Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 ON LINE
SPM: www.spm.uem.br/bspm

(3s.) **v. 2022 (40)** : 1–8.
ISSN-0037-8712 IN PRESS
doi:10.5269/bspm.49016

Existence of Solutions for a p-Laplacian System with a Nonresonance Condition Between the First and the Second Eigenvalues

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ABSTRACT: In this article, we study the existence of positive solutions for the quasilinear elliptic system

$$\begin{cases} -\Delta_p u(x) = f_1(x, v(x)) + h_1(x) & \text{in } \Omega, \\ -\Delta_p v(x) = f_2(x, u(x)) + h_2(x) & \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $f_i(x, s)$, $(i = 1, 2)$ locates between the first and the second eigenvalues of the p-Laplacian. To prove the existence of solutions, we use the Leray-Schauder degree.

Key Words: Quasi-elliptic equations, Degree-theoretic methods, Eigenvalues, Sobolev spaces.

Contents

1. Introduction

Systems of quasilinear elliptic equations present some new and interesting phenomena, which are not present in the study of a single equation. Many publications have appeared concerning quasilinear elliptic systems we refer the readers to ([4], [10]).

In recent years, the eigenvalue problems for p-Laplacian operators have been extensively studied (see [3], [6], [7], [8]). The main purpose of this article is to prove the existence of solutions for a quasilinear elliptic system when the second terms on the two equations $f_i(x, s)$, $(i = 1, 2)$ locates between the first and the second eigenvalue of the p-Laplacian. This result can be seen as a generalization of the result obtained by A. Anane and N. Tsouli in [3].

In this paper, we study the existence of positive solution for the nonlinear elliptic system

$$\begin{cases} -\Delta_p u(x) = f_1(x, v(x)) + h_1(x) & \text{in } \Omega, \\ -\Delta_p v(x) = f_2(x, u(x)) + h_2(x) & \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator with the exponent p, $1 < p < \infty$ and Ω is a smooth bounded region in $\mathbb{R}^n$ for $n \geq 1$.
Through this paper, $h_i \in W^{-1,p'}(\Omega)$ with $i = 1, 2$ and p' the Hölder conjugate of p. As to the nonlinearities f_i $(i = 1, 2)$, we assume that they are Carathéodory functions from $\Omega \times \mathbb{R}$ to $\mathbb{R}$ such that

$$\max_{|s| \leq R_i} |f_i(x, s)| \in L^{p'}(\Omega), \quad \forall R_i > 0, \tag{1.2}$$

$$\lambda_1 \underset{\neq}{\leq} l_i(x) \leq k_i(x) < \lambda_2 \quad \text{a.e. in } \Omega, \tag{1.3}$$

2010 *Mathematics Subject Classification:* 35K86, 35R35, 49J40.
Submitted August 04, 2019. Published October 14, 2019

where

$$l_i(x) = \lim_{s\to\pm\infty} \inf \frac{f_i(x,s)}{|s|^{p-2}s}, \quad k_i(x) = \lim_{s\to\pm\infty} \sup \frac{f_i(x,s)}{|s|^{p-2}s},$$

and λ_1 (resp., λ_2) is the first (resp., the second) eigenvalue of the problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

First inequality in (1.3) means: "less or equal almost everywhere with strict inequality on a set of positive measure". we also assume that the inequalities in (1.3) holds for $i = 1, 2$:

$$\begin{aligned} &\forall \varepsilon_i > 0, \quad \exists \eta(\varepsilon_i) > 0 : \lambda_1 - \varepsilon_i \le \tfrac{f_i(x,s)}{|s|^{p-2}s}, \quad \forall |s| \ge \eta(\varepsilon_i), \quad \text{a.e. in } \Omega, \\ &\forall \varepsilon_i > 0, \quad \exists \eta(\varepsilon_i) > 0 : \tfrac{f_i(x,s)}{|s|^{p-2}s} \le \lambda_2 + \varepsilon_i, \quad \forall |s| \ge \eta(\varepsilon_i), \quad \text{a.e. in } \Omega. \end{aligned} \tag{1.4}$$

Recently, A. Anane and N. Tsouli [3] study the existence of solutions for the Dirichlet problem $-\Delta_p u = f(x,u) + h(x)$ in Ω, $u = 0$ in $\partial\Omega$, when $f(x,u)$ locates between the first and the second eigenvalues of the p-Laplacian (Δ_p), using Leray-Schauder topological degree.
Their work is based on the absurd reasoning, they arrived at a contradiction by using different lemmas and the variation characterization of λ_2, more precisely the monotonicity of λ_2. Our work is based on the same method of proof.

