Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–18.
ISSN-0037-8712 in press
doi:10.5269/bspm.50822

Existence of Fixed Points in G-Metric Spaces

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Abstract: In this manuscript, we provide some new results for the existence of fixed points for a certain contractive condition of Geraghty type in the setting of partially ordered G-metric space. Also, we provide an example to illustrate the usability of results. Our results generalize or extend many well known results in the literature.

Key Words: G-Metric space, self-map, Banach Contraction, fixed point, poset, compatible mappings, comparable.

Contents

1. Introduction

In 2005 Mustafa and Sims [17] introduced the notion of new structure of metric spaces called $G-$metric spaces and derived some fixed point theorems in the setting of $G-$metric spaces. Thereafter, various researchers find the generalizations of contraction mappings in such spaces and obtained beautiful results. In this paper, we provide some new results for the existence of fixed points for a certain contractive condition of Geraghty type in the setting of partially ordered G-metric space. Also, we provide an example to illustrate the usability of results obtained. Our results generalize many well known results of Gordji [7], Aydi [4], Al-Mohiameed [23] and Sharma et al. [21] in the setting of partially ordered $G-$metric spaces.

2. Preliminaries

We begin with the definition of known known class of generalized metric spaces and two important known examples.

Definition 2.1. *[17] Let X be a non-empty set, and let $G : X \times X \times X \to R^+$, be a function satisfying the following*

1. $G(x, y, z) = 0$ *if* $x = y = z$,
2. $G(x, x, y) > 0$ *for all* $x, y \in X$, *with* $x \neq y$,
3. $G(x, x, y) \leq G(x, y, z)$, *for all* $x, y, z \in X$ *with* $y \neq z$,
4. $G(x, y, z) = G(y, z, x) = G(z, x, y) = \cdots$ *(symmetry in all three variables),*
5. $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, *for all* $x, y, z, a \in X$ *(rectangular inequality).*

Then the function G is called a generalized metric or more specifically a $G-$metric on X and the pair (X, G) is a $G-$metric space.

Example 2.2. *[27] If X is a non empty subset of R, then the function $G : X \times X \times X \to [0, \infty)$, given by $G(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in X$, is a symmetric $G-$metric on X.*

2010 *Mathematics Subject Classification:* 47H10, 54H25.
Submitted November 01, 2019. Published June 09, 2021

Example 2.3. *[27] Let $X = \{a, b\}$. Define $G(a, a, a) = G(b, b, b) = 0$, $G(a, a, b) = 1$, $G(a, b, b) = 2$, and extend G to X^3 by using the symmetry in the variables. Then it is clear that (X, G) is an asymmetric $G-$ metric space.*

Definition 2.4. *[23] Let (X, G) be a $G-$metric space, let $\{x_n\}$ be sequence of points of X, a point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$ and we say that the sequence $\{x_n\}$ is $G-$convergent to x. Thus, if $x_n \to x$ in a $G-$metric space (X, G), then for any $\epsilon > 0$, there exists a positive integer N such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$.*

Definition 2.5. *[19] Let (X, G) and $(X^{'}, G^{'})$ be $G-$metric spaces and let $f : (X, G) \to (X^{'}, G^{'})$ be a function, then f is said to be $G-$continous at a point $a \in X$ if and only if, given $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$; and $G(a, x, y) < \delta$ implies $G^{'}(f(a), f(x), f(y)) < \epsilon$. A function f is $G-$continuous on X if and only if it is a $G-$continuous at all $a \in X$.*

Definition 2.6. *[17] Let (X, G) be a $G-$metric space. The sequence $\{x_n\}$ is said to be $G-$Cauchy if for every $\epsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$.*

Definition 2.7. *[17] A $G-$metric space (X, G) is said to be $G-$Complete (or Complete $G-$metric space) if every $G-$Cauchy sequence in (X, G) is $G-$convergent in (X, G).*

Definition 2.8. *A partial order is a binary relation $\preceq$ over a set X satisfying the following properties. For all a, b and c in X,*

(1) Reflexivity : $a \preceq a$ (every element is related to itself)

(2) Antisymmetry : If $a \preceq b$ and $b \preceq a$ then $a = b$

(3) Transitivity : If $a \preceq b$ and $b \preceq c$ then $a \preceq c$

A set X with a partial order $\preceq$ is called a partially ordered set(also called a Poset).

