Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–10.
ISSN-0037-8712 in press
doi:10.5269/bspm.50821

Coupled fixed point theorems of JS-G-contraction on G-Metric Spaces

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ABSTRACT: Jaradat has proven some fixed point results using JS-G-contraction on G-metric spaces. Choudhury et al. were derived coupled fixed point theorems for the G-metric spaces. The purpose of this paper is to prove some coupled fixed point theorems of JS-G-contraction on G-metric spaces. Moreover, some example is presented to illustrate the validity of our results.

Key Words: G-metric space, coupled fixed point, JS-G-contraction.

Contents

1. Introduction

In theory of fixed point, Banach contraction principle is a simple and powerful result. These are several generalizations and extensions of the Banach contraction priciple in the existing literature. Jleli and Samet [7] established new contraction that is $\psi(d(fx, fy)) \leq [\psi(d(x, y))]^k$, where $k \in (0, 1)$ and $d(fx, fy) \neq 0, x, y \in X$ and $\psi \in \Psi$(For more details see [7], [8]). Jaradat and Mustafa [8] introduced new contraction called JS-G-contraction and they proved some fixed point results of such contraction in the setting of G-metric spaces. T.Gnana Bhaskar et al. [5] have derived the coupled fixed point theorems for metric spaces having mixed monotone property and Binayak S. Choudhury et al. [3] have generalized and obtained the results of Gnana Bhaskar et al. of coupled fixed point theorems for G-metric spaces. In this paper we derive the coupled fixed point theorems of JS-G-contraction on G- metric spaces.

2. Preliminaries

Definition 2.1. *[10] Let X be a non-empty set and $G : X \times X \times X \to R^+$ be a function satisfying the following*

1. $G(x, y, z) = 0$ *if* $x = y = z$,
2. $G(x, x, y) > 0$ *for all* $x, y \in X$, *with* $x \neq y$,
3. $G(x, x, y) \leq G(x, y, z)$, *for all* $x, y, z \in X$ *with* $y \neq z$,
4. $G(x, y, z) = G(y, z, x) = G(z, x, y) = \cdots$ *(symmetry in all three variables),*
5. $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, *for all* $x, y, z, a \in X$ *(rectangular inequality).*

Then the function G is called a generalized metric or more specifically a G-metric on X and the pair (X, G) is a G-metric space.

Example 2.2. *[10] If X is a non empty subset of R, then the function $G : X \times X \times X \to [0, \infty)$, given by $G(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in X$, is a G-metric on X.*

Example 2.3. *[19] Let $X = \{0, 1, 2\}$ and let $G : X \times X \times X \to [0, \infty)$ be the function given by the following table.*

2010 *Mathematics Subject Classification:* 54H25, 47H10.
Submitted November 01, 2019. Published May 02, 2021

(x,y,z)	G(x,y,z)
(0,0,0), (1,1,1), (2,2,2)	0
(0,0,1), (0,1,0), (1,0,0), (0,1,1), (1,0,1), (1,1,0)	1
(1,2,2), (2,1,2), (2,2,1)	2
(0,0,2), (0,2,0), (2,0,0), (0,2,2), (2,0,2), (2,2,0)	3
(1,1,2), (1,2,1), (2,1,1), (0,1,2), (0,2,1), (1,0,2)	4
(1,2,0), (2,0,1), (2,1,0)	4

Then G is a G-metric on X, but it is not symmetric because $G(1,1,2) = 4 \neq 2 = G(2,2,1)$.

Definition 2.4. *[12] Let (X, G) be a G-metric space, let $\{x_n\}$ be sequence of points of X, a point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$ and we say that the sequence $\{x_n\}$ is G-convergent to x. Thus, if $x_n \to x$ in a G-metric space (X, G), then for any $\epsilon > 0$, there exists a positive integer N such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$.*

Definition 2.5. *[15] Let (X, G) be a G-metric space. The sequence $\{x_n\}$ is said to be G-Cauchy if for every $\epsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_l) <\in$ for all $n, m, l \geq N$.*

Lemma 2.6. *[10] Let (X, G) be a G- metric space, then the following are equivalent:*

(1) $\{x_n\}$ is G-convergent to x.

(2) $G(x_n, x_n, x) \to 0$, as $n \to \infty$.

