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Cosine Families in GDP Quojection-Fréchet Spaces

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abstract: We prove that if the Quojection-Fréchet space X is a Grothendieck space with the Dunford-Pettis property, then every C_0-cosine family is necessarily uniformly continuous and therefore its infinitesimal generator is a continuous linear operator.

Key Words: Strongly continuous cosine families, Semi-groups of operators, Locally convex spaces, Quojection Fréchet spaces.

Contents

1. Introduction

A C_0-cosine family $\{C(t)\}_{t\geq 0}$ in a Banach space X is a family of bounded linear operators in X satisfying the D'Alembert functional equation (see Definition 1.1), $C(0) = I$ and $\lim\limits_{t\longrightarrow 0^+} C(t)x = x$, for all $x \in X$. This notion was introduced by M.Sova in 1966 [12], which associates to each C_0-cosine family an operator called the infinitesimal generator.

It is well-known, in the classical case (where X is a Banach space), that a C_0-cosine family $\{C(t)\}_{t\geq 0}$ is exponentially equicontinuous in X [4,8,10]. Therefore the family $\{C(t)\}_{t\geq 0}$ is strongly continuous in X [12], i.e. the map $C(.)x$ is continuous in $\mathbb{R}^+$, for all $x \in X$. This means that the notions "strongly continuous cosine family" and "C_0-cosine family" coincide. The infinitesimal generator A of a C_0-cosine family is a closed operator and is densely defined. The link between the family $\{C(t)\}_{t\geq 0}$ and its generator is given by the relation: $\lambda R(\lambda^2, A) = \int_0^\infty e^{-\lambda t} C(t)dt$, for some $\lambda \in \mathbb{C}$, with $R(\lambda^2, A) = (\lambda^2 I - A)^{-1}$ [4,8]. If in addition the family $\{C(t)\}_{t\geq 0}$ is uniformly continuous (i.e. $\parallel C(t) - I \parallel \longrightarrow 0$, as $t \longrightarrow 0^+$), then the infinitesimal generator is a bounded operator in X [9,13].

If X is a Hausdorff locally convex space, we say that $\{C(t)\}_{t\geq 0}$ is a strongly continuous cosine family if it satisfies the D'Alembert functional equation, $C(0) = I$ and $C(t) \longrightarrow C(t_0)$ in $\mathcal{L}_s(X)$ as $t \longrightarrow t_0$, for all $t_0 \geq 0$ [3]. If instead of the last condition, the family $\{C(t)\}_{t\geq 0}$ verifies $C(t) \longrightarrow I$ in $\mathcal{L}_s(X)$ as $t \longrightarrow 0^+$, we say that $\{C(t)\}_{t\geq 0}$ is a C_0-cosine family.

In the second section, we are interested studying in general the C_0-cosine families in Hausdorff locally convex spaces and we rely on the work of M.Sova, on the Banach spaces [12], to show that any locally equicontinuous C_0-cosine family in X (h.l.c.s.) is strongly continuous.

By a well-known analogy of K.Yosida [13] for semi-groups, we show in the third section the existence of the resolvent of the infinitesimal generator of an exponentially equicontinuous C_0-cosine family on X.

In the case of Fréchet spaces, we have given in [3] an example of a uniformly continuous cosine family whose infinitesimal generator is not defined everywhere in the space, and we showed that if the space is a Quojection, the infinitesimal generator of all uniformly continuous cosine family is a continuous operator. In the fourth section, we begin with a proposition that gives an answer of Conejero's question [6] in the case of cosine families in the space $\boldsymbol{\omega} = \mathbb{C}^{\mathbb{N}}$. Finally, we show that if X is a GDP Quojection-Fréchet space, then all C_0-cosine family in X is uniformly continuous, therefore its infinitesimal generator is a continuous linear operator on X.

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2. C_0-Cosine family of operators in locally convex spaces:

Let X be a sequentially complete locally convex Hausdorff space and Γ_X a system of continuous seminorms determining the topology of X. The strong topology τ_s in the space $\mathcal{L}(X)$, of all continuous linear operators from X into itself, is determined by the family of seminorms:

$$q_x(S) = q(Sx), \quad S \in \mathcal{L}(X),$$

for each $x \in X$ and $q \in \Gamma_X$, in this case we write $\mathcal{L}_s(X)$.

Denote by $B(X)$ the collection of all bounded subsets of X. The topology τ_b of uniform convergence on the elements of $B(X)$ is defined by the family of seminorms:

$$q_B(S) = \sup_{x\in B} q(Sx), \quad S \in \mathcal{L}(X),$$

for each $B \in B(X)$ and $q \in \Gamma_X$, in this case we write $\mathcal{L}_b(X)$.
For a Banach space X, τ_b is the operator norm topology in $\mathcal{L}(X)$. If Γ_X is countable and X is complete, then X is called a Fréchet space.

Definition 2.1. *Let $\{C(t)\}_{t\geq 0} \subseteq \mathcal{L}(X)$ be a family of operators verifying the following properties:*

1. $C(0) = I$.
2. $2C(t)C(s) = C(t+s) + C(t-s)$, *for all* $s, t \geq 0$, *with* $t \geq s$.
   (D'Alembert functional equation)

i) *We say that $\{C(t)\}_{t\geq 0}$ is a C_0-cosine family if:*

$$C(t) \longrightarrow I \text{ in } \mathcal{L}_s(X) \text{ as } t \longrightarrow 0^+.$$

ii) *We say that $\{C(t)\}_{t\geq 0}$ is a uniformly continuous C_0-cosine family if:*

$$C(t) \longrightarrow I \text{ in } \mathcal{L}_b(X) \text{ as } t \longrightarrow 0^+.$$

Definition 2.2. *Let $\{C(t)\}_{t\geq 0}$ be a family of continuous linear operators in X.*

1. *We say that $\{C(t)\}_{t\geq 0}$ is exponentially equicontinuous of order ω if the set $\{e^{-\omega t}C(t)\}_{t\geq 0}$ is equicontinuous, i.e. $\forall p \in \Gamma_X$, $\exists q \in \Gamma_X$, $\exists M \geq 0$ such that:*

$$p(C(t)x) \leq Me^{\omega t}q(x), \text{ for all } x \in X, \text{ and } t \geq 0.$$

2. *We say that $\{C(t)\}_{t\geq 0}$ is locally equicontinuous if for each $t_0 \geq 0$ the set $\{C(t), 0 \leq t \leq t_0\}$ is equicontinuous. i.e. $\forall p \in \Gamma_X$, $\exists q \in \Gamma_X$, $\exists M \geq 0$ such that:*

$$p(C(t)x) \leq Mq(x), \text{ for all } x \in X, \text{ and } t \in [0, t_0].$$

It's clear that every exponentially equicontinuous family is necessarily locally equicontinuous.