Example 2.9. *The set of natural numbers equipped with the relation of divisibility is a Poset.*

Definition 2.10. *[21] For a, b elements of a partially ordered set X, if $a \preceq b$ or $b \preceq a$, then a and b are comparable.*

Definition 2.11. *[21] Let $(X, \preceq)$ be a partially ordered set and $f, g : X \to X$ are said to be*
(2.8.1) Weakly increasing if $f(x) \preceq g(f(x))$ and $g(x) \preceq f(g(x))$ for all $x \in X$.
(2.8.2) Partially weakly increasing if $f(x) \preceq g(f(x))$ for all $x \in X$.
(2.8.3) Weakly increasing with respect to $H : X \to X$ if $f(X) \subseteq H(X)$, $g(X) \subseteq H(X)$, if and only if for all $x \in X$, $f(x) \preceq g(y)$, for all $y \in H^{-1}(f(x))$, $g(x) \preceq f(y)$, for all $y \in H^{-1}(g(x))$.

Example 2.12. *[4] Let $X = [0, 1]$ be endowed with usual ordering and $f, g : X \to X$ be define by $f(x) = x^2$ and $g(x) = \sqrt{x}$. Since $f(x) = x^2 \preceq g(f(x)) = g(x^2) = x$, it is easy to see that (f, g) is partially weakly increasing. But $g(x) = \sqrt{x} \npreceq x = f(g(x))$ for all $x \in X$ implies (g, f) is not partially weakly increasing.*

Definition 2.13. *[21] Let $(X, \preceq)$ be a partially ordered set and $f, g : X \to X$ then*
(2.9.1) f is called weak annihilator of g if $f(g(x)) \preceq x$ for all $x \in X$.
(2.9.2) f is called dominating if $x \preceq f(x)$ for all $x \in X$.

Example 2.14. *[21] Let $X = [0, 1]$ be endowed with usual ordering and $f, g : X \to X$ be define by $f(x) = x^2$ and $g(x) = x^3$. Since $f(g(x)) = x^6 \preceq x$, for all $x \in X$ thus f is a weak annihilator of g.*

Example 2.15. *[21] Let $X = [0, 1]$ be endowed with usual ordering and $f : X \to X$ be define by $f(x) = x^{\frac{1}{3}}$ since $x \preceq x^{\frac{1}{3}} = f(x)$ for all $x \in X$ so f is a dominating map.*

Definition 2.16. *[21] A subset W of a partially ordered set X is called well ordered if every pair of elements of W are comparable.*

Definition 2.17. *[21] Suppose f and g are self maps of a $G-$metric space (X, G). If $\lim_{n\to\infty} G(f(g(x_n)), g(f(x_n)), g(f(x_n))) = 0$ for every sequence $\{x_n\}$ in X with $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = u$ for some $u \in X$, then the pair (f, g) is said to be compatible.*
Weakly compatible *if they commute at their coincidence points, that is, if $f(x) = g(x)$ for some $x \in X$ then $f(g(x)) = g(f(x))$.*

Now, we define a more general class of functions.

Definition 2.18. *$\mathcal{F}$ is the family of functions $\alpha : R^+ \to [0, 1)$ with*

(1) $R^+ = \{t \in R : t > 0\}$,

(2) $\alpha(t_n) \to 1$ implies $t_n \to 0$.

Note 1. *(1) We do not assume that α is continuous in any sense.*

(2) We only require that if α gets near one, it does so only near zero.

3. Main Results

Our first new result is the next:

Theorem 3.1. *Let $(X, \preceq)$ be a partially ordered set and there exists a metric G in X such that (X, G) is a complete metric space and $f : X \to X$ be a non-decreasing self mapping such that there exists $x_0 \in X$ with $x_0 \preceq f(x_0)$ satisfying*
(3.1.1) $G(f(x), f(y), f(z)) \leq \alpha(G(x, y, z))G(x, y, z)$, for all $x, y, z \in X$ with $x \preceq y \preceq z$;
(3.1.2) either f is continuous or there exist a non-decreasing sequence $\{x_n\}$ in X such that $x_n \to x$ then $x_n \preceq x$, for all $n \in \mathbb{N}$. Then f has a fixed point in X. Further, if assume the following
(3.1.3) For any $x, y, z \in X$, there exists $u \in X$ which is comparable to x, y, z. Then f has a unique fixed point in X.