(3) $G(x_n, x, x) \to 0$, as $n \to \infty$.

(4) $G(x_m, x_n, x) \to 0$, as $m, n \to \infty$.

Lemma 2.7. *[10] If (X, G) be a G-metric space, then the following are equivalent:*

(1) $\{x_n\}$ is G-Cauchy.

(2) for every $\epsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_m) <\in$ for all $n, m \geq N$.

Lemma 2.8. *[6] If (X, G) be a G-metric space, then $G(x, y, z) \leq 2G(x, y, z)$ for all $x, y \in X$.*

Lemma 2.9. *[5] If (X, G) be a G-metric space, then The sequence $\{x_n\}$ is a $G-$Cauchy sequence if and only if for every $\epsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_m) < \epsilon$ for all $m > n \geq N$.*

Definition 2.10. *[13] Let (X, G) and (X', G') be two G-metric spaces and $f : (X, G) \to (X', G')$ be a function, then f is said to be G-continous at a point $a \in X$ if and only if it is G sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G-convergent to $f(x)$.*

Definition 2.11. *[6] A G metric space (X, G) is called symmetric G-metric space if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.*

Definition 2.12. *[10] A $G-$metric space (X, G) is said to be G-complete (or complete G-metric space) if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).*

Definition 2.13. *[5] An element $(x, y) \in X \times X$; when X is any non empty set, is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.*

Definition 2.14. *[3] Let (X, G) be a G-metric space. A mapping $F : X \times X \to X$ is said to be continuous if for any two G-convergent sequences $\{x_n\}$ and $\{y_n\}$ converging to x and y respectively, $F(x_n, y_n)$ is G-convergent to $F(x, y)$.*

Jleli and Samet [7] introduced a new type of contraction which involves the following set of all functions $\psi : (0,\infty) \to (1,\infty)$ satisfying the conditions:
(ψ_1) ψ is non decreasing;
(ψ_2) for each sequence $t_n \subseteq (0,\infty)$, $\lim_{n\to\infty} \psi(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;
(ψ_3) there exist $r \in (0,1)$ and $L \in (0,\infty]$ such that $\lim_{t\to 0^+} \frac{\psi(t)-1}{t^r} = L$.
To be consistent with Jleli and Samet, we denote by Ψ the set of all functions $\psi : (0,\infty) \to (1,\infty)$ satisfying the conditions $(\psi_1 - \psi_3)$.
Also, they established the following result as a generalization of Banach contraction principle.

Theorem 2.15. *Let (X,d) be a complete metric space and $f : X \to X$ be a mapping. Suppose that there exist $\psi \in \Psi$ and $k \in (0,1)$ such that $x, y \in X, d(fx, fy) \neq 0$ implies $\psi(d(fx, fy)) \leq [\psi(d(x,y))]^k$. Then f has a unique fixed point.*

In 2015, Hussain et al. [6] customized the above family of functions and proved a fixed point theorem as a generalization of [6]. They customized the family of functions $\psi : (0,\infty) \to (1,\infty)$ to be as follows:
(ψ_1) ψ is non decreasing and $\psi(t) = 1$ if and only if $t = 0$;
(ψ_2) for each sequence $\{t_n\} \subseteq (0,\infty)$, $\lim_{n\to\infty} \psi(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;
(ψ_3) there exist $r \in (0,1)$ and $L \in (0,\infty]$ such that $\lim_{t\to 0^+} \frac{\psi(t)-1}{t^r} = L$;
(ψ_4) $\psi(u+v) \leq \psi(u).\psi(v)$ for all $u, v > 0$.

To be consistent with Hussain et al [6], we denote by Ψ the set of all functions $\psi : (0,\infty) \to (1,\infty)$ satisfying the conditions $(\psi_1 - \psi_4)$.

Definition 2.16. *[2] Let (X,G) be a G-metric space, and $g : X \to X$ be a self mapping. Then g is said to be a JS-G-contraction whenever there exist a function $\psi \in \Psi$ and positive real numbers r_1, r_2, r_3, r_4 with $0 \leq r_1 + 3r_2 + r_3 + 2r_4 < 1$ such that*

$$\psi(G(gx, gy, gz)) \leq [\psi(G(x,y,z))]^{r_1} [\psi(G(x, gx, gz))]^{r_2} [\psi(G(y, gy, gz))]^{r_3} [\psi(G(x, gy, gy) + G(y, gx, gx))]^{r_4} \tag{2.1}$$

for all $x, y, z \in X$

Jaradat et al. [8] proved the following theorem.