It is known that every C_0-cosine family in a Banach space is necessary exponentially equicontinuous [8,12]. For Fréchet spaces this need not be the case. For example, in the sequence space $\boldsymbol{\omega} = \mathbb{C}^{\mathbb{N}}$, the family:

$$C(t)x = (cosh(nt)x_n)_{n\geq 1} = (\frac{e^{nt}+e^{-nt}}{2}x_n)_{n\geq 1}, \text{ for all } x = (x_n)_{n\geq 1} \in \boldsymbol{\omega}, \text{ and } t \geq 0,$$

defines a C_0-cosine family in $\boldsymbol{\omega}$ which is not exponentially equicontinuous.

Based on M. Sova's theorem ([12], Theorem 2.7), which shows that for every C_0-cosine family, in a Banach space, the map $C(.)x$ is continuous in $\mathbb{R}^+$ for every $x \in X$. The following proposition generalizes this result in locally convex spaces.

Proposition 2.3. *Let $\{C(t)\}_{t\geq 0}$ be a locally equicontinuous C_0-cosine family in locally convex space X. Then, for all $x \in X$, and $t_0 \geq 0$:*

$$\lim_{t\longrightarrow t_0} C(t)x = C(t_0)x.$$

i.e. $\{C(t)\}_{t\geq 0}$ is strongly continuous.

Proof. Suppose that there exist $x_0 \in X$ and $t_0 \geq 0$ such that $C(.)x_0$ is not continuous at t_0, i.e. $\exists \varepsilon > 0$, $\exists q \in \Gamma_X$, $\forall \eta > 0$, $\exists t \geq 0$ such that $|t - t_0| \leq \eta$ and $q(C(t)x_0 - C(t_0)x_0) \geq \varepsilon$.
For each $n \in \mathbb{N}^*$, put:

$$K_n = \sup\{q(C(t)x_0 - C(s)x_0) : \quad |t - t_0| \leq \frac{t_0}{8n}, \quad |s - t_0| \leq \frac{t_0}{8n}, \quad s, t \in \mathbb{R}^+ \text{ with } t \neq s\}$$

1°/ There is $K > 0$ such that $K_n \geq K$, for all $n \in \mathbb{N}^*$. Indeed:
First, we have $(K_n)_{n=1}^{\infty}$ is a decreasing sequence of positive numbers, therefore there exists $K \in \mathbb{R}^+$ such that $K_n \longrightarrow K$. According to the hypothesis, we obtain that $K > 0$, hence the result.
2°/ There exist $(\tau_n)_n \subset \mathbb{R}^+$ and $(\sigma_n)_n \subset \mathbb{R}^+$, with $\tau_n < \sigma_n$, for all $n \in \mathbb{N}^*$, such that:

$$|\tau_n - t_0| \leq \frac{t_0}{8n}, \quad |\sigma_n - t_0| \leq \frac{t_0}{8n}, \text{ and } q(C(\tau_n)x_0 - C(\sigma_n)x_0) \geq K_n - \frac{1}{n}, \text{ for all } n \in \mathbb{N}^*.$$

3°/ For all $n \in \mathbb{N}^*$, $2\tau_n - \sigma_n \geq 0$. Indeed:
For each $n \in \mathbb{N}^*$ we have $\sigma_n - \tau_n \leq \frac{t_0}{4n}$ and $\tau_n \geq t_0 - \frac{t_0}{8n} = \frac{8n-1}{8n} t_0 \geq \frac{t_0}{2n} > \frac{t_0}{4n}$.
Which implies $2\tau_n - \sigma_n \geq 0$, for all $n \in \mathbb{N}^*$.
4°/For all $n \in \mathbb{N}^*$, $q(C(\sigma_{4n})x_0 - C(2\tau_{4n} - \sigma_{4n})x_0) \leq K_n$. Indeed:
Just show that for all $n \in \mathbb{N}^*$ we have $|\sigma_{4n} - t_0| \leq \frac{t_0}{8n}$ and $|2\tau_{4n} - \sigma_{4n} - t_0| \leq \frac{t_0}{8n}$.
Let $n \in \mathbb{N}^*$ we have:
$|2\tau_{4n} - \sigma_{4n} - t_0| \leq |\sigma_{4n} - \tau_{4n}| + |\tau_{4n} - t_0| \leq \frac{t_0}{16n} + \frac{t_0}{32n} \leq \frac{t_0}{8n}$, hence the result.
5°/ $\lim_{n\longrightarrow\infty} K_n = 0$. Indeed:
Let $s, t \in \mathbb{R}^+$ with $s \leq t$, we have:

$$2C(t)(C(s) - I) = C(t - s) - C(t + s) + 2(C(t + s) - C(t)).$$

On the other hand, for all $p \in \Gamma_X$, and $x \in X$ we have:

$$2p(C(t + s)x - C(t)x) \leq 2p(C(t)(C(s) - I)x) + p(C(t + s)x - C(t - s)x).$$

Now, for each $n \in \mathbb{N}^*$ we take: $t = \tau_{4n}$, $s = \sigma_{4n} - \tau_{4n}$, $x = x_0$, and $p = q$, we obtain:

$$2q(C(\sigma_{4n})x_0 - C(\tau_{4n})x_0) \leq 2q(C(\tau_{4n})(C(\sigma_{4n} - \tau_{4n}) - I)x_0) + q(C(\sigma_{4n})x_0 - C(2\tau_{4n} - \sigma_{4n})x_0).$$

According to 2°/ and 4°/ we have:

$$2(K_{4n} - \frac{1}{4n}) \leq 2q(C(\tau_{4n})(C(\sigma_{4n} - \tau_{4n}) - I)x_0) + K_n, \text{ for all } n \in \mathbb{N}^*.$$

Therefore

$$K_{4n} \leq 2q(C(\tau_{4n})(C(\sigma_{4n} - \tau_{4n}) - I)x_0) + \frac{1}{2n} + (K_n - K_{4n}), \text{ for all } n \in \mathbb{N}^*.$$

Since $\{C(t)\}_{t\geq 0}$ is locally equicontinuous in X, there exist $q^* \in \Gamma_X$, and $M \geq 0$ such that:

$$K_{4n} \leq 2Mq^*((C(\sigma_{4n} - \tau_{4n}) - I)x_0) + \frac{1}{2n} + (K_n - K_{4n}), \text{ for all } n \in \mathbb{N}^*.$$

As $\lim_{n\longrightarrow\infty} \tau_{4n} = t_0$, $\lim_{n\longrightarrow\infty} \sigma_{4n} = t_0$ and $\lim_{t\longrightarrow 0} C(t)x_0 = x_0$, thus $\lim_{n\longrightarrow\infty} K_n = 0$.
Which is contradict 1°/, hence the result.

