3.1, there exists sequences (x_n) and (y_n) in A satisfying the following conditions.

$$d(x_{n+1}, F(x_n, y_n)) = d(A, B) \quad \text{with } x_n \preceq x_{n+1}, \forall n \in \mathbb{N} \tag{3.4}$$

$$d(y_{n+1}, F(y_n, x_n)) = d(A, B) \quad \text{with } y_n \succeq y_{n+1}, \forall n \in \mathbb{N}. \tag{3.5}$$

Moreover, x_n converges to x and y_n converges to y in A. From (2.15) and (2.16) we get, $x_n \preceq x$ and $y_n \succeq y$. Note that the sequences (x_n) and (y_n) are in A_0 and A_0 is closed, we get $(x, y) \in A_0 \times A_0$. Since $F(A_0 \times A_0) \subseteq B_0$, we have $F(x, y), F(y, x) \in B_0$. Therefore there exists $(u, v) \in A_0 \times A_0$ such that,

$$d(u, F(x, y)) = d(A, B) \quad \text{and} \tag{3.6}$$

$$d(v, F(y, x)) = d(A, B) \tag{3.7}$$

Since $x_n \preceq x$ and $y_n \succeq y$ and using the proximally $\mathscr{Z}$-coupled contraction condition on A for (3.4), (3.5) and (3.6), (3.7), we get

$$0 \leq \zeta(max\{d(x_{n+1}, u), d(y_{n+1}, v)\}, max\{d(x_n, x), d(y_n, y)\}) \tag{3.8}$$

Case I: Suppose that $max\{d(x_{n+1}, u), d(y_{n+1}, v)\} = 0$, for some $n \in \mathbb{N}$, we get $x_{n+1} = u$ and $y_{n+1} = v$, which implies that $u \preceq x_{n+2}$ and $v \succeq y_{n+2}$. Note that $(x, y_{n+1}) \in A_0 \times A_0$, since $F(A_0 \times A_0) \subseteq B_0$ we have $F(x^*, F(x, y_{n+1})) = d(A, B)$ for some $x^* \in A_0$, $x_{n+1} \preceq x$ and $y_{n+1} \succeq y$ implies that

$$d(x_{n+2}, F(x_{n+1}, y_{n+1})) = d(A, B)$$
$$d(x^*, F(x, y_{n+1})) = d(A, B)$$

$$\Rightarrow x_{n+2} \preceq x^*$$

$$d(x^*, F(x, y_{n+1})) = d(A, B)$$
$$d(u, F(x, y)) = d(A, B)$$

$\Rightarrow x^* \preceq u$. Hence we have $x_{n+2} = u$. We can show that this is true for all $m \geq n$. Since $x_n \to x$, by uniqueness of the limit we get $x = u$, similarly $y = v$. Hence the claim.

In (3.8) suppose that $max\{d(x_n, x), d(y_n, y)\} = 0$ for some $n \in \mathbb{N}$. We get $x = x_n \preceq x_{n+1} \preceq x$ also, $y = y_n \succeq y_{n+1} \succeq y$, implies $x_{n+1} = x$ and $y_{n+1} = y$. We can show that this is true for all $m \geq n$. From (3.4) and (3.5) we get the conclusion.

Case II: In (3.8) suppose that $max\{d(x_{n+1}, u), d(y_{n+1}, v)\} \neq 0$, and $max\{d(x_n, x), d(y_n, y)\} \neq 0$ for all $n \in \mathbb{N}$, then we have

$$0 < max\{d(x_n, x), d(y_n, y)\} - max\{d(x_{n+1}, u), d(y_{n+1}, v)\}$$

As $n \to \infty$, the previous inequality becomes,

$$\lim_{n\to\infty} max\{d(x_{n+1}, u), d(y_{n+1}, v)\} \leq 0,$$

which implies that $x_{n+1} \to u$ and $y_{n+1} \to v$, since $x_n \to x$ and $y_n \to y$ and by the uniqueness of the limits we have, $x = u$ and $y = v$. From (3.6) and (3.7) we get $d(x, F(x, y)) = d(A, B)$ and $d(y, F(y, x)) = d(A, B)$. Hence the proof of the theorem. □