Theorem 2.17. *Let (X,G) be a complete G-metric space and $g : X \to X$ be a JS-G-contraction. Then g has a unique fixed point.*

Our first result is the following;

3. Main Results

Theorem 3.1. *Let (X,G) be a G-metric space, and let $f : X \times X \to X$ be a mapping. Suppose there exist a function $\psi \in \Psi$ and positive real numbers r_1, r_2, r_3, r_4 with $0 \leq r_1 + 3r_2 + r_3 + 2r_4 < 1$ such that*

$$\psi(G(f(x,u), f(y,v), f(z,w)) \leq [\psi(G(x,y,z))]^{r_1} [\psi(G(x, f(x,u), f(z,w)))]^{r_2} [\psi(G(y, f(y,v), f(z,w)))]^{r_3} [\psi(G(x, f(y,v), f(y,v)) + G(y, f(x,u), f(x,u)))]^{r_4} \tag{3.1}$$

for all $x, y, z, u, v, w \in X$. Then f has a unique coupled fixed point.

Proof. Let $x_0 \in X$ be arbitrary. For $x_0 \in X$, we define the sequence $\{x_n\}$ by $x_n = f^n(x_0, u_0) = f(x_{n-1}, u_{n-1})$. If there exist $n_0 \in N$ such that $(x_{n_0}, u_{n_0}) = (x_{n_0+1}, u_{n_0+1})$, then (x_{n_0}, u_{n_0}) is a fixed point of f, and we have nothing to prove. Thus we suppose that $x_n \neq x_{n+1}$ that is $G(f(x_n, u_n), f(x_n, u_n), f(x_n, u_n)) > 0$ for all $n \in N$. Now, we will prove that $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$. from (3.1), we get that

$$\begin{aligned}
1 < \psi(G(x_n, x_{n+1}, x_{n+1})) =& \psi(G(f(x_{n-1}, u_{n-1}), f(x_n, u_n), f(x_n, u_n))\\
&\leq [\psi(G(x_{n-1}, x_n, x_n))]^{r_1}\\
&[\psi(G(x_{n-1}, f(x_{n-1}, u_{n-1}), f(x_n, u_n)))]^{r_2}\\
&[\psi(G(x_n, f(x_n, u_n), f(x_n, u_n)))]^{r_3}\\
&[\psi(G(x_{n-1}, f(x_n, u_n), f(x_n, u_n))\\
&+ G(x_n, f(x_{n-1}, u_{n-1}), f(x_{n-1}, u_{n-1})))]^{r_4}\\
&= [\psi(G(x_{n-1}, x_n, x_n))]^{r_1} [\psi(G(x_{n-1}, x_n, x_{n+1}))]^{r_2}\\
&[\psi(G(x_n, x_{n+1}, x_{n+1}))]^{r_3} [\psi(G(x_{n-1}, x_{n+1}, x_{n+1})\\
&+ G(x_n, x_n, x_n))]^{r_4}\\
&= [\psi(G(x_{n-1}, x_n, x_n))]^{r_1} [\psi(G(x_{n-1}, x_n, x_{n+1}))]^{r_2}\\
&[\psi(G(x_n, x_{n+1}, x_{n+1}))]^{r_3} [\psi(G(x_{n-1}, x_{n+1}, x_{n+1}))]^{r_4}
\end{aligned}$$

using (G_5) and (ψ_4), we get

$$\begin{aligned}
\psi(G(x_{n-1}, x_n, x_{n+1})) &\leq \psi(G(x_{n-1}, x_n, x_n) + G(x_n, x_n, x_{n+1}))\\
&\leq \psi(G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1}))\\
&\leq \psi(G(x_{n-1}, x_n, x_n)) + \psi(2G(x_n, x_{n+1}, x_{n+1}))\\
&= \psi(G(x_{n-1}, x_n, x_n))\psi(G(x_n, x_{n+1}, x_{n+1})\\
&+ G(x_n, x_{n+1}, x_{n+1}))\\
&\leq \psi(G(x_{n-1}, x_n, x_n))[\psi(G(x_n, x_{n+1}, x_{n+1}))]^2
\end{aligned}$$

and

$$\begin{aligned}
\psi(G(x_{n-1}, x_{n+1}, x_{n+1})) &\leq \psi(G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}))\\
&\leq \psi(G(x_{n-1}, x_n, x_n))\psi(G(x_n, x_{n+1}, x_{n+1}))
\end{aligned}$$