□

The previous proposition show that if the C_0-cosine family $\{C(t)\}_{t\geq 0}$ is locally equicontinuous, then the following conditions are equivalent:

$$i)\quad C(t) \longrightarrow I \text{ in } \mathcal{L}_s(X) \text{ as } t \longrightarrow 0^+.$$

$$i')\quad C(t) \longrightarrow C(t_0) \text{ in } \mathcal{L}_s(X) \text{ as } t \longrightarrow t_0, \text{ for all } t_0 \geq 0.$$

In this case, the notions "C_0-cosine family" and "strongly continuous cosine family" coincide.

By the same procedure, and under the condition of locally equicontinuity, we have the equivalence between ii) of Definition 1. and the condition:

$$ii')\quad C(t) \longrightarrow C(t_0) \text{ in } \mathcal{L}_b(X) \text{ as } t \longrightarrow t_0, \text{ for all } t_0 \geq 0.$$

Remark 2.4. *Every locally equicontinuous C_0-cosine family $\{C(t)\}_{t\geq 0}$ commute. Indeed:*
Let $t > 0$, According to Definition 1.(ii) we have $C(t) = 2(C(\frac{t}{2}))^2 - I$, which implies that for all $n \geq 0$, $C(t)$ is polynomial in $C(\frac{t}{2^n})$, so they commute.

Assume that $C(t)$ commutes with $C(\frac{rt}{2^n})$ for $r = 1, 2, ..., k$ for some $k \geq 1$. Since

$$C(\frac{(k+1)t}{2^n}) = 2C(\frac{kt}{2^n})C(\frac{t}{2^n}) - C(\frac{(k-1)t}{2^n}).$$

It follows that $C(t)$ commutes with $C(\frac{(k+1)t}{2^n})$. Then $C(t)$ commutes with $C(\frac{rt}{2^n})$ for all integers $r, n \geq 1$.

Now, let $s \geq 0$ we take $r = [\frac{s2^n}{t}]$. Then $C(t)$ commute with $C([\frac{s2^n}{t}]\frac{t}{2^n})$.

Since $[\frac{s2^n}{t}]\frac{t}{2^n} \longrightarrow s$ as $n \longrightarrow \infty$ and $\{C(t)\}_{t\geq 0}$ is strongly continuous by Proposition 2.1, then $C(t)$ commute with $C(s)$. Hence the result.

Definition 2.5. *Let $\{C(t)\}_{t\geq 0}$ be a C_0-cosine family in X. The infinitesimal generator is the linear operator A defined in $D(A)$ by:*

$$Ax = \lim_{t \longrightarrow 0^+} \frac{2}{t^2}(C(t)x - x)$$

Where $D(A) = \{x \in X / \lim_{t\longrightarrow 0} \frac{2}{t^2}(C(t)x - x)$ exist in $X\}$.

Proposition 2.6. *Let $\{C(t)\}_{t\geq 0}$ be a locally equicontinuous C_0-cosine family in X and A its infinitesimal generator. And let $x, y \in X$, then:*

1. *For all $t \geq 0$ it holds $\lim\limits_{h\longrightarrow 0^+} \frac{2}{h^2}\int_t^{t+h}(t + h - s)C(s)xds = C(t)x$.*
2. *For all $t \geq 0$ we have $\int_0^t (t - s)C(s)xds \in D(A)$ and $A\int_0^t (t - s)C(s)xds = C(t)x - x$.*
3. *$D(A)$ is dense in X and $A : D(A) \to X$ is a closed operator.*
4. *$x \in D(A)$ and $Ax = y$ if and only if $C(t)x - x = \int_0^t (t - s)C(s)yds$ for all $t \geq 0$.*
5. *For $x \in D(A)$ the mapping $[0, \infty) \to X$, $t \mapsto C(t)x$ is twice continuous differentiable, $C(t)x \in D(A)$, and $AC(t)x = C(t)Ax = \frac{d^2}{dt^2}C(t)x$ for all $t \geq 0$.*

Proof. While 1. is [Proposition 1. [3]] it follows from Proposition 2.3 that $\{C(t)\}_{t\geq 0}$ is strongly continuous. Thus, 2. follows from [Corollary 1. [3]], 3. is a consequence of [Proposition 2. and Corollary 2. [3]] while 4. is implied by [Proposition 4. [3]]. Finally, if $x \in D(A)$ we have: $C(t)x = x + t\int_0^t C(s)Axds - \int_0^t sC(s)Axds$ by 4. which yields 5. □

3. The Resolvent of infinitesimal generator:

In this section we study some spectral properties of the infinitesimal generator, which are important for our main result.

We begin with the following lemma for ease of reading.

Lemma 3.1. *Let $\{C(t)\}_{t\geq 0}$ be a C_0-cosine family in X.*
Then, for all $\lambda > 0$ and $x \in X$ We have:

$$\lim_{h\longrightarrow 0^+} \frac{1}{h^2}\int_0^h (e^{-\lambda(h-t)} - e^{-\lambda(t-h)})C(t)xdt = -\lambda x.$$

Proof. Let $\lambda > 0$. For all $0 < h < 1$ put:

$$B_h = \frac{1}{h^2}\int_0^h (e^{-\lambda(h-t)} - e^{-\lambda(t-h)})dt = -\frac{1}{\lambda}(\frac{2}{h^2}(cosh(\lambda h) - 1)).$$

