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Existence Results for a Nonlocal Superlinear Problems Involving $p(x)-$Laplacian Near to Zero

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ABSTRACT: In this work, by using a variational approach, we give a result on the existence and multiplicity of solutions concerned a class of nonlocal elliptic problems with variable exponent.

Key Words: $p(x)-$Kirchhoff, $p(x)-$Laplacian , Critical point.

Contents

1. Introduction

Partial differential equations related to a kind of Kirchhoff problem has been a very active field of investigation in recent years. This type of equations describes many natural phenomena like elastic mechanics, image restoration, electrorheological fluids, biological systems where such solution modelizes a process depending on the average of itself. See for example, [7,21,23] and its references.

Such equation is related to the stationary version of a model in general version introduced by Kirchhoff [16]. More precisely, Kirchhoff suggested a model

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0,$$

where ρ_0, ρ, L and h are constants associated to the effects of the changes in the length of strings during the vibrations. It is an extension of the classical D'Alembert's wave equation.

The authors in [4] have investigated the Kirchhoff type equation involving the $p(x)-$Laplacian of the form

$$u_{tt}-M(\int_\Omega\frac{1}{p(x)}\mid\nabla u\mid^{p(x)}dx)\Delta_{p(x)}u+Q(t,x,u,u_t)+f(x,u)=0.$$

They have introduced the asymptotic stability, as time tends to infinity. This type of Kirchhoff problems with stationary process has received considerable attention by several researchers, see for example [1,2,3,4,6,5,8,12,14,17,18,19] and the references therein.

We consider the following nonlocal problem

$$\begin{gathered} h\left(\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\left(-\Delta_{p(x)}u\right)=f(x,u)\quad\text{in }\Omega,\\ u=0\quad\text{on }\partial\Omega, \end{gathered}\tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary, $h:\ \mathbb{R}_+\to\mathbb{R}$ is a continuous function and the operator $\Delta_{p(x)}u:=div(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)-$Laplacian with $p\in C(\overline{\Omega})$ satisfies

$$1<p^-:=\inf_{x\in\Omega}p(x)\le p(x)\le p^+:=\sup_{x\in\Omega}p(x)<\infty$$

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and $N \geq 2, q \in C_+(\overline{\Omega})$, with $C_+(\overline{\Omega})$ is defined by

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) \ and \ \inf_{x\in\overline{\Omega}} p(x) > 1\}.$$

For the function h we suppose the following conditions:
(h_1) There exist $h_3 > h_1 > 0, h_2 > 0$ and $\beta > 1$ such that

$$h_1 \leq h(t) \leq h_2|t|^\beta + h_3.$$

(h_2) There exists $\lambda \in (0, 1]$ such that

$$H(t) \leq \lambda h(t)t$$

where $H(t) = \int_0^t h(s)\, ds$.

We also give the appropriate assumptions on f in order to state the basic result of this paper,
(f_1) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and there exists $1 < q(x) < p^*(x)$, such that

$$|f(x, t)| \leq C_1 + C_2|t|^{q(x)-1}$$

for a.e $x \in \Omega, t \in \mathbb{R}$, with C_1 and C_2 are positives constants and

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} \ if \ p(x) < N, \\ +\infty \ if \ p(x) \geq N. \end{cases}$$

(f_2) $\lim\limits_{|t|\to 0} \dfrac{f(x,t)}{|t|^{p^- -2}t} = \infty.$

(f_3) $f(x, 0) = 0,\ L(x, t) = p^- F(x, t) - f(x, t)t > 0$ for every $x \in \Omega,\ |t| \leq \delta$ and $t \neq 0$ where

$$F(x, t) = \int_0^t f(x, s)\, ds,$$

$\delta > 0$ be small enough.

(f_4) When t is small, f is odd with respect to the second argument.

Now, we present our main result,

Theorem 1.1. *Under the assumptions* $(f_1) - (f_4)$, $(h_1), (h_2)$, *the problem* (1.1) *has a sequence of weak solutions such that* $\|u_n\|_{L^\infty} \to 0$ *as* $n \to \infty$.

It is known that the $p(x)-$laplacian operator possesses more complicated nonlinearities than the $p-$Laplacian operator, mainly due to the fact that it is not homogeneous. As far as we are aware, contributions discussed a nonlocal problem involving $p(x)-$laplacian operator have seldom been studied. So it is necessary for us to investigate the related problems deeply. Here, a distinguishing feature that we have assumed some conditions only at zero, however, there are no conditions imposed on f at infinity which is necessary in many works, we borrow the main ideas from [20], Tan and Fang in [22].