Therefore

$$\begin{aligned}
1 < \psi(G(x_n, x_{n+1}, x_{n+1})) \leq& [\psi(G(x_{n-1}, x_n, x_n))]^{r_1} [\psi(G(x_{n-1}, x_n, x_n))]^{r_2}\\
&[\psi(G(x_n, x_{n+1}, x_{n+1}))]^{2r_2} [\psi(G(x_n, x_{n+1}, x_{n+1}))]^{r_3}\\
&[\psi(G(x_{n-1}, x_n, x_n))]^{r_4} [\psi(G(x_n, x_{n+1}, x_{n+1}))]^{r_4}
\end{aligned}$$

by recording the product terms of the above inequality, then using the induction, we get that

$$\begin{aligned}
1 < \psi(G(x_n, x_{n+1}, x_{n+1})) &\leq [\psi(G(x_{n-1}, x_n, x_n))]^{\frac{r_1+r_2+r_4}{1-2r_2-r_3-r_4}}\\
&\quad .\\
&\quad .\\
&\quad .\\
&\leq [\psi(G(x_0, x_1, x_1))]^{(\frac{r_1+r_2+r_4}{1-2r_2-r_3-r_4})^n} \qquad (3.2)
\end{aligned}$$

Taking limit as $n \to \infty$, and noting that $\frac{r_1+r_2+r_4}{1-2r_2-r_3-r_4} < 1$, we get

$$\lim_{n\to\infty} \psi(G(x_n, x_{n+1}, x_{n+1})) = 1 \tag{3.3}$$

which implies by ψ_2 that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{3.4}$$

From the condition ψ_3, there exist $0 < r < 1$ and $L \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\psi(G(x_n, x_{n+1}, x_{n+1})) - 1}{[G(x_n, x_{n+1}, x_{n+1})]^r} = L.$$

Suppose that $L < \infty$. In this case, let $B_1 = \frac{L}{2} > 0$. From the definition of the limit, there exist $n_0 \in N$ such that

$$|\frac{\psi(G(x_n, x_{n+1}, x_{n+1})) - 1}{[G(x_n, x_{n+1}, x_{n+1})]^r} - L| \leq B_1,$$

for all $n > n_0$. This implies that

$$\frac{\psi(G(x_n, x_{n+1}, x_{n+1})) - 1}{[G(x_n, x_{n+1}, x_{n+1})]^r} \geq L - B_1 = \frac{L}{2} = B_1,$$

for all $n > n_0$. Then

$$n.[G(x_n, x_{n+1}, x_{n+1})]^r \leq A_1.n.[\psi(G(x_n, x_{n+1}, x_{n+1})) - 1],$$

where $A_1 = \frac{1}{B_1}$.

Now for $L = \infty$, let $B_2 > 0$ be an arbitrary number, from the definition of the limit, there exist $n_1 \in N$ such that

$$|\frac{\psi(G(x_n, x_{n+1}, x_{n+1})) - 1}{[G(x_n, x_{n+1}, x_{n+1})]^r}| \geq B_2,$$

for all $n > n_1$. Then

$$n.[G(x_n, x_{n+1}, x_{n+1})]^r \leq A_2.n.[\psi(G(x_n, x_{n+1}, x_{n+1})) - 1],$$

where $A_2 = \frac{1}{B_2}$.