With $cosh(\lambda h) = \frac{e^{\lambda h}+e^{-\lambda h}}{2}$, $\forall \lambda \in \mathbb{C}$. Then $\lim\limits_{h\longrightarrow 0^+} B_h = -\lambda$.
On the other hand, let $p \in \Gamma_X$and $x \in X$, we have:

$$\begin{aligned} p(\frac{1}{h^2}\int_0^h (e^{-\lambda(h-t)} - e^{-\lambda(t-h)})C(t)xdt + \lambda x) &= p(\frac{1}{h^2}\int_0^h (e^{-\lambda(h-t)} - e^{-\lambda(t-h)})(C(t)x - x)dt + B_h x + \lambda x) \\ &\leq (\frac{1}{h^2}\int_0^h |e^{-\lambda(h-t)} - e^{-\lambda(t-h)}|dt) \sup_{t\in[0,h]} p(C(t)x - x) \\ &+ p(B_h x + \lambda x) \\ &\leq -B_h \sup_{t\in[0,h]} p(C(t)x - x) + p(B_h x + \lambda x) \end{aligned}$$

Since $\forall x \in X$, $\lim\limits_{t\longrightarrow 0^+} C(t)x = x$ and $B_h \longrightarrow -\lambda$ as $h \longrightarrow 0^+$,

$$\lim_{h\longrightarrow 0^+} \frac{1}{h^2}\int_0^h (e^{-\lambda(h-t)} - e^{-\lambda(t-h)})C(t)xdt = -\lambda x, \text{ for all } x \in X.$$

□

For semigroups, the two next Theorems is due to K.Yosida [13].

Theorem 3.2. *Let $\{C(t)\}_{t\geq 0}$ be an exponentially equicontinuous C_0-cosine family of order ω in X.*
For all $\lambda > \omega$, consider the linear operator C_λ:

$$C_\lambda x = \int_0^\infty e^{-\lambda t}C(t)xdt, \text{ for all } x \in X.$$

Then, the following properties hold:

1. *C_λ is a continuous linear operator in X.*
2. *$Im(C_\lambda) \subseteq D(A)$, and for all $x \in X$ we have: $AC_\lambda x = \lambda^2 C_\lambda x - \lambda x$.*
3. *for all $x \in X$: $\lim\limits_{\lambda\longrightarrow\infty} \lambda C_\lambda x = x$.*

Proof. 1. Let $p \in \Gamma_X$, $x \in X$ and $\lambda > \omega$, then:

$$\begin{aligned} p(C_\lambda x) = p(\int_0^\infty e^{-\lambda t}C(t)xdt) &\leq \int_0^\infty p(e^{-\lambda t}C(t)x)dt \\ &\leq \int_0^\infty e^{-\lambda t}p(C(t)x)dt \end{aligned}$$

Since $\{C(t)\}_{t\geq 0}$ is exponentially equicontinuous of order ω, there exist $M \geq 0$, and $q \in \varGamma_X$ such that:

$$p(C_\lambda x) \leq \int_0^\infty e^{-\lambda t} M e^{\omega t} q(x)dt \leq \frac{M}{\lambda - \omega} q(x).$$

Then, $C_\lambda \in \mathcal{L}(X)$.

2. Let $x \in X$, and $h > 0$, then:

$$\begin{aligned}
\frac{2}{h^2}(C(h)-I)\int_0^\infty e^{-\lambda t}C(t)xdt &= \frac{2}{h^2}\int_0^\infty e^{-\lambda t}C(h)C(t)xdt - \frac{2}{h^2}\int_0^\infty e^{-\lambda t}C(t)xdt \\
&= \frac{2}{h^2}\int_0^h e^{-\lambda t}C(h)C(t)xdt + \frac{2}{h^2}\int_h^\infty e^{-\lambda t}C(h)C(t)xdt - \frac{2}{h^2}\int_0^\infty e^{-\lambda t}C(t)xdt \\
&= \frac{1}{h^2}\int_0^h e^{-\lambda t}C(h+t)xdt + \frac{1}{h^2}\int_0^h e^{-\lambda t}C(h-t)xdt + \frac{1}{h^2}\int_h^\infty e^{-\lambda t}C(t+h)dt \\
&+ \frac{1}{h^2}\int_h^\infty e^{-\lambda t}C(t-h)xdt - \frac{2}{h^2}\int_0^\infty e^{-\lambda t}C(t)xdt \\
&= \frac{1}{h^2}\int_0^\infty e^{-\lambda t}C(t+h)xdt + \frac{1}{h^2}\int_0^h e^{-\lambda t}C(h-t)xdt + \frac{1}{h^2}\int_h^\infty e^{-\lambda t}C(t-h)xdt \\
&- \frac{2}{h^2}\int_0^\infty e^{-\lambda t}C(t)xdt \\
&= \frac{1}{h^2}\int_h^\infty e^{-\lambda(t-h)}C(t)xdt + \frac{1}{h^2}\int_0^h e^{-\lambda(h-t)}C(t)xdt + \frac{1}{h^2}\int_0^\infty e^{-\lambda(t+h)}C(t)xdt \\
&- \frac{2}{h^2}\int_0^\infty e^{-\lambda t}C(t)xdt \\
&= \frac{e^{\lambda h}}{h^2}\int_h^\infty e^{-\lambda t}C(t)xdt + \frac{e^{-\lambda h}}{h^2}\int_0^h e^{\lambda t}C(t)xdt + \frac{e^{-\lambda h}}{h^2}\int_0^\infty e^{-\lambda t}C(t)xdt - \frac{2}{h^2}\int_0^\infty e^{-\lambda t}C(t)xdt \\
&= \frac{e^{\lambda h}+e^{-\lambda h}-2}{h^2}\int_0^\infty e^{-\lambda t}C(t)xdt + \frac{1}{h^2}\int_0^h (e^{-\lambda(h-t)} - e^{-\lambda(t-h)})C(t)xdt.
\end{aligned}$$

According to Lemma 3.1, as $h \longrightarrow 0^+$, we have: $Im(C_\lambda) \subseteq D(A)$ and $AC_\lambda x = \lambda^2 C_\lambda x - \lambda x$.

3. Let $\lambda > \omega$, since $\int_0^\infty \lambda e^{-\lambda t}dt = 1$, $\lambda C_\lambda x - x = \int_0^\infty \lambda e^{-\lambda t}(C(t)x - x)dt$, then for all $p \in \varGamma_X$:

$$p(\lambda C_\lambda x - x) = \int_0^\infty \lambda e^{-\lambda t} p(C(t)x - x)dt = I_1 + I_2$$

with $I_1 = \int_0^\delta \lambda e^{-\lambda t} p(C(t)x - x)dt$ and $I_2 = \int_\delta^\infty \lambda e^{-\lambda t} p(C(t)x - x)dt$ where $\delta > 0$ is a positive number.