Proof. Assume that x_0 be an arbitrary point in X with $x_0 \preceq f(x_o)$. Define $x_n = f^n(x_0), n = 1, 2, 3, \cdots$. Since f is a non-decreasing function, by induction we obtain that

$$x_0 \preceq f(x_0) \preceq f^2(x_0) \preceq \cdots \preceq f^n(x_0) \preceq f^{n+1}(x_0) \preceq \cdots . \tag{3.1}$$

Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$ then by (3.1.1), we have

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+3}) &= G(x_{n+1}, x_{n+2}, x_{n+3}) \\ &= G(f^{n+1}x_0, f^{n+2}x_0, f^{n+3}x_0) \\ &\leq \alpha(G(f^n x_0, f^{n+1}x_0, f^{n+2}x_0))G(f^n x_0, f^{n+1}x_0, f^{n+2}x_0) \\ &\leq \alpha(G(x_n, x_{n+1}, x_{n+2}))G(x_n, x_{n+1}, x_{n+2}) \\ &\leq G(x_n, x_{n+1}, x_{n+2}). \end{aligned}$$

Therefore, the sequence $\{G(x_n, x_{n+1}, x_{n+2})\}$ is non-increasing and bounded below. Thus there exists $t \geq 0$ such that $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+2}) = t$. Since $\alpha(t) \to 1$ implies $t \to 0$ then there exist $r \in [0, 1)$ and $\epsilon > o$ such that $\alpha(t) \to r$ for all $t \in [t, t + \epsilon]$. We can take $\nu \in \mathbb{N}$ such that $t \leq G(x_n, x_{n+1}, x_{n+2}) \leq t + \epsilon$, for all $n \in \mathbb{N}$ with $n \geq \nu$. Since

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+3}) &\leq \alpha(G(x_n, x_{n+1}, x_{n+2}))G(x_n, x_{n+1}, x_{n+2}) \\ &\leq rG(x_n, x_{n+1}, x_{n+2}), \end{aligned} \tag{3.2}$$

for all $n \in \mathbb{N}$ with $n \geq \nu$. Therefore, we have

$$\begin{aligned} \sum_{n=1}^{\infty}(G(x_n, x_{n+1}, x_{n+2})) \leq& \sum_{n=1}^{\nu}(G(x_n, x_{n+1}, x_{n+2}) \\ &+ \sum_{n=1}^{\infty} r^n(G(x_\nu, x_{\nu+1}, x_{\nu+2}) < \infty. \end{aligned}$$

Hence $\{x_n\}$ is a $G-$ Cauchy sequence. Since X is complete, $\{x_n\}$ converges to some point of $u \in X$. Now, we prove that u is a fixed point of f. If f is continuous, then

$$u = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f^n(x_0) = \lim_{n\to\infty} f^{n+1}(x_0) = f(\lim_{n\to\infty} f^n(x_0)) = f(u). \tag{3.3}$$

Hence $u = f(u)$.

Suppose that there exists a non-decreasing sequence $\{x_n\}$ in X such that $x_n \to x$. Consider

$$\begin{aligned} G(f(u), u, u) &\le G(f(u), f(x_n), f(x_n)) + G(f(x_n), f(x_n), u) \\ &\le \alpha(G(x_n, x_n, u))G(x_n, x_n, u) + G(x_{n+1}, x_{n+1}, u) \\ &\le G(x_n, x_n, u) + G(x_{n+1}, x_{n+1}, u). \end{aligned}$$

since $G(x_n, x_n, u) \to 0$ then we get $f(u) = u$.

To prove the uniqueness of the fixed point, assume that w ($w \neq u$) is another fixed point of f. From (3.1.3), there exists $x \in X$ which is comparable to u and w. Monotonically implies that $f^n(x)$ is comparable to $f^n(u) = u$ and $f^n(w) = w$ for $n = 0, 1, 2, \cdots$

Moreover,

$$\begin{aligned} G(u, w, f^n(x)) &= G(f^n(u), f^n(w), f^n(x)) \\ &\le \alpha(G(f^{n-1}(u), f^{n-1}(w), f^{n-1}(x))) \\ &\quad G(f^{n-1}(u), f^{n-1}(w), f^{n-1}(x)) \\ &\le G(f^{n-1}(u), f^{n-1}(w), f^{n-1}(x)) \\ &= G(u, w, f^{n-1}(x)) \end{aligned}$$

Consequently, the sequence $\xi_n^u = G(u, w, f^n(x))$ is non-negative and decreasing and so $\lim_{n\to\infty} G(u, u, f^n(x)) = \xi_u \in X$. Similarly we can show that the sequence $\xi_n^w = G(w, w, f^n(x))$ is non negative and decreasing and so $\lim_{n\to\infty} G(w, w, f^n(x)) = \xi_w \in X$.