Thus, in both cases, there exist $A = max\{A_1, A_2\} > 0$ and $n_p = max\{n_0, n_1\} \in N$ such that

$n.[G(x_n, x_{n+1}, x_{n+1})]^r \leq A.n.[[\psi(G(x_n, x_{n+1}, x_{n+1}))]^{\alpha^n} - 1]$, where , $\alpha = \frac{r_1+r_2+r_4}{1-2r_2-r_3-r_4}$. But,

$$
\begin{aligned}
&\lim_{n\to\infty} n.[[\psi(G(x_0,x_1,x_1))]^{\alpha^n}-1] \\
&= \lim_{n\to\infty} \frac{[[\psi(G(x_0,x_1,x_1))]^{\alpha^n}-1]}{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{\alpha^n.ln(\alpha).ln(\psi(G(x_0,x_1,x_1)))[[\psi(G(x_n,x_{n+1},x_{n+1}))]^{\alpha^n}]}{\frac{-1}{n^2}} \\
&= \lim_{n\to\infty} (-n^2).\alpha^n.ln(\alpha).ln(\psi(G(x_0,x_1,x_1)))[[\psi(G(x_n,x_{n+1},x_{n+1}))]^{\alpha^n}] \\
&= \lim_{n\to\infty} \frac{(-n^2).ln(\alpha).ln(\psi(G(x_0,x_1,x_1)))[[\psi(G(x_n,x_{n+1},x_{n+1}))]^{\alpha^n}]}{\alpha_1^n} \\
&= \lim_{n\to\infty} \frac{-n^2}{\alpha_1^n}. \lim_{n\to\infty} ln(\alpha).ln(\psi(G(x_0,x_1,x_1)))[[\psi(G(x_n,x_{n+1},x_{n+1}))]^{\alpha^n}] \\
&= 0.ln(\alpha).ln(\psi(G(x_0,x_1,x_1))) \\
&= 0
\end{aligned}
$$

where $\alpha_1 = \frac{1}{\alpha}$. Which implies that $\lim_{n\to\infty} n.[G(x_n,x_{n+1},x_{n+1})]^r = 0$, thus there exist $n_2 \in N$ such that $G(x_n,x_{n+1},x_{n+1}) \leq \frac{1}{n^{\frac{1}{r}}}$, for all $n > n_2$. Now, for $m > n > n_2$, we have

$$G(x_n,x_m,x_m) \leq \sum_{i=n}^{m-1} G(x_i,x_{i+1},x_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{r}}} \sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{r}}}.$$

Since $0 < r < 1$, then $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{r}}}$ is G-convergent and hence $G(x_n,x_m,x_m) \to 0$ as $m,n \to \infty$. Thus, we proved that $\{x_n\}$ is a G-**Cauchy** sequence. Completeness of (X,G) ensures that there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Now we shall show that (x^*,u^*) is a coupled fixed point of f. Using (G_5) we get that

$$
\begin{aligned}
G(x^*,x^*,f(x^*,u^*)) \leq & G(x^*,x^*,x_{n+1}) + G(x_{n+1},x_{n+1},f(x^*,u^*)) \\
& G(x^*,x^*,x_{n+1}) + G(f(x_n,u_n),f(x_n,u_n),f(x^*,u^*))
\end{aligned} \tag{3.5}
$$

and

$$G(x_n,x_{n+1},f(x^*,u^*)) \leq G(x_n,x_{n+1},x^*) + G(x^*,x^*,f(x^*,u^*)) \tag{3.6}$$

Hence, by the properties of ψ we get that

$$\psi(G(x^*,x^*,f(x^*,u^*))) \leq \psi(G(x^*,x^*,x_{n+1}))\psi(G(x_{n+1},x_{n+1},f(x^*,u^*))) \tag{3.7}$$

$$\psi(G(x_n,x_{n+1},f(x^*,u^*))) \leq \psi(G(x_n,x_{n+1},x^*))\psi(G(x^*,x^*,f(x^*,u^*))) \tag{3.8}$$

Thus,

$$
\begin{aligned}
[\psi(G(x_n,x_{n+1},f(x^*,u^*)))]^{r_2+r_3} \leq & [\psi(G(x_n,x_{n+1},x^*))]^{r_2+r_3} \\
& [\psi(G(x^*,x^*,f(x^*,u^*)))]^{r_2+r_3}
\end{aligned} \tag{3.9}
$$