By continuity of $C(.)x$ en 0, for any $\varepsilon > 0$ we can choose $\delta > 0$ such that:
$p(C(t)x - x) \leq \epsilon$ for $0 \leq t \leq \delta$. Then

$$I_1 \leq \varepsilon\lambda \int_0^\delta e^{-\lambda t}dt \leq \varepsilon\lambda \int_0^\infty e^{-\lambda t}dt = \varepsilon.$$

For the previous $\delta > 0$, by the exponentially equicontinuity of $\{C(t)\}_{t\geq 0}$,
(i.e. $\exists \omega \geq 0$, $\forall p \in \varGamma_X$ $\exists q \in \varGamma_X$ and $\exists M \geq 0$ such that $p(C(t)x) \leq Me^{\omega t}q(x)$, $\forall x \in X$, and $t \geq 0$),

we have:

$$\begin{aligned} I_2 &\le \lambda \int_\delta^\infty e^{-\lambda t}(p(C(t)x) + p(x))dt \\ &\le \lambda \int_\delta^\infty e^{-\lambda t} p(C(t)x)dt + \lambda \int_\delta^\infty e^{-\lambda t} p(x)dt \\ &\le \lambda \int_\delta^\infty e^{-\lambda t} M e^{\omega t} q(x)dt + \lambda \int_\delta^\infty e^{-\lambda t} p(x)dt \\ &\le \lambda M q(x) \int_\delta^\infty e^{(\omega-\lambda)t}dt + \lambda p(x) \int_\delta^\infty e^{-\lambda t}dt \\ &\le \frac{\lambda}{\lambda-\omega} e^{(\omega-\lambda)\delta} M q(x) + e^{-\lambda\delta} p(x) \longrightarrow 0 \text{ as } \lambda \longrightarrow \infty. \end{aligned}$$

Hence the result.

□

Theorem 3.3. *let $\{C(t)\}_{t\ge 0}$ be an exponentially equicontinuous C_0-cosine family of order ω in X. For every $\lambda > \omega$, $(\lambda^2 I - A)^{-1}$ exist in $\mathcal{L}(X)$.*
Moreover, for all $x \in X$:

$$(\lambda^2 I - A)^{-1} x = \frac{1}{\lambda} \int_0^\infty e^{-\lambda t} C(t)x dt.$$

Proof. Let $\lambda > \omega$. Suppose there is $x_0 \in D(A)$, with $x_0 \neq 0$, such that $(\lambda^2 I - A)x_0 = 0$. Since $x_0 \in D(A)$, $C(t)x_0 = x_0 + t\int_0^t C(s)Ax_0 ds - \int_0^t sC(s)Ax_0 ds$, for all $t \ge 0$, by 5. of Proposition 2.2 and $\frac{d}{dt}C(t)x_0 = \int_0^t C(s)Ax_0 ds$.

On the other hand, as $x_0 \neq 0$, $\exists f \in X'$ suth that $f(x_0) = 1$. Put $g(t) = f(C(t)x_0)$, for all $t \ge 0$, since $C(.)x_0$ is continuous in $\mathbb{R}^+$ (be cause $\{C(t)\}_{t\ge 0}$ is exponentially equicontinuous in X), $g(.)$ is continuous in $\mathbb{R}^+$. Moreover, $\frac{d^2}{dt^2}g(t) = \frac{d^2}{dt^2}f(C(t)x_0) = f(\frac{d^2}{dt^2}C(t)x_0) = f(C(t)Ax_0) = \lambda^2 g(t)$, $g(0) = 1$ and $g'(0) = 0$. Thus

$$g(t) = \frac{e^{\lambda t} + e^{-\lambda t}}{2}, \quad t \ge 0.$$

Since $\{C(t)\}_{t\ge 0}$ is exponentially equicontinuous of order ω, there exists $M_0 \ge 0$ such that: $|g(t)| \le M_0 e^{\omega t}$, for all $t \ge 0$. Which is absurd (because $\lambda > \omega$). Thus $(\lambda^2 I - A) : D(A) \to X$ is injective. Moreover, according to Theorem 3.1 we have $Im(C_\lambda) \subseteq D(A)$ and $(\lambda^2 - A)C_\lambda x = \lambda x$ for every $x \in X$ so that $(\lambda^2 I - A)$ is also surjective and $(\lambda^2 I - A)^{-1} = \frac{1}{\lambda}C_\lambda$ so that $(\lambda^2 I - A)^{-1} \in \mathcal{L}(X)$ by 1. of Theorem 3.1.
□

4. C_0-Cosine family in a GDP Quojection-Fréchet space:

It is known that a Fréchet space X is a projective limit of a sequence of Banach spaces $(X_k)_k$ with respect to the projective operators $P_k : X_{k+1} \longrightarrow X_k$. A Fréchet space is a quojection if it is isomorphic to a projective limit of Banach spaces with surjective projective operators [5].

Recall that a locally convex Hausdorff space is a Grothendieck space, if every sequence in X', which convergent for $\sigma(X', X)$, is also convergent for $\sigma(X', X'')$ [1].

A locally convex Hausdorff space X is said to have the Dunford-Pettis property if for all $T \in \mathcal{L}(X, Y)$, for Y any quasicomplete locally convex Hausdorff space, which transforms elements of $B(X)$ into relatively $\sigma(Y, Y')$-compact subsets of Y, also transforms $\sigma(X, X')$-compact subsets of X into relatively compact subsets of Y [1,9].

A Grothendieck locally convex Hausdorff space X with the Dunford-Pettis property is called, briefly, a GDP space.

We begin this section with the following lemma [Lemma 2.4 [2]] which plays an important role in the main results:

Lemma 4.1. *Let X be a barrelled GDP space and $\{S_n\}_{n\geq 1} \subseteq \mathcal{L}(X)$ be a sequence of operators satisfying the following properties:*

1. *For all $m, n \in \mathbb{N}^*$, $S_m S_n = S_n S_m$.*
2. *For all $m \in \mathbb{N}^*$, $\lim_{n\longrightarrow\infty}(S_n - I)S_m = 0$, in $\mathcal{L}_b(X)$.*
3. *$\lim_{n\longrightarrow\infty} S_n = I$, in $\mathcal{L}_s(X)$.*

Then:

$$\lim_{n\longrightarrow\infty}(S_n - I)^2 = 0, \quad \text{in} \quad \mathcal{L}_b(X).$$

If, in addition, X is a Quojection Fréchet space and there exists a fundamental sequence $\{q_j\}_{j\geq 1}$, of seminorms generating the locally convex topology of X which satisfy:
For each $j \in \mathbb{N}$ there exists $c_j > 0$ such that:

$$q_j(S_n x) \leq c_j q_j(x), \text{ for all } x \in X \text{ and } n \in \mathbb{N}^*.$$

Then:

$$(S_n - I) \longrightarrow 0, \quad \text{in} \quad \mathcal{L}_b(X), \quad \text{as} \quad n \longrightarrow \infty.$$

We recall our Theorem [3] about the cosine family in Quojection space.