Now similarly the above method we can choose $r_1, r_2 \in [0, 1)$ and $t \in \mathbb{N}$ such that

$$\begin{aligned} G(u, u, f^n(x)) &\le \alpha(G(u, u, f^{n-1}(x)))G(u, u, f^{n-1}(x)) \\ &\le r_1 G(u, u, f^{n-1}(x)) \\ G(w, w, f^n(x)) &\le \alpha(G(w, w, f^{n-1}(x)))G(w, w, f^{n-1}(x)) \\ &\le r_2 G(w, w, f^{n-1}(x)), \end{aligned} \tag{3.4}$$

for all $n \in \mathbb{N}$ with $n > t_1$. Finally

$$\begin{aligned} G(u, u, w) &\le G(u, u, f^n(x)) + G(f^n(x), w, w) \\ &\le r_1^{n-t_1} G(u, u, f^{t_1}(x_0)) + r_2^{n-t_1} G(w, w, f^{t_1}(x_0)) \end{aligned} \tag{3.5}$$

for all $n \in \mathbb{N}$ with $n > t_1$. Therefore by taking $n \to \infty$ in (3.5), we have $G(u, u, w) = 0$. Therefore $u = w$.
□

Now we announce our second new result:

Theorem 3.2. *Let $(X, \preceq)$ be a partially ordered set and there exists a metric G on X such that (X, G) is a complete $G-$ metric space. Let f, g, h, S, T and $U : X \to X$ be given mappings satisfying*
(3.2.1) $f(X) \subseteq U(X)$, $g(X) \subseteq T(X)$ and $h(X) \subseteq S(X)$,
(3.2.2) for every comparable elements $x, y, z \in X$,
$G(f(x), g(y), h(z)) \le \alpha(G(S(x), T(y), U(z)))G(S(x), T(y), U(z))$ where $\alpha \in \mathcal{F}$
(3.2.3) The pairs $(U, f), (T, g)$and (S, h) are partially weakly increasing,
(3.2.4) f, g, h are dominating and weak annihilator maps of U, T and S respectively.

(3.2.5) there exist a non decreasing sequence $\{x_n\}$ *with* $x_n \preceq y_n$ *for all* n *and* $y_n \to u$ *implies* $x_n \preceq u$.
(3.2.6) either
(3.2.6.1) (f, S), (g, T) *are compatible, pair* (h, U) *is weakly compatible and* f, g *or* S, T *are continuous maps.*

(OR)

(3.2.6.2) (g, T), (h, U) *are compatible, pair* (f, S) *is weakly compatible and* g, h *or* T, U *are continuous maps.*

(OR)

(3.2.6.3) (h, U), (f, S) *are compatible, pair* (g, T) *is weakly compatible and* h,f *or* U, S *are continuous maps. Then* f, g, h, S, T *and* U *have a common fixed point. Moreover,*f, g, h, S, T *and* U *have one common fixed point if and only if the set of common fixed point of* f, g, h, S, T *and* U *is well ordered.*

Proof. Let x_0 be an arbitrary point in X. Since $f(X) \subseteq U(X)$, $g(X) \subseteq T(X)$ and $h(X) \subseteq S(X)$, we can construct sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that $y_{3n-1} = f(x_{3n-2}) = U(x_{3n-1})$, $y_{3n} = g(x_{3n-1}) = T(x_{3n})$ and $y_{3n+1} = h(x_{3n}) = S(x_{3n+1})$ for all $n \in \mathbb{N}$. From (3.2.4), we write

$$x_{3n-2} \preceq f(x_{3n-2}) = U(x_{3n-1}) \preceq f(U(x_{3n-1})) \preceq x_{3n-1},$$

$$x_{3n-1} \preceq g(x_{3n-1}) = T(x_{3n}) \preceq T(g(x_{3n})) \preceq x_{3n}$$

and

$$x_{3n} \preceq h(x_{3n}) = S(x_{3n+1}) \preceq S(h(x_{3n+1})) \preceq x_{3n+1}.$$