The main result of this paper is the following theorem.

Theorem 1.1. *For $i = 1, 2$, assume that f_i satisfies (1.2), (1.3) and (1.4). Then for any $h_i \in W^{-1,p'}(\Omega)$, (1.1) admits a weak solution (u, v) in $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$.*

As usual, a weak solution of system (1.1) is any $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi_1 dx + \int_\Omega |\nabla v|^{p-2}\nabla v \nabla\varphi_2 dx = &\int_\Omega f_1(x,v)\varphi_1 dx + \int_\Omega f_2(x,u)\varphi_2 dx \\ &+\langle h_1, \varphi_1\rangle + \langle h_2, \varphi_2\rangle, \end{aligned}$$

for every $\varphi_i \in W^{-1,p'}(\Omega)$, $(i = 1, 2)$, where $\langle ., .\rangle$ denotes the duality product between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$.

Next, let us define by $(T_t)_{t\in[0,1]}$ the family of operators from $W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ to $W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ defined by

$$T_t(u,v) = \begin{pmatrix} T_{1t}(u,v) \\ T_{2t}(u,v) \end{pmatrix} = \begin{pmatrix} -\Delta_p^{-1} & 0 \\ 0 & -\Delta_p^{-1} \end{pmatrix} \times \begin{pmatrix} (1-t)\alpha_1|u|^{p-2}u + tf_1(x,v) + th_1 \\ (1-t)\alpha_2|v|^{p-2}v + tf_2(x,u) + th_2 \end{pmatrix}, \tag{1.5}$$

where α_i, $i = 1, 2$ are some fixed numbers with $\lambda_1 < \alpha_i < \lambda_2$.
We consider the space $U = W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ endowed with the norm

$$\|(u,v)\|_U = \|u\|^p_{W_0^{1,p}(\Omega)} + \|v\|^p_{W_0^{1,p}(\Omega)}, \tag{1.6}$$

$V = L^p(\Omega) \times L^p(\Omega)$, $Y = L^{p'}(\Omega) \times L^{p'}(\Omega)$ and $Z = W^{-1,p'}(\Omega) \times W^{-1,p'}(\Omega)$. In the sequel, $\|.\|_{L^p(\Omega)}$ and $\|.\|_{L^{p'}(\Omega)}$ will denote the usual norms on $L^p(\Omega)$ and $L^{p'}(\Omega)$, respectively.

Remark 1.2. *Hypotheses (1.2) and (1.4) give us the growth conditions*

$$|f_i(x,s)| \le a_i|s|^{p-1} + b_i(x) \quad \forall |s| \in \mathbb{R}, \ \textit{a.e. in } \Omega, \tag{1.7}$$

where $a_i > 0$ and $b_i(.) \in L^{p'}(\Omega)$.

Remark 1.3. *Equations (1.2) and (1.4) imply*

$$\begin{gathered}\forall\varepsilon_i>0,\quad \exists b_{\varepsilon_i}\in L^{p'}(\Omega)\ \textit{such that}\\ |s|^p(\lambda_1-\varepsilon_i)-b_{\varepsilon_i}(x)\le sf_i(x,s)\le |s|^p(\lambda_2+\varepsilon_i)-b_{\varepsilon_i}(x),\\ \forall s\in\mathbb{R},\quad \textit{a.e. in }\Omega.\end{gathered}\tag{1.8}$$

Lemma 1.4. *T_t is continuous and compact.*

Proof. We have, $T_t : U \to U$; to prove the Lemma, we have

$$U\hookrightarrow V\underset{A}{\rightarrow} Y\hookrightarrow Z\underset{S}{\rightarrow} U,\tag{1.9}$$

such that the Nemytskii operator

$$\begin{aligned}A:\quad & V && \to Y\\ & (u,v) && \mapsto (f_1(x,v),f_2(x,u)),\end{aligned}$$

and

$$\begin{aligned}S:\quad & Z && \to U\\ & \begin{pmatrix}f_1\\ f_2\end{pmatrix} && \mapsto \begin{pmatrix}-\Delta_p^{-1} & 0\\ 0 & -\Delta_p^{-1}\end{pmatrix}\begin{pmatrix}f_1(x,v)\\ f_2(x,u)\end{pmatrix}=\begin{pmatrix}u\\ v\end{pmatrix},\end{aligned}$$

are continuous and compact. □

2. A priori estimate

To prove theorem (1.1), we first establish the following estimate:

$$\exists R>0 \text{ such that } \forall t\in[0,1],\forall(u,v)\in\partial B(0,R) \text{ such that } [I-T_t](u,v)\neq 0,$$

where $B(0,R)$ denotes the ball of center 0 and radius R in U.
For, we assume by contradiction that

$$\begin{gathered}\forall n>0,\quad \exists t_n\in[0,1],\quad \exists(u_n,v_n)\in U \text{ with}\\ \|(u_n,v_n)\|_{1,p}=n \text{ such that } T_{t_n}(u_n,v_n)=(u_n,v_n).\end{gathered}\tag{2.1}$$