Theorem 4.2. *Let X be a quojection.*
The infinitesimal generator of every uniformly continuous cosine family is continuous, (i.e. $A \in \mathcal{L}(X)$). Moreover, for all $x \in X$ and $t \geq 0$ we have:

$$C(t)x = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} A^k x.$$

On the other hand, we consider the space $\boldsymbol{\omega} = \mathbb{C}^{\mathbb{N}}$ of all sequence equipped with its topology of coordinates convergence (i.e. $p_k(x) = \max_{0\leq j\leq k} |x_j|$, $(x_j)_{j\in\mathbb{N}} \in \boldsymbol{\omega}$, for each $k \in \mathbb{N}$). It's well known that, the space $\boldsymbol{\omega}$ is a Quojection, because it is a product of countable copies of the Banach space $\mathbb{C}$, Moreover $\boldsymbol{\omega}$ is a Montel space, therefore the strong convergence and the uniform convergence coincide on bounded sets, then every C_0-cosine family is actually uniformly continuous.

L. Frerick, E. Jorda, T. Kalmes and J. Wengenroth [11] given a satisfactory answer of Conejero's question [6] of whether every C_0-semigroup on $\boldsymbol{\omega}$ is of the form $\{e^{tA}\}_{t\geq 0}$, for some $A \in \mathcal{L}(\boldsymbol{\omega})$.

The following proposition gives a version of the previous result in the case of cosine families.

Proposition 4.3. *Every locally equicontinous C_0-cosine family on the space $\boldsymbol{\omega} = \mathbb{C}^{\mathbb{N}}$ has a continuous infinitesimal generator A and is of the form:*

$$C(t) = cosh(tA) = \frac{e^{tA} + e^{-tA}}{2}, \text{ for all } t \geq 0.$$

Let $\{C(t)\}_{t\geq 0}$ be an exponentially equicontinous C_0-cosine family of order ω in locally convex Hausdorff space X, i.e. for all $p \in \Gamma_X$, there exist $q \in \Gamma_X$ and $M \geq 0$ such that:

$$p(C(t)x) \leq Me^{\omega t} q(x), \text{ for all } t \geq 0, \text{ and } x \in X.$$

For each $p \in \Gamma_X$, we define $\tilde{p}$ in X by:

$$\tilde{p}(x) = \sup_{t\geq 0} p(e^{-\omega t} C(t)x), \text{ for all } x \in X.$$

Each $\tilde{p}$ is a continuous seminorm in X. Moreover, since $\{C(t)\}_{t\geq 0}$ is exponentially equicontinuous of order ω in X, $\tilde{p}$ satisfies:

$$p(x) \leq \tilde{p}(x) \leq Mq(x) \leq M\tilde{q}(x), \text{ for all } x \in X.$$

Then, the family of seminorms $\tilde{\Gamma}_X = \{\tilde{p},\ p \in \Gamma\}$ is also a system of continuous seminorms generating the locally convex topology of X.

Proposition 4.4. *For each seminorm $\tilde{p} \in \tilde{\Gamma}_X$, we have:*

1. *For all $t \geq 0$ and $x \in X$:*
$$\tilde{p}(C(t)x) \leq 2e^{\omega t}\tilde{p}(x).$$
2. *For all $x \in X$ and $\lambda > \omega$:*
$$\tilde{p}(\lambda^2 R(\lambda^2, A)x) \leq \frac{2\lambda}{\lambda - \omega}\tilde{p}(x).$$

Proof. 1. Let $t \geq 0$, $x \in X$, then for each $\tilde{p} \in \tilde{\Gamma}_X$ we have:

$$\begin{aligned}
2\tilde{p}(C(t)x) &= 2\sup_{s\geq 0} p(e^{-\omega s}C(s)C(t)x) \leq 2\sup_{s\leq t} p(e^{-\omega s}C(s)C(t)x) + 2\sup_{s>t} p(e^{-\omega s}C(s)C(t)x) \\
&\leq \sup_{s\leq t} p(e^{-\omega s}(C(t+s)x + C(t-s)x)) + \sup_{s>t} p(e^{-\omega s}(C(s+t)x + C(s-t)x)) \\
&\leq \sup_{s\leq t} p(e^{-\omega s}C(t+s)x) + \sup_{s\leq t} p(e^{-\omega s}C(t-s)x) + \sup_{s>t} p(e^{-\omega s}C(s+t)x) \\
&+ \sup_{s>t} p(e^{-\omega s}C(s-t)x) \\
&\leq e^{\omega t}\sup_{s\leq t} p(e^{-\omega(t+s)}C(t+s)x) + e^{\omega t}\sup_{s\leq t} e^{-2\omega s} p(e^{-\omega(t-s)}C(t-s)x) \\
&+ e^{\omega t}\sup_{s>t} p(e^{-\omega(s+t)}C(s+t)x) + e^{-\omega t}\sup_{s>t} p(e^{-\omega(s-t)}C(s-t)x) \\
&\leq e^{\omega t}\sup_{k\leq 2t} p(e^{-\omega k}C(k)x) + e^{\omega t}\sup_{k\leq t} p(e^{-\omega k}C(k)x) + e^{\omega t}\sup_{k>2t} p(e^{-\omega k}C(k)x) \\
&+ e^{-\omega t}\sup_{k>0} p(e^{-\omega k}C(k)x) \\
&\leq 4e^{\omega t}\sup_{k\geq 0} p(e^{-\omega k}C(k)x) = 4e^{\omega t}\tilde{p}(x).
\end{aligned}$$

2. For each $\lambda > \omega$, according to the section 3, we have:
$$\lambda^2 R(\lambda^2, A)x = \lambda^2(\lambda^2 I - A)^{-1}x = \lambda \int_0^\infty e^{-\lambda t}C(t)x dt, \text{ for all } x \in X.$$
Then, for all seminorm $\tilde{p} \in \tilde{\Gamma}_X$ we have:
$$\tilde{p}(\lambda^2 R(\lambda^2, A)x) \leq \frac{2\lambda}{\lambda - \omega}\tilde{p}(x), \text{ for all } x \in X.$$
In particular if $\lambda > \omega + 1$, then for each $\tilde{p} \in \tilde{\Gamma}_X$ we obtain:
$$\tilde{p}(\lambda^2 R(\lambda^2, A)x) \leq \frac{2\lambda}{\lambda - \omega}\tilde{p}(x) \leq 2(1+\omega)\tilde{p}(x), \text{ for all } x \in X.$$

□

For semigroups, the following Theorem is due to A.A. Albanese, J. Bonet, W.J. Ricker [2].