Thus, we have $x_n \preceq x_{n+1}$, for all $n \geq 1$. Put $x = x_{3n}, y = x_{3n+1}, z = x_{3n+2}$ in (3.2.2), we get

$$G(f(x_{3n}), g(x_{3n+1}), h(x_{3n+2})) \leq \alpha(G(S(x_{3n}), T(x_{3n+1}), U(x_{3n+2})))G(S(x_{3n}), T(x_{3n+1}), U(x_{3n+2})),$$

that is,

$$G(y_{3n+1}, y_{3n+2}, y_{3n+3}) \leq \alpha(G(y_{3n}, y_{3n+1}, y_{3n+2}))G(y_{3n}, y_{3n+1}, y_{3n+2}). \tag{3.6}$$

If there exist $n \in \mathbb{N}$ such that $G(y_{3n}, y_{3n+1}, y_{3n+2}) = 0$ then it follows from (3.6) that $y_{3n} = y_{3n+1} = y_{3n+2}$. This leads to $y_m = y_{3n+1}$ for any $m \geq 3n$. This implies that $\{y_m\}$ is a $G-$ Cauchy sequence.

We shall show that

$$\lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+2}) = 0 \tag{3.7}$$

Now suppose that $G(y_{3n}, y_{3n+1}, y_{3n+2}) > 0$ for all $n \in \mathbb{N}$. Put $x = x_{3n+2}, y = x_{3n+1}$ *and* $z = x_{3n}$ in (3.2.2), we get

$$\begin{aligned} G(f(x), g(y), h(z)) &= G(f(x_{3n+2}), g(x_{3n+1}), h(x_{3n})) \\ &= G(y_{3n+3}, y_{3n+2}, y_{3n+1}) \\ &\leq \alpha(G(S(x_{3n+2}), T(x_{3n+1}), U(x_{3n})))G(S(x_{3n+2}), T(x_{3n+1}), U(x_{3n})) \\ &\leq \alpha(G(y_{3n+2}, y_{3n+1}, y_{3n}))G(y_{3n+2}, y_{3n+1}, y_{3n}). \end{aligned} \tag{3.8}$$

Using $0 \leq \alpha < 1$, we get

$$G(y_{3n+3}, y_{3n+2}, y_{3n+1}) \leq G(y_{3n+2}, y_{3n+1}, y_{3n}). \tag{3.9}$$

similarly we write

$$G(y_{3n+2}, y_{3n+1}, y_{3n}) \leq G(y_{3n+1}, y_{3n}, y_{3n-1}). \tag{3.10}$$

for $x = x_{3n}$, $y = x_{3n+1}$ and $z = x_{3n+2}$ in (3.2.2). From (3.9) and (3.10), for any $n \in \mathbb{N}$, we get

$$G(y_{n+2}, y_{n+1}, y_n) \leq G(y_{n+1}, y_n, y_{n-1}). \tag{3.11}$$

Therefore the sequence $\{G(y_{n+1}, y_n, y_{n-1})\}$ is monotonic decreasing. Hence, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} G(y_{n+1}, y_n, y_{n-1}) = r \tag{3.12}$$

From (3.8), we have

$$\frac{G(y_{3n+3}, y_{3n+2}, y_{3n+1})}{G(y_{3n+2}, y_{3n+1}, y_{3n})} \leq \alpha(G(y_{3n+2}, y_{3n+1}, y_{3n})) < 1.$$

Letting $n \to \infty$ in the above inequality, and using (3.12), we get $\lim_{n\to\infty} \alpha(G(y_{3n+2}, y_{3n+1}, y_{3n})) = 1$. By the property of α, it follows that $r = 0$, hence (3.7) holds.

Now to check that $\{y_n\}$ is a $G-$ Cauchy sequence. It suffices to prove that $\{y_{3n}\}$ is a $G-$ Cauchy sequence. To do this, we proceed by contradiction. Suppose that $\{y_{3n}\}$ is not a $G-$ Cauchy sequence. Then for any $\epsilon > 0$, there exists three sequences of positive integers $m(k)$ and $n(k)$such that for all positive integers k, $m(k) > n(k) > k$, we have

$$\begin{aligned} G(y_{3m(k)}, y_{3n(k)}, y_{3n(k)}) &> \epsilon, \\ G(y_{3m(k)}, y_{3n(k)-2}, y_{3n(k)-2}) &\leq \epsilon. \end{aligned} \tag{3.13}$$

