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The Algebraic Face of the Alexandroff One Point Compactification *

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To the memory of Professor Jorge Medina Sancho

ABSTRACT: We give a characterization, in algebraic terms, of the Alexandroff one point compactification of a locally compact Hausdorff space. In fact, we prove that if (X, τ) is a locally compact Hausdorff space, then $(\widetilde{X}, \widetilde{\tau})$ is its Alexandroff one point compactification if and only if τ is a maximal ideal of $\widetilde{\tau}$.

Key Words: Semiring, Maximal ideal, Alexandroff one point compactification, Completely regular topological space.

Contents

1. Introduction

In 1934 Vandiver introduced the term *semiring* and its structure [6], though the early examples of semirings had appeared in the works of Dedekind in 1894, when he worked on the algebra of the ideals of commutative rings [2]. But it was in the late 1960s that semiring theory was considered an important topic for research, when real applications were found for semirings related with automata theory and computer science. Nowadays, semiring applications to diverse areas of mathematics have diversified a lot. Golan [3] gives an extensive account of semiring theory and its applications in diverse disciplines.

Let (X, τ) be a topological space. We consider τ as a commutative semiring, where the adition and multiplication are the union and intersection, respectively, and the identities of the adition and multiplication are the empty set and the whole set X, respectively. In [1], properties of the topologies viewed as semirings are studied by connecting the algebraic properties of τ with the topological properties of (X, τ).

The Collatz conjecture (1936) states that, starting from any positive integer n, repeated application of the function f will eventually produce the number 1, where $f : \mathbb{N} \to \mathbb{N}$ is defined as follows

$$f(n) = \begin{cases} \frac{n}{2} & \text{for} \quad n \text{ even} \\ \\ 3n+1 & \text{for} \quad n \text{ odd} \end{cases}$$

Recently, Guale and Vielma [4] obtained an algebraic equivalent proposition to the Collatz conjecture by considering the topology $\tau_f = \{\theta \subseteq \mathbb{N} : f^{-1}(\theta) \subseteq \theta\}$ on $\mathbb{N}$, as a semiring. In fact, they proved that the Collatz conjecture is true if and only if τ_f is a local semiring.

It is well known that a completely regular Hausdorff topological space is locally compact if and only if it is an open subset in any of its compactifications. One of the well known compactifications of a locally compact Hausdorff space (X, τ) is the so called Alexandroff one point compactification $(\widetilde{X}, \widetilde{\tau})$. In this work, we prove that if $(\widetilde{X}, \widetilde{\tau})$ is a Hausdorff compactification of a completely regular Hausdorff space (X, τ), then $(\widetilde{X}, \widetilde{\tau})$ is the Alexandroff one point compactification of (X, τ) if and only if τ is a maximal ideal of $\widetilde{\tau}$.

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2. Basic facts

Recall that a *semiring* (commutative with non zero identity) is an algebra $(R, +, \cdot, 0, 1)$, where R is a non empty set with $0, 1 \in R$, and $+$ and $\cdot$ are binary operations on R called adition and multiplication, respectively, which satisfy the following:

(1) $(R, +, 0)$ and $(R, \cdot, 1)$ are commutative monoids with $1 \neq 0$.

(2) $a \cdot (b + c) = a \cdot b + a \cdot c$, for every $a, b, c \in R$.

(3) $a \cdot 0 = 0$, for every $a \in R$.

A non empty subset I of R is called an *ideal* of R if $a, b \in I$ and $r \in R$ imply $a + b \in I$ and $r \cdot a \in I$. A *prime ideal* of R is a proper ideal P of R in which $x \in P$ or $y \in P$ whenever $xy \in P$. A *maximal ideal* of R is a proper ideal M of R such that, if I is a proper ideal of R with $M \subseteq I$, then $I = M$.

In what follows, $|A|$ will denote the cardinality of the set A.

Remember that a subset of X is said to be compact if it is compact as a subspace. X is *locally compact*, if every point x of X has a compact neighbourhood.

A *Hausdorff compactification* of a topological space (X, τ) is a pair $(\widetilde{X}, \widetilde{\tau})$ such that:

(1) $(\widetilde{X}, \widetilde{\tau})$ is a Hausdorff compact space.

(2) There exist a homeomorphism $\iota : (X, \tau) \to (\widetilde{X}, \widetilde{\tau})$ onto $\iota(X)$.

(3) $\iota(X)$ is dense in $\widetilde{X}$.

It is well known that every locally compact non compact Hausdorff topological space has a compactification [5], called the *Alexandroff one point compactification*, denoted by $(\tilde{X}, \tilde{\tau})$, where $\tilde{X} = X \cup \{\infty\}$, $\infty \notin X$ and $\tilde{\tau}$ is the family of all open subsets of X and all subsets U of $\tilde{X}$ such that $\infty \in U$ and $\tilde{X} \setminus U$ is a compact subset of X.

The following result will be useful in the next section

Lemma 2.1. *[5] A topological space X is homeomorphic to an open subspace of a compact Hausdorff space if and only if X is a locally compact Hausdorff space.*

3. Main results

Lemma 3.1. *Let (X, τ) be a Hausdorff topological space. For $x \in X$ let*

$$\phi(x) = \{\theta \in \tau : x \notin \theta\}.$$

Then $\phi(x)$ is a maximal ideal of τ.