Let $w_n=(w_{1n},w_{2n})=(\frac{u_n}{n},\frac{v_n}{n})$. We still denoted by (w_n) the subsequence of (w_n) which converges weakly in U, strongly in V and a.e. in Ω to w.
We can also suppose that t_n converges to $t\in[0,1]$. That to reach a contradiction, we need the following lemmas.

Lemma 2.1. *If the sequence $g_n=(g_{1n},g_{2n})$ are defined by*

$$g_{in}=\frac{f_i(x,nw_{i+(-1)^{i+1}n})}{n^{p-1}},\quad i=1,2,\tag{2.2}$$

then g_{in} are bounded in $L^{p'}(\Omega)$, and they admit subsequences g_{in} converging weakly to some g_i in $L^{p'}(\Omega)$.

Proof. From (1.7), we have

$$|f_i(x,s)|\le a_i|s|^{p-1}+b_i(x),$$

then

$$|g_{in}(x)|\le a_i|w_{i+(-1)^{i+1}n}|^{p-1}+\frac{b_i(x)}{n^{p-1}};$$

as $b_i(x)$ in $L^{p'}(\Omega)$ and $|w_{i+(-1)^{i+1}n}|^{p-1}\in L^{p'}(\Omega)$, so g_{in} become bounded in $L^{p'}(\Omega)$.
Consequently, there exists a subsequence, still denoted by g_{in} converging weakly to g_i in $L^{p'}(\Omega)$. □

Lemma 2.2. *$w_i \neq 0$, $i = 1, 2$.*

Proof. We have that w_n verifies

$$\int_\Omega |\nabla w_{1n}|^p dx + \int_\Omega |\nabla w_{2n}|^p dx = (1-t_n)\big[\alpha_1 \int_\Omega |w_{1n}|^p dx + \alpha_2 \int_\Omega |w_{2n}|^p dx\big] + t_n\big[\int_\Omega g_{1n}(x) w_{1n} dx + \int_\Omega g_{2n}(x) w_{2n} dx + \frac{1}{n^{p-1}} < h_1, w_{1n} > + \frac{1}{n^{p-1}} < h_2, w_{2n} > \big]. \tag{2.3}$$

We get from lemma (2.1)

$$1 = (1-t)\big[\alpha_1 \int_\Omega |w_1|^p dx + \alpha_2 \int_\Omega |w_2|^p dx\big] + t\big[\int_\Omega g_1(x) w_1 dx + \int_\Omega g_2(x) w_2 dx\big]; \tag{2.4}$$

from the diffrent properties of the weak and strong convergences we get that $w_i \neq 0$, $i = 1, 2$. □

Lemma 2.3. *Let $A = \{x \in \Omega : w_i(x) \neq 0, \quad (i = 1, 2)\}$, then*

$$g_i = 0 \text{ a.e. in } \Omega \setminus A \text{ where } i = 1, 2.$$

Proof. The inequality (1.7) gives us for every i $(i = 1, 2)$

$$|g_{in}(x)| \leq a_i |w_{i+(-1)^{i+1}n}|^{p-1} + \frac{b_i(x)}{n^{p-1}} \quad \text{a.e. in } \Omega \setminus A, \tag{2.5}$$

so

$$\|g_{in}\|_{L^{p'}(\Omega \setminus A)} \leq a_i \|w_{i+(-1)^{i+1}n}\|^{\frac{p}{p'}}_{L^p(\Omega \setminus A)} + \frac{1}{n^{p-1}} \|b_i\|_{L^{p'}(\Omega \setminus A)}. \tag{2.6}$$

From lemma (2.2), we have

$$\lim_{n \to +\infty} \|g_{in}\|_{L^{p'}(\Omega \setminus A)} = 0. \quad (i = 1, 2) \tag{2.7}$$