Therefore we use (3.13) and triangular inequality, we get

$$\begin{aligned} \epsilon &< G(y_{3m(k)}, y_{3n(k)}, y_{3n(k)}) \\ &\leq G(y_{3m(k)}, y_{3n(k)-2}, y_{3n(k)-2}) + G(y_{3n(k)-2}, y_{3n(k)-1}, y_{3n(k)-1}) + G(y_{3n(k)-1}, y_{3n(k)-1}, y_{3n(k)}) \\ &\leq \epsilon + G(y_{3n(k)-2}, y_{3n(k)-1}, y_{3n(k)-1}) + G(y_{3n(k)-1}, y_{3n(k)}, y_{3n(k)}). \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (3.7) we get

$$\lim_{k\to\infty} G(y_{3m(k)}, y_{3n(k)}, y_{3n(k)}) = \epsilon. \tag{3.14}$$

Again using the triangular inequality, we have
$G(y_{3n(k)}, y_{3m(k)}, y_{3n(k)}) \leq G(y_{3n(k)}, y_{3m(k)-1}, y_{3m(k)-1}) + G(y_{3m(k)-1}, y_{3m(k)}, y_{3n(k)})$
$|G(y_{3n(k)}, y_{3m(k)-1}, y_{3m(k)-1}) - G(y_{3n(k)}, y_{3m(k)}, y_{3n(k)})| \leq G(y_{3m(k)}, y_{3m(k)-1}, y_{3n(k)})$.
Letting again $k \to \infty$ in the above inequality and using (3.7) and (3.14) we get

$$\lim_{k\to\infty} G(y_{3n(k)}, y_{3m(k)-1}, y_{3m(k)-1}) = \epsilon. \tag{3.15}$$

On the other hand, we have

$$\begin{aligned} G(y_{3n(k)}, y_{3m(k)}, y_{3n(k)}) &\leq G(y_{3n(k)}, y_{3n(k)+1}, y_{3n(k)+1}) + G(y_{3n(k)+1}, y_{3m(k)}, y_{3n(k)}) \\ &= G(y_{3n(k)}, y_{3n(k)+1}, y_{3n(k)+1}) + G(fx_{3n(k)}, gx_{3m(k)-1}, hx_{3n(k)-1}) \end{aligned}$$

using (3.7) and (3.14) and letting $k \to \infty$ in the above inequality, we get

$$\epsilon \leq \lim_{k\to\infty} G(fx_{3n(k)}, gx_{3m(k)-1}, hx_{3n(k)-1}). \tag{3.16}$$

Choose $x = x_{3n(k)}$ and $y = x_{3m(k)-1}$ and $z = x_{3m(k)-1}$ in (3.2.2), we get

$$\begin{aligned} &G(fx_{3n(k)}, gx_{3m(k)-1}, hx_{3m(k)-1}) \leq \alpha(G(S(x_{3n(k)}), T(x_{3m(k)-1}), U(x_{3m(k)-1}))) \\ &G(S(x_{3n(k)}), T(x_{3m(k)-1}), U(x_{3m(k)-1})) \\ &< G(S(x_{3n(k)}), T(x_{3m(k)-1}), U(x_{3m(k)-1})). \end{aligned}$$

Letting again $k \to \infty$ in the above inequality and using (3.7) and (3.15), we get

$$\lim_{k\to\infty} G(fx_{3n(k)}, gx_{3m(k)-1}, hx_{3m(k)-1}) \leq \epsilon. \tag{3.17}$$

Combining (3.16) and (3.17), we get $\lim_{k\to\infty} G(fx_{3n(k)}, gx_{3m(k)-1}, hx_{3m(k)-1}) = \epsilon$. Since $y_{3n(k)} \neq y_{3m(k)-1}$, then

$$\frac{G(fx_{3n(k)}, gx_{3m(k)-1}, hx_{3m(k)-1})}{G(S(x_{3n(k)}), T(x_{3m(k)-1}), U(x_{3m(k)-1}))} \leq \alpha(G(S(x_{3n(k)}), T(x_{3m(k)-1}), U(x_{3m(k)-1}))) < 1.$$

Using the fact

$$\epsilon = \lim_{k\to\infty} G(fx_{3n(k)}, gx_{3m(k)-1}, hx_{3m(k)-1}) = \lim_{k\to\infty} G(S(x_{3n(k)}), T(x_{3m(k)-1}), U(x_{3m(k)-1})).$$