Proof. Let $A, B \in \phi(x)$, then $x \notin A$ and $x \notin B$, so $x \notin A \cup B$ and $A \cup B \in \phi(x)$. Now, let $A \in \tau$ and $B \in \phi(x)$, then $x \notin B$ thus $x \notin A \cap B$, so $A \cap B \in \phi(x)$. Therefore $\phi(x)$ is an ideal of τ. Let P be an ideal of τ such that $\phi(x) \subsetneq P \subseteq \tau$ and let $\theta \in P \setminus \phi(x)$, then $x \in \theta$. Note that $\{x\}$ is closed since X is Hausdorff, implying that $X \setminus \{x\} \in \phi(x) \subseteq P$. Thus $X = \theta \cup X \setminus \{x\} \in P$, in consequence, $P = \tau$ and $\phi(x)$ is a maximal ideal of τ. □

Lemma 3.2. *Let (X, τ) be a Hausdorff topological space. If Q is a prime ideal of τ, then*

$$\mid X \setminus \cup_{\theta \in Q} \theta \mid \leq 1.$$

Proof. Suppose that $\mid X \setminus \cup_{\theta \in Q} \theta \mid > 1$. Let $p, q \in X \setminus \cup_{\theta \in Q} \theta$ with $p \neq q$. Since (X, τ) is Hausdorff, there exists θ_p, $\theta_q \in \tau$, disjoint, such that $p \in \theta_p$ and $q \in \theta_q$. Now, since Q is a prime ideal of τ and $\emptyset = \theta_p \cap \theta_q \in Q$, either θ_p or θ_q belong to Q. Therefore, p or q belong to $\cup_{\theta \in Q} \theta$, which is a contradiction. □

Lemma 3.3. *Let (X, τ) be a Hausdorff compact topological space. If Q is a prime ideal of τ , then*

$$\mid X \setminus \cup_{\theta \in Q} \theta \mid = 1.$$

Proof. If $\mid X \setminus \cup_{\theta \in Q} \theta \mid = 0$, then $X = \cup_{\theta \in Q} \theta$. Since X is compact, we can find $\theta_1, \theta_2, \ldots, \theta_n \in Q$ such that $X = \cup_{i=1}^{n} \theta_i$, implying that $X \in Q$, which is a contradiction, since Q is a prime ideal. By Lemma 3.2, the result follows. □

From the above lemmas the following question seems to be interesting.

Question: Is there a completely regular Hausdorff topological space (X, τ) and a prime ideal Q of τ such that $X = \cup_{\theta \in Q} \theta$?

Theorem 3.4. *Let (X, τ) be a non compact completely regular Hausdorff topological space, and let $(\widetilde{X}, \widetilde{\tau})$ be a Hausdorff compactification of (X, τ). Then $(\widetilde{X}, \widetilde{\tau})$ is the Alexandroff one point compactification of (X, τ) if and only if τ is a maximal ideal of $\widetilde{\tau}$.*

Proof. Suppose that $(\widetilde{X}, \widetilde{\tau})$ is the Alexandroff one point compactification of (X, τ), that is, $\widetilde{X} = X \cup \{\infty\}$. Note that τ is a prime ideal of $\widetilde{\tau}$, in fact, τ is closed under finite unions. If θ is an open set in $\widetilde{\tau}$ containig ∞ and $\omega \in \tau$, we have that $\theta \cap \omega = (\theta \cap X) \cap \omega$. But X is $\widetilde{\tau}$-open and τ-open, then $\theta \cap \omega \in \tau$. Now suppose that $\omega_1, \omega_2 \in \widetilde{\tau}$ such that $\omega_1 \cap \omega_2 \in \tau$, then necessarily one of them does not contain ∞, let us say ω_1, thus $\omega_1 \in \tau$. Therefore τ is a prime ideal of $\widetilde{\tau}$.

Now, let us see that τ is a maximal ideal of $\widetilde{\tau}$. Since $(\widetilde{X}, \widetilde{\tau})$ is a Hausdorff compact space, by Lemma 3.3, $\widetilde{X} \setminus \bigcup_{\theta \in \tau} \theta = \{\infty\}$ for the element $\infty \in \widetilde{X}$ so, $\tau \subseteq \phi(\infty)$. Now if $\omega \in \phi(\infty)$, we have that $\omega \in \widetilde{\tau}$ and $\infty \notin \omega$, so $\omega \in \tau$. Thus, $\tau = \phi(\infty)$ and τ is a maximal ideal, by Lemma 3.1.

Conversely, suppose that τ is a maximal ideal of $\widetilde{\tau}$. Then τ is a prime ideal of $\widetilde{\tau}$. Moreover, $X \in \widetilde{\tau}$ and, by Lemma 2.1, X is locally compact. Since $(\widetilde{X}, \widetilde{\tau})$ is a Hausdorff compact space, by Lemma 3.3, $\widetilde{X} \setminus \bigcup_{\theta \in \tau} \theta = \{x_\tau\}$ for some $x_\tau \in \widetilde{X}$. Hence, $\widetilde{X} \setminus X = \{x_\tau\}$. If $\theta \in \widetilde{\tau}$ and $x_\tau \in \theta$, then $\widetilde{X} \setminus \theta$ is a compact subset of X. Therefore, $(\widetilde{X}, \widetilde{\tau})$ is the Alexandroff one point compactification of (X, τ). □

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