This paper is organized as follows. In Section 2, we recall some preliminaries on variable exponent spaces. In Sections 3, we give the proof of result and we establish the existence and multiplicity of solutions via a variational structure.

2. Preliminaries

To deal with problem (1.1), we need some theory of variable exponent Lebesgue-Sobolev spaces. For convenience, we only recall some basic facts which will be used later. For more details, we refer to [9,13]. Let Ω be a bounded domain of $\mathbb{R}^N$, denote

$$C_+(\overline{\Omega}) = \left\{p(x) : p(x) \in C(\overline{\Omega}),\ p(x) > 1,\ \text{for all } x \in \overline{\Omega}\right\},$$

$$p^+ = \max\{p(x) : x \in \overline{\Omega}\}, \quad p^- = \min\{p(x); x \in \overline{\Omega}\},$$

$$L^{p(x)}(\Omega) = \left\{u : u \text{ is a measurable real-valued function, } \int_\Omega |u(x)|^{p(x)}dx < \infty\right\},$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \le 1\}$$

becomes a Banach space . We also define the space

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u(x)|_{L^{p(x)}(\Omega)} + |\nabla u(x)|_{L^{p(x)}(\Omega)}.$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Of course the norm $\|u\| = |\nabla u|_{L^{p(x)}(\Omega)}$ is an equivalent norm in $W_0^{1,p(x)}(\Omega)$. In this paper, we denote by $X = W_0^{1,p(x)}(\Omega)$.

Proposition 2.1. *(cf. [9]) (i) The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\int_\Omega |uv| dx \le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)|u|_{p(x)}|v|_{p'(x)} \le 2|u|_{p(x)}|v|_{p'(x)};$$

(ii) If $p_1(x), p_2(x) \in C_+(\overline{\Omega})$ and $p_1(x) \le p_2(x)$ for all $x \in \overline{\Omega}$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.

Proposition 2.2. *(cf. [13]) Set $\rho(u) = \int_\Omega |\nabla u(x)|^{p(x)} dx$, then for $u \in X$ and $(u_k) \subset X$, we have*

(1) $\|u\| < 1$ (respectively$= 1; > 1$) if and only if $\rho(u) < 1$ (respectively$= 1; > 1$);

(2) for $u \ne 0$, $\|u\| = \lambda$ if and only if $\rho(\frac{u}{\lambda}) = 1$;

(3) if $\|u\| > 1$, then $\|u\|^{p^-} \le \rho(u) \le \|u\|^{p^+}$;

(4) if $\|u\| < 1$, then $\|u\|^{p^+} \le \rho(u) \le \|u\|^{p^-}$;

(5) $\|u_k\| \to 0$ (respectively $\to \infty$) if and only if $\rho(u_k) \to 0$ (respectively $\to \infty$).

For $x \in \Omega$, let us define

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \ge N. \end{cases}$$

Proposition 2.3. *([13]) If $q \in C_+(\overline{\Omega})$ and $q(x) \le p^*(x)$ $(q(x) < p^*(x))$ for $x \in \overline{\Omega}$, then there is a continuous (compact) embedding $X \hookrightarrow L^{q(x)}(\Omega)$.*

Definition 2.4. *We say that $u \in X$ is a weak solution of (1.1), if*

$$h\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx\right) \int_\Omega (|\nabla u|^{p(x)-2} \nabla u \nabla v\, dx - \int_\Omega f(x,u) v\, dx = 0,$$

$\forall v \in X$.

In order to study (1.1) by means of variational methods, we introduce the functional associated

$$\phi(u) = H\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx\right) - \int_\Omega F(x,u) dx,$$

for $u \in X$.

3. Proof of the main result

Firstly, we shall recall the following interesting proposition:

Proposition 3.1. *[15] Let $\phi \in C^1(X,\mathbb{R})$ where X is a Banach space. Assume ϕ satisfies the (P.S.) condition, is even and bounded from below, and $\phi(0)=0$. If for any $n \in \mathbb{N}$, there exists a k-dimensional subspace X_n and $\rho_n > 0$ such that*

$$\sup_{X_n \cap S_{\rho_n}} \phi < 0;$$

where $S_\rho := \{u \in X : \|u\| = \rho\}$, then ϕ has a sequence of critical values $c_n < 0$ satisfying $c_n \to 0$ as $n \to \infty$.