Hence we get $\lim_{k\to\infty} \alpha(G(S(x_{3n(k)}), T(x_{3m(k)-1}), U(x_{3m(k)-1}))) = 1$. Using the property of α, we get $\lim_{k\to\infty} G(S(x_{3n(k)}), T(x_{3m(k)-1}), U(x_{3m(k)-1})) = 0$. Hence $\lim_{k\to\infty} G(y_{3n(k)}, y_{3m(k)-1}, y_{3m(k)-1}) = 0$, which is a contradiction with (3.14). Therefore, $\{y_{3n}\}$ is a $G-$ Cauchy sequence. Since (X, G) is a complete $G-$ metric space, there exists $u \in X$ such that

$$\lim_{n\to\infty} y_{3n} = u. \tag{3.18}$$

Therefore

$$\lim_{n\to\infty} y_{3n+1} = \lim_{n\to\infty} U(x_{3n+1}) = \lim_{n\to\infty} f(x_{3n}) = u. \tag{3.19}$$

$$\lim_{n\to\infty} y_{3n+2} = \lim_{n\to\infty} T(x_{3n+2}) = \lim_{n\to\infty} g(x_{3n+1}) = u. \tag{3.20}$$

$$\lim_{n\to\infty} y_{3n+3} = \lim_{n\to\infty} S(x_{3n+3}) = \lim_{n\to\infty} h(x_{3n+2}) = u. \tag{3.21}$$

Assume that S is continuous. Since f, S are compatible, we have

$$\lim_{n\to\infty} f(S(x_{3n+2})) = \lim_{n\to\infty} S(f(x_{3n+2})) = S(u). \tag{3.22}$$

Also, $x_{3n+1} \preceq gx_{3n+1} = Tx_{3n+2}$. Now,

$$\begin{aligned} G(f(S(x_{3n+2})), g(x_{3n+1}), h(x_{3n+1})) \leq \alpha(G(S(S(x_{3n+2})), T(x_{3n+1}), U(x_{3n+1}))) \\ G(S(S(x_{3n+2})), T(x_{3n+1}), U(x_{3n+1})). \end{aligned}$$

Letting $n \to \infty$ in the above inequality and using (3.20) and (3.22)
we get $G(S(u), u, u) < G(S(u), u, u)$, hence $S(u) = u$. Now $(x_{3n+1}) \preceq g(x_{3n+1})$ and $g(x_{3n+1}) \to u$ as $n \to \infty$, $x_{3n+1} \preceq u$ and (3.2.2) becomes

$$G(f(u), g(x_{3n+1}), h(x_{3n+1})) \leq \alpha(G(S(u), T(x_{3n+1}), U(x_{3n+1})))G(S(u), T(x_{3n+1}), U(x_{3n+1})).$$

Letting $n \to \infty$ in the above inequality and using (3.19) we get $G(f(u), u, u) < G(S(u), u, u)$, hence $f(u) = u$. Since $f(X) \subseteq U(X)$, there exist a point $v \in X$ such that $f(u) = U(v)$. Suppose that $h(v) \neq U(v)$. Since $u \preceq f(u) = U(v) \preceq f(U(v)) \preceq v$ implies $u \preceq v$. From (3.2.2) we obtain $G(U(v), h(v), h(v)) = G(f(u), h(v), h(v)) \leq G(S(u), U(v), U(v)) = G(u, u, u) = 0$, which is a contradiction. Hence $U(v) = h(v)$. Since h and U are weakly compatible, therefore $h(u) = h(f(u)) = h(U(v)) = U(h(v)) = U(f(u)) = U(u)$. Thus u is a coincidence point of h and U. Now, since $x_{3n} \preceq f(x_{3n})$ and $f(x_{3n}) \to u$ as $n \to \infty$, implies that $x_{3n} \preceq u$. From (3.2.2) we get $G(f(x_{3n}), h(u), h(u)) \leq \alpha(G(S(x_{3n}), U(u), U(u)))G(S(x_{3n}), U(u), U(u))$. If $\lim_{n\to\infty} G(S(x_{3n}), U(u), U(u)) = 0$ then $G(u, U(u), U(u)) = 0$. Hence $U(u) = u$.
If $\lim_{n\to\infty} G(S(x_{3n}), Uu), U(u)) \neq 0$. Letting $n \to \infty$ in the above inequality and using (3.21), we get

$$1 = \frac{G(u, h(u), h(u))}{G(u, U(u), U(u))} = \frac{\lim_{n\to\infty} G(f(x_{3n}), h(u), h(u))}{\lim_{n\to\infty} G(S(x_{3n}), U(u), U(u))} \leq \lim_{n\to\infty} \alpha(G(S(x_{3n}), U(u), U(u))) \leq 1.$$