Let $D = \{x \in \Omega \setminus A : g_i \neq 0, \quad (i = 1, 2)\}$. By lemma (2.1) we get, for $\phi_i(x) = sign[g_i(x)]\chi_D(x) \in L^p(D)$ such that

$$\chi_D(x) = \begin{cases} 0 & ; x \notin D, \\ 1 & ; x \in D, \end{cases}$$

that

$$\lim_{n \to +\infty} \int_D g_{in}(x) \phi_i(x) dx = \int_D g_i(x) \phi_i(x) dx = \int_D |g_i(x)| dx, \tag{2.8}$$

but, we have by (2.7)

$$\int_D |g_i(x)| dx = 0, \quad (i = 1, 2) \tag{2.9}$$

consequently, $meas(D) = 0$ which implies

$$g_i = 0 \text{ a.e. in } \Omega \setminus A \text{ where } i = 1, 2.$$

□

Lemma 2.4. *Let $i = 1, 2$ and*

$$\tilde{g}_i(x) = \begin{cases} \frac{g_i(x)}{|w(x)_{i+(-1)^{i+1}}|^{p-2} w(x)_{i+(-1)^{i+1}}} & \text{on } A, \\ \beta_i & \text{on } \Omega \setminus A, \end{cases} \tag{2.10}$$

where β_i are fixed numbers such that $\lambda_1 < \beta_i < \lambda_2$, then

$$\lambda_1 \underset{\neq}{\leq} \tilde{g}_i(x) < \lambda_2 \quad \text{a.e. in } \Omega. \tag{2.11}$$

Proof. For $i = 1, 2$, firstly we define new subsets us follow

$$B_{l_i} = \{x \in A : w_{i+(-1)^{i+1}}(x)g_i(x) < l_i(x)|w_{i+(-1)^{i+1}}(x)|^p\},$$
$$B_{k_i} = \{x \in A : w_{i+(-1)^{i+1}}(x)g_i(x) > k_i(x)|w_{i+(-1)^{i+1}}(x)|^p\},$$

then we prove that $meas(B_{l_i}) = meas(B_{k_i}) = 0$.
By remark (1.3), we have that $\forall \varepsilon_i > 0, \quad \exists b_{\varepsilon_i} \in L^{p'}(\Omega)$ such that

$$|w_{i+(-1)^{i+1}n}|^p(l_i - \varepsilon_i) - \frac{b_{\varepsilon_i}}{n^p} \leq w_{i+(-1)^{i+1}n}g_{in} \leq |w_{i+(-1)^{i+1}n}|^p(k_i + \varepsilon_i) + \frac{b_{\varepsilon_i}}{n^p}. \tag{2.12}$$

By integrating in the first inequality and letting $n \to \infty$, then $\varepsilon \to 0$, we deduce

$$\int_{B_{l_i}} [w_{i+(-1)^{i+1}}(x)g_i(x) - |w_{i+(-1)^{i+1}}(x)|^p l_i(x)]dx \geq 0, \tag{2.13}$$

and from the definition of the subset B_{l_i}, we get

$$\int_{B_{l_i}} [w_{i+(-1)^{i+1}}(x)g_i(x) - |w_{i+(-1)^{i+1}}(x)|^p l_i(x)]dx < 0. \tag{2.14}$$

Whereupon

$$\int_{B_{l_i}} [w_{i+(-1)^{i+1}}(x)g_i(x) - |w_{i+(-1)^{i+1}}(x)|^p l_i(x)]dx = 0, \tag{2.15}$$

which implies $meas(B_{l_i}) = 0$. The second inequality give us $meas(B_{k_i}) = 0$.
In the second step, from the definition of $\tilde{g}_i$, we obtain

$$l_i(x) \leq \tilde{g}_i(x) \leq k_i(x) \text{ a.e. in } A, \tag{2.16}$$

and hypothesis (1.3) allow us to write

$$\lambda_1 \leq \tilde{g}_i(x) < \lambda_2 \text{ a.e. in } A. \tag{2.17}$$

Since $\tilde{g}_i = \beta_i$ in $\Omega \setminus A$, then

$$\lambda_1 < \tilde{g}_i < \lambda_2 \text{ in } \Omega \setminus A. \tag{2.18}$$

The inequalities (2.17) and (2.18) leads to

$$\lambda_1 \leq \tilde{g}_i(x) < \lambda_2 \text{ a.e. in } \Omega. \tag{2.19}$$

From (2.18), (2.19) and the fact that $mes(\Omega \setminus A) \neq 0$, we obtain

$$\lambda_1 \underset{\neq}{\leq} \tilde{g}_i(x) < \lambda_2 \quad \text{a.e. in } \Omega.$$

□

Lemma 2.5. *If $i = 1, 2$, then w_i is a solution of*

$$\begin{cases} -\Delta_p w_i = m_i |w_i|^{p-2} w_i & \text{in } \Omega, \\ w_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.20}$$

where $m_i(x) = (1-t)\alpha_i + t\tilde{g}_{i+(-1)^{i+1}}(x)$.