We split the proof of the above result into three claims as follows.
Claim 1: There exist $\delta > 0$ and $\widetilde{f} \in C(\Omega \times \mathbb{R})$ such that $\widetilde{f}$ is odd and

$$\widetilde{f}(x,t) = f(x,t) \ \text{ for } |t| < \frac{\delta}{2}. \tag{3.1}$$

$$p^- \widetilde{F}(x,t) - \widetilde{f}(x,t)t \geq 0, \ \text{ for } (x,t) \in \Omega \times \mathbb{R} \tag{3.2}$$

and

$$p^- \widetilde{F}(x,t) - \widetilde{f}(x,t)t = 0 \quad \text{for} \quad |t| > \delta (or\ t = 0), \tag{3.3}$$

where

$$\widetilde{F}(x,t) = \int_0^t \widetilde{f}(x,s)ds.$$

Proof: Firstly let define

$$\widetilde{F}(x,t) = g(t)F(x,t) + (1 - g(t))d|t|^{p^-}$$

where d is a positive constant and g is a cut-off function presented as follows:

$$g(t) = \begin{cases} 1\ if\ |t| \leq \frac{\delta}{2}, \\ 0\ if\ |t| \geq \delta, \end{cases}$$

and

$$g'(t)t \leq 0, \ |g'(t)| \leq \frac{4}{\delta}.$$

For $|t| \leq \frac{\delta}{2}$, easily (3.1) holds.

Meanwhile, we have

$$\widetilde{f}(x,t) = \frac{\partial}{\partial t}\widetilde{F}(x,t) = g'(t)F(x,t) + g(t)f(x,t) + (1-g(t))(d|t|^{p^-})' - dg'(t)|t|^{p^-}.$$

It is immediate to see that when $|t| \geq \delta$

$$p^- \widetilde{F}(x,t) = p^- d|t|^{p^-},$$

therefore (3.3) is satisfied. In the rest, from (f_2), we may choose $\delta > 0$ sufficiently small in order to get $F(x,t) \geq d|t|^{p^-}$ when $t \in [\frac{\delta}{2}, \delta]$ and due to the fact that $g'(t)t \leq 0$ we obtain the formula (3.2).□
We introduce the functional

$$\widetilde{\phi}(u) = H\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx\right) - \int_\Omega \widetilde{F}(x,u)dx,$$

for $u \in X$.

Claim 2: If $\widetilde{\phi}(u)=\widetilde{\phi}'(u).u=0$ then $u=0$. Otherwise, $u\neq 0$. From our assumption,

$$\int_\Omega \widetilde{F}(x,u)dx = H\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)$$

and

$$h\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\left(\int_\Omega |\nabla u|^{p(x)}dx\right)=\int_\Omega f(x,u)u\,dx,$$

so we get

$$\begin{aligned} p^-\int_\Omega F(x,u)dx &= p^-H\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\\ &\leq p^-\lambda h\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\\ &\leq p^-\frac{1}{p^-}\lambda h\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\left(\int_\Omega |\nabla u|^{p(x)}dx\right)\\ &< \int_\Omega \widetilde{f}(x,u)u\,dx, \end{aligned} \tag{3.4}$$

which contradicts the condition (3.2).

Claim 3: The functional $\widetilde{\phi}$ satisfies the Palais-Smale condition (P.S):

According to previous Claim 2, it is easy to check that $\widetilde{\phi}$ is even and $\widetilde{\phi}\in C(X,\mathbb{R})$. Meanwhile, for $\|u\|>1$ we have

$$\begin{aligned} \widetilde{\phi}(u) &= H\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)-\int_\Omega \widetilde{F}(x,u)\,dx\\ &\geq h_1\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)-A\int_\Omega |u|^{p^-}\,dx\\ &\geq \frac{h_1}{p^+}\|u\|^{p^-}-B\|u\|^{p^-}, \end{aligned} \tag{3.5}$$

with A and B are two positive constants. Then $\widetilde{\phi}$ is coercive i.e $\widetilde{\phi}(u)\to\infty$ when $\|u\|\to\infty$. Hence any $(P.S)_c$ sequence of $\widetilde{\phi}$ is bounded. So by a standard argument shows that $\widetilde{\phi}$ verifies $(P.S)_c$ condition on X for all c.