Theorem 3.4. *Adding condition (2.18) to the hypothesis of Theorem 3.1(resp. Theorem 3.3) we obtain the uniqueness of the coupled fixed point of F.*

Proof. Suppose that (x, y) and (z, t) are coupled best proximity points of F, that is

$$d(x, F(x, y)) = d(A, B),\ d(y, F(y, x)) = d(A, B) \text{ and}$$
$$d(z, F(z, t)) = d(A, B),\ d(t, F(t, z)) = d(A, B).$$

Let (u, v) be an element in $A_0 \times A_0$ which is comparable to (x, y) and (z, t). Suppose that $(u, v) \preceq (x, y)$ (similar proof holds for other cases also).

Since $F(A_0 \times A_0) \subseteq B_0$ and let $u_0 = u$ and $v_0 = v$, there exists $(u_1, v_1) \in A_0 \times A_0$ such that,

$$d(u_1, F(u_0, v_0)) = d(A, B) \text{ and}$$
$$d(v_1, F(v_0, u_0)) = d(A, B)$$

We claim that $(u_n, v_n) \preceq (x, y)$ for each $n \in \mathbb{N}$. From lemma 1.13 and lemma 1.14 we get,

$$\left.\begin{array}{l} u_0 \preceq x \text{ and } v_0 \succeq y, \\ d(u_1, F(u_0, v_0)) = d(A, B) \\ d(x, F(x, y)) = d(A, B) \end{array}\right\} \Rightarrow u_1 \preceq x$$

and

$$\left.\begin{array}{l} u_0 \preceq x \text{ and } v_0 \succeq y, \\ d(v_1, F(v_0, u_0)) = d(A, B) \\ d(y, F(y, x)) = d(A, B) \end{array}\right\} \Rightarrow v_1 \succeq y.$$

From the above two inequalities we get, $(u_1, v_1) \preceq (x, y)$. Continuing this process, we get sequences (u_n) and (v_n) such that $d(u_{n+1}, F(u_n, v_n)) = d(A, B)$ and $d(v_{n+1}, F(v_n, u_n)) = d(A, B)$ with $(u_n, v_n) \preceq (x, y)$ for all $n \in \mathbb{N}$. Using the proximally $\mathscr{Z}$-coupled contraction condition on A, we get

$$\left.\begin{array}{l} u_n \preceq x \text{ and } v_n \succeq y, \\ d(u_{n+1}, F(u_n, v_n)) = d(A, B) \\ d(x, F(x, y)) = d(A, B) \\ d(v_{n+1}, F(v_n, u_n)) = d(A, B) \\ d(y, F(y, x)) = d(A, B) \end{array}\right\}$$
$$\Rightarrow 0 \leq \zeta(max\{d(u_{n+1}, x), d(v_{n+1}, y)\}, max\{d(u_n, x), d(v_n, y)\}) \tag{3.9}$$

Suppose that $max\{d(u_{n+1}, x), d(v_{n+1}, y)\} = 0$, for some $n \in \mathbb{N}$ implies that $u_{n+1} = x$ and $v_{n+1} = y$. Note that,

$$F(u_{n+2}, F(x, y)) = d(A, B)$$
$$F(x, F(x, y)) = d(A, B)$$

using the proximal mixed monotone property $u_{n+2} = x$ and $v_{n+2} = y$, this is true for all $m \geq n$. Using the argument in Theorem 2.6 we get, $x = z$ and $y = t$.

Suppose that in (3.9), $max\{d(u_{n+1}, x), d(v_{n+1}, y)\} \neq 0$ and $max\{d(u_n, x), d(v_n, y)\} \neq 0$, then we have

$$0 < max\{d(u_n, x), d(v_n, y)\} - max\{d(u_{n+1}, x), d(v_{n+1}, y)\}$$

which implies that the sequence $(max\{d(u_n, x), d(v_n, y)\})$ is a decreasing sequence. Now apply the same argument in Theorem 2.6 we get, $x = z$ and $y = t$. Hence our claim. □

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