However, by using (3.1), (ψ_4) and (3.9) we have

$$\begin{aligned}\psi(G(x_n, x_{n+1}, f(x^*, u^*))) &= \psi(G(f(x_n, u_n), f(x_n, u_n), f(x^*, u^*)))\\ &\leq [\psi(G(x_n, x_n, x^*))]^{r_1}[\psi(G(x_n, x_{n+1}, f(x^*, u^*)))]^{r_2}\\ &[\psi(G(x_n, x_{n+1}, f(x^*, u^*)))]^{r_2}\\ &[\psi(G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1}))]^{r_4}\\ &= [\psi(G(x_n, x_n, x^*))]^{r_1}\\ &[\psi(G(x_n, x_{n+1}, f(x^*, u^*)))]^{r_2+r_3}\\ &[\psi(G(x_n, x_{n+1}, x_{n+1}))]^{2r_4}\\ &\leq [\psi(G(x_n, x_n, x^*))]^{r_1}[\psi(G(x_n, x_{n+1}, x^*))]^{r_2+r_3}\\ &[\psi(G(x^*, x^*, f(x^*, u^*)))]^{r_2+r_3}\\ &[\psi(G(x_n, x_{n+1}, x_{n+1}))]^{2r_4}\end{aligned} \tag{3.10}$$

Now, substituting (3.10) in (3.7) we get that

$$\begin{aligned}\psi(G(x^*, x^*, f(x^*, u^*))) \leq \psi(G(x^*, x^*, x_{n+1}))&[\psi(G(x_n, x_n, x^*))]^{r_1}\\ &[\psi(G(x_n, x_{n+1}, x^*))]^{r_2+r_3}\\ &[\psi(G(x^*, x^*, f(x^*, u^*)))]^{r_2+r_3}\\ &[\psi(G(x_n, x_{n+1}, x_{n+1}))]^{2r_4}\end{aligned} \tag{3.11}$$

Hence,

$$\begin{aligned}1 \leq [\psi(G(x^*, x^*, f(x^*, u^*)))]^{1-r_2-r_3} \leq \psi(G(x^*, x^*, x_{n+1}))&[\psi(G(x_n, x_n, x^*))]^{r_1}\\ &[\psi(G(x_n, x_{n+1}, x^*))]^{r_2+r_3}\\ &[\psi(G(x_n, x_{n+1}, x_{n+1}))]^{2r_4}\end{aligned} \tag{3.12}$$

By taking the limit as $n \to \infty$ and using (3.4), (ψ_2), proposition (1.3) and the convergence of $\{x_n\}$ to x^* in the above equation we get that

$$\psi(G(x^*, x^*, f(x^*, u^*))) = 1 \tag{3.13}$$

which implies by (ψ_1) that $G(x^*, x^*, f(x^*, u^*)) = 0$ and so $x^* = f(x^*, u^*)$. Thus (x^*, u^*) is a coupled fixed point of f. Finally to show the uniqueness, assume that there exist $(x^*, u^*) \neq (x', u')$ such that $x' = f(x', u')$. By (G_2), $G(x', x', x^*) = G(f(x', u'), f(x', u'), f(x^*, u^*)) > 0$. Thus, by (3.1) we get

$$\begin{aligned}\psi(G(x', x', x^*)) &= \psi(G(f(x', u'), f(x', u'), f(x^*, u^*)))\\ &\leq [\psi(G(x', x', x^*))]^{r_1}[\psi(G(x', f(x', u'), f(x^*, u^*)))]^{r_2}\\ &[\psi(G(x', f(x', u'), f(x^*, u^*)))]^{r_3}\\ &[\psi(G(x', f(x', u'), f(x', u')) + G(x', f(x', u'), f(x', u')))]^{r_4}\\ &= [\psi(G(x', x', x^*))]^{r_1}[\psi(G(x', x', x^*))]^{r_2}\\ &[\psi(G(x', x', x^*))]^{r_3}\\ &[\psi(G(x', x', x')) + G(x', x', x'))]^{r_4}\\ &= [\psi(G(x', x', x^*))]^{r_1+r_2+r_3}\end{aligned}$$

which leads to a contraction because $r_1 + r_2 + r_3 < 1$. Therefore, f has a unique coupled fixed point. □

The following result is a direct consequence of theorem 3.1 by taking $\psi(t) = e^{\sqrt{t}}$ in (3.1)