Proof of Theorem 1.1:

We modify and extend $f(x,u)$ to get $\widetilde{f}(x,u)\in C(\Omega\times\mathbb{R},\mathbb{R})$ verifying the mentioned properties of Proposition 3.1. .

For any $k\in\mathbb{N}$ we get k independent smooth functions ψ_i for $i=1,...k$ and define the subspace $X_k:=span\{\psi_1,...,\psi_k\}$. Meanwhile, form the Claim 1, for $\|u\|<1$ we can obtain

$$\begin{aligned} \widetilde{\phi}(u) &\leq \frac{h_2}{\beta}\left(\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^\beta+h_3\left(\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)-C\int_\Omega |u|^{p^-}\,dx\\ &\leq \frac{h_2}{\beta(p^-)^\beta}\|u\|^{\beta p^-}+\frac{h_3}{p^-}\|u\|^{p^-}-C\int_\Omega |u|^{p^-}\,dx. \end{aligned} \tag{3.6}$$

Taking (3.6) in account and as all norms in X_k are equivalent, so for sufficiently small $\rho_k>0$ and for C sufficiently large, we get

$$\sup_{X_k\cap S_{\rho_n}}\widetilde{\phi}<0.$$

We see that the assertions of proposition 3.1 are fulfilled, and then there exist a sequence of negative critical values c_k for the functional $\widetilde{\phi}$ verifies $c_k\to 0$ when k is large enough.

Thereby, for any $u_k \in X$ satisfying $\widetilde{\phi}(u_k) = c_k$ and $\widetilde{\phi}'(u_k) = 0$, we have $(u_k)_k$ is $(P.S)_0$ sequence of $\widetilde{\phi}(u)$. Passing, if necessary, to a subsequence still denoted by (u_k), we may assume that $(u_k)_k$ has a limit.

According to Claim 1 and Claim 2, we know that the value 0 is the only critical point when the energy is zero what implies that the subsequence of (u_k) converges to 0. By the results of regularity in [10,11],

$$(u_k) \in C(\overline{\Omega}) \quad \text{and} \quad |u_k|_{L^\infty} \to 0 \qquad \text{as} \quad k \to \infty.$$

So from the Claim 1, we have

$$|u_n|_{C(\Omega)} \leq \frac{\delta}{2}.$$

Thereby the sequence $(u_k)_k$ are solutions of the problem (1.1). □

Remark 3.2. *Assume the following hypothesis:*
(f_2') $f \in C(\Omega \times (-\delta, \delta), \mathbb{R})$ *such that*

$$\lim_{|t| \to 0} \frac{F(x,t)}{|t|^{p^-}} = \infty$$

instead of (f_2)*, so by the same method, with slightly modified lines, we may obtain the above result. An example of function satisfies our hypotheses is*

$$F(x,t) = \frac{c(x)}{p^-}|t|^{p^-} \; log\left(\frac{1}{|t|}\right) \; c(x) > 0.$$

Assume that
(g) $g(x,u)$ is continuous, odd in u in a neighborhood of 0 and there exists

$$1 < r(x) < p^*(x),$$

such that

$$|g(x,t)| \leq C(1 + |t|^{r(x)-1})$$

for a.e $x \in \Omega, t \in \mathbb{R}$, with C be positive constant and we have

$$\liminf_{t \to 0} \frac{G(x,t)}{F(x,t)} > -1$$

and

$$\liminf_{t \to 0} \frac{L_0(x,t)}{L(x,t)} > -1$$

uniformly in Ω, where

$$L_0(x,t) = p^- G(x,t) - g(x,t)t > 0$$

for every $x \in \Omega$ and $t \neq 0$ and $G(x,t) = \int_0^t g(x,s)\,ds$.

As an application of Theorem 1.1, we have the following result.

Corollary 3.3. *Let us consider the following problem*

$$\begin{gathered} h\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} dx\right)\left(-\Delta_{p(x)} u\right) = f(x,u) + g(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.7}$$

Under the assumptions $(f_1) - (f_4)$, $(h_1), (h_2)$, *and* (g), *the problem* (3.7) *has a sequence of weak solutions such that* $\|u_n\|_{L^\infty} \to 0$ *as* $n \to \infty$.

proof: From the condition (g), we can see that the new nonlinear term $f(x,t) + g(x,t)$ still satisfies the above conditions used in the main result, so by the same lines in proof of Theorem 1.1, we infer that the problem (3.7) has a sequence of solutions with $\|u_n\|_{L^\infty} \to 0$ as $n \to \infty$.

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