Proof. We first prove that w_i $(i=1,2)$ is a solution of

$$\begin{cases} -\Delta_p w_i = (1-t)\alpha_i |w_i|^{p-2} w_i + t g_{i+(-1)^{i+1}} & \text{in } \Omega, \\ w_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.21}$$

From [3], we have that w_{in} $(i=1,2)$ satisfies

$$\begin{cases} -\Delta_p w_{in} = (1-t_n)|w_{in}|^{p-2} w_{in} + t_n \big[g_{i+(-1)^{i+1}n} + \frac{1}{n^{p-1}} h_i \big] & \text{in } \Omega, \\ w_{in} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.22}$$

We know that for $i=1,2$, $(-\Delta_p)(w_{in})$ are bounded in $W^{-1,p'}(\Omega)$, so we can extract from it a subsequence (w_{in}) (for simplicity of the notation), and a distribution $L_i \in W^{-1,p'}$ such that

$$(-\Delta_p)(w_{in}) \xrightarrow[weak]{} L_i,$$

in particular

$$\lim_{n\to+\infty} < -\Delta_p w_{in}, w_i > = < L_i, w_i > .$$

Since

$$\begin{aligned} < -\Delta_p w_{in}, w_{in} - w_i > &= (1-t_n)\alpha_i \int_\Omega |w_{in}|^{p-2} w_{in}(w_{in} - w_i) dx \\ &\quad + t_n \Big[\int_\Omega g_{i+(-1)^{i+1}n}(w_{in} - w_i) dx + \frac{1}{n^{p-1}} < h_i, w_{in} - w_i > \Big], \end{aligned}$$

it holds

$$\lim_{n\to+\infty} < -\Delta_p w_{in}, w_{in} - w_i > = 0.$$

But, we have

$$\begin{aligned} \lim_{n\to+\infty} < -\Delta_p w_{in}, w_{in} - w_i > &= \lim_{n\to+\infty} < -\Delta_p w_{in}, w_{in} > - \lim_{n\to+\infty} < -\Delta_p w_{in}, w_i > \\ &= \lim_{n\to+\infty} < -\Delta_p w_{in}, w_{in} > - < L_i, w_i > \\ &= 0, \end{aligned}$$

consequently

$$\lim_{n\to+\infty} < -\Delta_p w_{in}, w_{in} > = < L_i, w_i > .$$

We also know that $(-\Delta_p)$ is an operator of type (M), so we get

$$L_i = -\Delta_p w_i.$$

Passing to the limit in (2.22) gives (2.21), but by lemma (2.3), we have

$$(1-t)\alpha_i |w_i|^{p-2} + t g_{i+(-1)^{i+1}} = m_i |w_i|^{p-2} w_i \quad \text{a.e. in } \Omega,$$

which implies that w_i is a solution of (2.20) for every i sush that $i=1,2$. □

Now, we can prove our estimate.

To reach the contradiction, we set $\lambda_1(\Omega, m_i(x))$ (resp., $\lambda_2(\Omega, m_i(x))$ to be the first (resp., the second) eigenvalue of the problem with weight

$$\begin{cases} -\Delta_p u = \lambda m_i(x) |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

For $i = 1, 2$, we use lemma (2.4) and the fact that $\lambda_1 < \alpha_i < \lambda_2$, to get

$$\lambda_1 \underset{\neq}{\leq} m_i(x) < \lambda_2 \quad \text{a.e. in } \Omega;$$

now, by the strict monotonicity property of the first eigenvalue [9] and the second eigenvalue [2], we have

$$\lambda_1(\Omega, m_i) < \lambda_1(\Omega, \lambda_1) = 1,$$

and

$$1 = \lambda_2(\Omega, \lambda_2) < \lambda_2(\Omega, m_i),$$

so clearly

$$\lambda_1(\Omega, m_i) < 1 < \lambda_2(\Omega, m_i).$$

But by lemmas (2.2) and (2.5), for every i (sush that $i = 1, 2$), 1 is an eigenvalue of $(-\Delta_p)$ for the weights m_i, which contradicts the definition of the second eigenvalues $\lambda_2(\Omega, m_i)$.
From above we deduce that the estimation holds true.

3. Proof of the main result

Using the homotopy invariance of the degree map, which through the homotopy T_t yields

$$deg(I - T_0, B(0, R), 0) = deg(I - T_1, B(0, R), 0).$$

As T_0 is odd, so following the theory of Borsuk, we get that $deg(I - T_0, B(0, R), 0)$ is an odd integer and so nonzero. This implies that there exists $(u, v) \in B(0, R)$ such that $T_1(u, v) = (u, v)$. Hence, system (1.1) has a positive solution.

This completes the proof.

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