Theorem 4.5. *Let X be a GDP Quojection-Fréchet space and $\{C(t)\}_{t\geq 0}$ be an exponentially equicontinuous C_0-cosine family of order ω in X.*

Then, $\{C(t)\}_{t\geq 0}$ is uniformly continuous and its infinitesimal generator is a linear continuous operator in X.

Proof. According to the discussion before Proposition 4.2 there is a fundamental increasing sequence $\{q_j\}_{j\in\mathbb{N}^*}$ of continuous seminorms on X such that for all $j \in \mathbb{N}^*$:
$$q_j(C(t)x) \leq 2e^{\omega t}q_j(x), \text{ for all } x \in X, \text{ and } t \geq 0. \qquad (*)$$

For all $j \in \mathbb{N}^*$, put $X_j = X/q_j^{-1}(\{0\})$ endowed with the quotient locally convex topology. We note by $Q_j : X \longrightarrow X_j$ the canonical surjective quotient map, so that $Ker(Q_j) = q_j^{-1}(\{0\})$, thus X_j is Fréchet space whose locally convex topology is generating by the family of seminorms giving by:

$$(\hat{q}_j)_k(Q_jx) = inf\{q_k(y), \quad with \quad Q_jy = Q_jx\}, \text{ for all } x \in X.$$

First, we have:

$$(\hat{q}_j)_k(Q_jx) \leq q_k(x), \text{ for all } x \in X, \text{ and } k \in \mathbb{N}.$$

Moreover,

$$(\hat{q}_j)_j(Q_jx) = q_j(x),, \text{ for all } x \in X, \text{ and } j \in \mathbb{N}.$$

Which implies that, $(\hat{q}_j)_j$ is a continuous norm in X_j, therefore, $(\hat{q}_j)_k$ is a continuous norm in X_j, for all $k \geq j$. Since X is a Quojection and according to [Proposition 3. [1]] X_j is a Banach space.

On the other hand, for each $k \geq j$ we have $(\hat{q}_j)_k(Q_jx) \leq (\hat{q}_j)_{k+1}(Q_jx)$, for all $x \in X$, which implies that the norms $\{(\hat{q}_j)_k\}_{k\geq j}$ are equivalent. Then, for each $j \in \mathbb{N}^*$, we take $k(j) \geq j$ such that $(\hat{q}_j)_{k(j)}$ is a norm generating the Banach topology of X_j. Consequently, X is isomorphic to the projective limit of the sequence $(X_j, (\hat{q}_j)_{k(j)}))_{j\geq 1}$ of Banach spaces, with respect to the surjective linking maps $Q_{j,j+1} : X_{j+1} \longrightarrow X_j$ defined by $Q_{j,j+1}(Q_{j+1}x) = Q_jx$, for all $x \in X$.

On the other hand, let $(\lambda_n)_n \subseteq \mathbb{R}^+$, with each $\lambda_n > \omega + 1$ and $\lim_{n\longrightarrow\infty} \lambda_n = \infty$, put the sequence of operators defined in X by: $S_n = \lambda_n^2 R(\lambda_n^2, A)$, for all $n \in \mathbb{N}$.

First, for each $n, m \in \mathbb{N}$, we have $S_nS_m = S_mS_n$, and according to Theorem 1 we have $S_n \longrightarrow I$ in $\mathcal{L}_s(X)$, as $n \longrightarrow \infty$, and according to 2. of Proposition 2.2 for each seminorm q_j we have:

$$q_j(S_nx) \leq 2(1 + \omega)q_j(x), \text{ for all } x \in X, \text{ and } n \in \mathbb{N}.$$

Let $n, m \in \mathbb{N}$ such that $\lambda_n \neq \lambda_m$, we have:

$$\begin{aligned}
(S_n - I)S_m &= (\lambda_n^2 R(\lambda_n^2, A) - I)\lambda_m^2 R(\lambda_m^2, A) \\
&= \lambda_n^2\lambda_m^2 R(\lambda_n^2, A)R(\lambda_m^2, A) - \lambda_m^2 R(\lambda_m^2, A) \\
&= \lambda_m^2(\lambda_n^2 R(\lambda_n^2, A)R(\lambda_m^2, A) - R(\lambda_m^2, A)) \\
&= \lambda_m^2(\lambda_n^2 R(\lambda_n^2, A)R(\lambda_m^2, A) + (\lambda_m^2 - \lambda_n^2)R(\lambda_n^2, A)R(\lambda_m^2, A) - R(\lambda_n^2, A)) \\
&= \lambda_m^2(\lambda_m^2 R(\lambda_n^2, A)R(\lambda_m^2, A) - R(\lambda_n^2, A)) \\
&= \lambda_m^2(\lambda_m^2(\lambda_n^2 - \lambda_m^2)^{-1}(R(\lambda_n^2, A) - R(\lambda_m^2, A)) - R(\lambda_n^2, A)) \\
&= \lambda_m^2(\lambda_m^2 - \lambda_n^2)^{-1}(\lambda_m^2(R(\lambda_n^2, A) - R(\lambda_m^2, A)) - (\lambda_m^2 - \lambda_n^2)R(\lambda_n^2, A)) \\
&= \lambda_m^2(\lambda_m^2 - \lambda_n^2)^{-1}(\lambda_n^2 R(\lambda_n^2, A) - \lambda_m^2 R(\lambda_m^2, A)) \\
&= \lambda_m^2(\lambda_m^2 - \lambda_n^2)^{-1}(S_n - S_m).
\end{aligned}$$

Which implies

$$\lim_{n\longrightarrow\infty}(S_n - I)S_m = 0, \text{ in } \mathcal{L}_b(X), \text{ for all } m \in \mathbf{N}.$$

Then according to Lemme 4.1., we have $S_n \longrightarrow I$, in $\mathcal{L}_b(X)$, as $n \longrightarrow \infty$.