Corollary 3.2. *Let (X,G) be a G-metric space, and let $f : X \times X \to X$ be a mapping. Suppose there exist a nonnegative real numbers r_1, r_2, r_3, r_4 with $0 \le r_1 + 3r_2 + r_3 + 2r_4 < 1$ such that*

$$\begin{aligned} &\sqrt{G(f(x,u), f(y,v), f(z,w))} \\ &\le r_1.\sqrt{G(x,y,z)} + r_2.\sqrt{G(x, f(x,u), f(z,w))} \\ &+ r_3.\sqrt{G(y, f(y,v), f(z,w))} \\ &+ r_4.\sqrt{G(x, f(y,v), f(y,v)) + G(y, f(x,u), f(x,u))} \end{aligned} \tag{3.14}$$

for all $x,y,z,u,v,w \in X$. Then f has a unique coupled fixed point.

Remark 3.3. *Note that condition (3.14) is equivalent to*

$$\begin{aligned} &G(f(x,u), f(y,v), f(z,w)) \\ &\le r_1^2.G(x,y,z) + r_1^2.G(x, f(x,u), f(z,w)) \\ &+ r_3^2.G(y, f(y,v), f(z,w)) \\ &+ r_4^2.[G(x, f(y,v), f(y,v)) + G(y, f(x,u), f(x,u))] \\ &+ 2r_1r_2\sqrt{G(x,y,z)G(x, f(x,u), f(z,w))} \\ &+ 2r_1r_3\sqrt{G(x,y,z)G(y, f(y,v), f(z,w))} \\ &+ 2r_1r_4\sqrt{G(x,y,z)[G(x, f(y,v), f(y,v)) + G(y, f(x,u), f(x,u))]} \\ &+ 2r_2r_3\sqrt{G(x,y,z)[G(x, f(x,u), f(z,w)) + G(y, f(y,v), f(z,w))]} \\ &+ 2r_2r_4\sqrt{G(x, f(x,u), f(z,w))[G(x, f(y,v), f(y,v)) + G(y, f(x,u), f(x,u))]} \\ &+ 2r_3r_4\sqrt{G(y, f(y,v), f(z,w))[G(x, f(y,v), f(y,v)) + G(y, f(x,u), f(x,u))]} \end{aligned}$$

Next, by taking $r_2 = r_3 = r_4 = 0$ in corollary (3.1), we obtain the following corollary.

Corollary 3.4. *Let (X,G) be a G-metric space, and let $f : X \times X \to X$ be a mapping. Suppose there exists a positive real number $0 < r_1 < 1$ such that $G(f(x,u), f(y,v), f(z,w)) \le r_1^2 G(x,y,z)$ for all $x,y,z,u,v,w \in X$. Then f has a unique coupled fixed point.*

Finally, by taking $\psi(t) = e^{\sqrt[n]{t}}$ in (3.1), we get the following corollary.

Corollary 3.5. *Let (X,G) be a G-metric space, and let $f : X \times X \to X$ be a mapping. Suppose there exist a positive real numbers r_1, r_2, r_3, r_4 with $0 \le r_1 + 3r_2 + r_3 + 2r_4 < 1$ such that*

$$\begin{aligned} \sqrt[n]{G(f(x,u), f(y,v), f(z,w))} \le\ & r_1.\sqrt[n]{G(x,y,z)} + r_2.\sqrt[n]{G(x, f(x,u), f(z,w))} \\ &+ r_3.\sqrt[n]{G(y, f(y,v), f(z,w))} \\ &+ r_4.\sqrt[n]{G(x, f(y,v), f(y,v)) + G(y, f(x,u), f(x,u))} \end{aligned}$$

for all $x,y,z,u,v,w \in X$. Then f has a unique coupled fixed point.

Remark 3.6. *By specifying $r_i = 0$ for some $i \in \{1,2,3,4\}$ in remark (3.1) and corollary (3.1), we can get several results.*

Example 3.7. *Let $X = [0,\infty)$ and let $G(x,y,z) = max\{|x-y|, |y-z|, |z-x|\}$ for all $x,y,z \in X$. Then (X,G) is a G-metric space. Let $f(x,y) = \frac{x+y}{8}$ and $\psi(t) = e^{\sqrt{t}}$. Then clearly all conditions of theorem 3.1 are satisfied with $r_i = \frac{1}{\sqrt{8}}$; $i = 1,2,3,4$, and $(x,y) = (0,0)$ is a coupled fixed point of f.*

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