On the other hand, for each $j \in \mathbb{N}^*$ define a one parameter family $\{C_j(t)\}_{t\geq 0}$ by:

$$C_j(t)Q_jx = Q_jC(t)x, \text{ for all } x \in X, \text{ and } t \geq 0.$$

Each $C_j(t)$, for $t \geq 0$, is a well defined linear continuous operator on X_j. Indeed, let $x \in X$ such that $Q_jx = 0$, which implies that $x \in Ker(Q_j)$, so $q_j(x) = 0$. Then according to $(*)$, we have $q_j(C(t)x) = 0$ and hence $C(t)x \in q_j^{-1}(\{0\})$; therefore $Q_jC(t)x = 0$.

Actually, for each $j \in \mathbb{N}^*$, the family $\{C_j(t)\}_{t\geq 0}$ is a C_0-cosine family in X_j. Note by A_j its infinitesimal generator.

Moreover $\{C_j(t)\}_{t\geq 0}$ is exponentially equicontinuous of order ω. Indeed, for all $x \in X$, we have:

$$(\hat{q}_j)_{k(j)}(C_j(t)Q_jx) = (\hat{q}_j)_{k(j)}(Q_jC(t)x) \leq q_{k(j)}(C(t)x).$$

According to (*) we obtain:

$$(\hat{q}_j)_{k(j)}(C_j(t)Q_jx) \leq 2e^{\omega t}q_{k(j)}(x), \text{ for all } x \in X.$$

Taking the infimum with respect to $y \in Q_j^{-1}(\{Q_jx\})$, if follows that $(\hat{q}_j)_{k(j)}(C_j(t)\hat{x_j}) \leq 2e^{\omega t}(\hat{q}_j)_{k(j)}(\hat{x_j})$, for all $\hat{x_j} \in X_j$ with $\hat{x_j} = Q_jx$.

Let $\lambda > \omega$, then we have: $R(\lambda^2, A_j)Q_jx = Q_jR(\lambda^2, A)x$, $\forall x \in X$. Indeed, for all $x \in X$ we have:

$$\begin{aligned}
&C_j(t)Q_jx = Q_jC(t)x.\\
\Rightarrow\ &e^{-\lambda t}C_j(t)Q_jx = e^{-\lambda t}Q_jC(t)x = Q_je^{-\lambda t}C(t)x.\\
\Rightarrow\ &\int_0^\infty e^{-\lambda t}C_j(t)Q_jxdt = \int_0^\infty Q_je^{-\lambda t}C(t)xdt = Q_j\int_0^\infty e^{-\lambda t}C(t)xdt.
\end{aligned}$$

Moreover, we have $Q_j(D(A)) \subseteq D(A_j)$. Indeed, let $x \in X$ and $h > 0$ we have: $\frac{2}{h^2}(C_j(t)Q_jx - Q_jx) = Q_j\frac{2}{h^2}(C(t)x - x)$, then, as $h \longrightarrow 0^+$, we obtain $A_jQ_jx = Q_jAx$. For each $j \in \mathbb{N}^*$, Put $S_n^{(j)} = \lambda_n^2R(\lambda_n^2, A_j)$, for all $n \in \mathbb{N}^*$. $\{S_n^{(j)}\}_{n=1}^\infty$ is a sequence of bounded operator in X_j.

Since $S_n \longrightarrow I$, in $\mathcal{L}_b(X)$, as $n \longrightarrow \infty$, then $S_n^{(j)} \longrightarrow I$, in $\mathcal{L}_b(X_j)$, as $n \longrightarrow \infty$.

Indeed, let B_j the unit ball of X_j, since X is a Quojection, $\exists B \in B(X)$ such that $B_j \subseteq Q_j(B)$ [7], and we have:

$$\begin{aligned}
\sup_{x_j \in B_j}(\tilde{q}_j)_{k(j)}(S_n^{(j)}x_j - x_j) &\leq \sup_{x_j \in Q_j(B)}(\tilde{q}_j)_{k(j)}(S_n^{(j)}x_j - x_j)\\
&\leq \sup_{x \in B}(\tilde{q}_j)_{k(j)}(S_n^{(j)}Q_jx - Q_jx)\\
&\leq \sup_{x \in B}(\tilde{q}_j)_{k(j)}(Q_j(S_nx - x))\\
&\leq \sup_{x \in B}q_{k(j)}(S_nx - x).
\end{aligned}$$

Hence the result. Consequently, there is $\eta_j > 0$ such that $(\tilde{q}_j)_{k(j)}(S_n^{(j)}x_j - x_j) < 1$, $\forall n \geq \eta_j$.

Then for each $j \in \mathbb{N}^*$, $\exists n(j) \in \mathbb{N}$ with $n(j) \geq \eta_j$, such that $S_{n(j)}^{(j)}$ is invertible in $\mathcal{L}(X_j)$. Therefore, $Im(S_{n(j)}^{(j)}) = X_j = D(A_j)$. Moreover $A_j = \lambda_{n(j)}(I - (S_{n(j)}^{(j)})^{-1}) \in \mathcal{L}(X_j)$, which implies that $\{C_j(t)\}_{t\geq 0}$ is an uniformly continuous cosine family in X_j.

Finally, let $j \in \mathbb{N}^*$ and $B \in B(X)$ we have:

$$\begin{aligned}
\sup_{x \in B}q_j(C(t)x - x) &= \sup_{x \in B}(\tilde{q}_j)_j(Q_jC(t)x - Q_jx)\\
&\leq \sup_{x \in B}(\tilde{q}_j)_{k(j)}(C_j(t)Q_jx - Q_jx)\\
&\leq \sup_{y_j \in Q_j(B)}(\tilde{q}_j)_{k(j)}(C_j(t)y_j - y_j).
\end{aligned}$$

Since $\{C_j(t)\}_{t\geq 0}$ is uniformly continuous in X_j and $Q_j(B)$ is a bounded subset of X_j, then $\{C(t)\}_{t\geq 0}$ is uniformly continuous in X.

Then, according to Theorem 4.1. the infinitesimal generator A of the family $\{C(t)\}_{t\geq 0}$ is a continuous linear operator in X ($A \in \mathcal{L}(X)$). □

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