Using property of α, we get $\lim_{n\to\infty} G(S(x_{3n}), U(u), U(u)) = 0$, which is a contradiction. Hence $u = U(u)$. Therefore, $f(u) = h(u) = S(u) = U(u) = u$. Assume that T is continuous since g and T are compatible, we have

$$\lim_{n\to\infty} g(T(x_{3n+3})) = \lim_{n\to\infty} T(g(x_{3n+3})) = T(u).$$

Also $(x_{3n+2}) \preceq h(x_{3n+2}) = S(x_{3n+3})$. Now

$$\begin{aligned} G(f(x_{3n+2}), g(T(x_{3n+3})), h(x_{3n+2})) \leq & \alpha(G(S(x_{3n+2}), T(T(x_{3n+3})), U(x_{3n+2}))) \\ & G(G(S(x_{3n+2}), T(T(x_{3n+3})), U(x_{3n+2}))). \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get $G(u, T(u), u) < G(u, T(u), u)$, hence $T(u) = u$. From (3.2.2) becomes $G(f(u), g(u), h(u)) \leq \alpha(G(S(u), T(u), U(u)))G(S(u), T(u), U(u))$ implies $G(u, gu, u) \leq G(u, u, u) \leq 0$. Hence $g(u) = u$. Therefore, from above, we have $f(u) = g(u) = h(u) = S(u) = T(u) = U(u) = u$. Similarly, the result follows when (3.2.6.2) holds, (3.2.6.3) holds. Suppose that the set of common fixed points of f, g, h, S, T and U is well ordered and u and v ($u \neq v$) are any two fixed points of f, g, h, S, T and U. From (3.2.2), we have

$$\begin{aligned} & G(f(u), g(v), g(v)) \leq \alpha(G(S(u), T(v), T(v)))G(S(u), T(v), T(v)) \\ & G(u, v, v) \leq \alpha(G(u, v, v))G(u, v, v) \\ & G(u, v, v) < G(u, v, v), \end{aligned}$$

which is a contradiction. Therefore $u = v$ that is unique common fixed point of f, g, h, S, T and U. Conversely, if f, g, h, S, T and U have only one common fixed point then the set of common fixed point of f, g, h, S, T and U being singleton is well ordered. □

On the similar lines of Theorem 3.2, we have the following results.

Theorem 3.3. *Let $(X, \preceq)$ be a partially ordered set and there exists a metric G on X such that (X, G) is a complete $G-$ metric space. Let f, g, h, S, T and $U : X \to X$ be given mappings satisfying*
(3.3.1) $f(X) \subseteq U(X)$, $g(X) \subseteq T(X)$ and $h(X) \subseteq S(X)$,
(3.3.2) for every comparable elements $x, y, z \in X$,
$G(f(x), g(y), h(z)) \leq \alpha(G(S(x), T(y), U(z)))G(S(x), T(y), U(z))$, where $\alpha \in \mathcal{F}$
(3.3.3) The pairs $(U, f), (T, g)$ and (S, h) are partially weakly increasing,
(3.3.4) f, g, h are dominating and weak annihilator maps of U, T and S respectively.
(3.3.5) there exist a non decreasing sequence $\{x_n\}$ with $x_n \preceq y_n$ for all n and $y_n \to u$ implies $x_n \preceq u$.
(3.3.6) either
(3.3.6.1) (f, S), (g, T) are compatible, (h, U) is weakly compatible and f, g or S, T are continuous maps.

(OR)

(3.3.6.2) (g, T), (h, U) are compatible, (f, S) is weakly compatible and g, h or T, U are continuous maps.

(OR)

(3.3.6.3) (h, U), (f, S) are compatible, (g, T) is weakly compatible and h,f or U, S are continuous maps. and (3.3.7) $G(f(x), g(y), h(z)) \leq \alpha(M(x, y, z))M(x, y, z)$, where

$$\begin{aligned} M(x, y, z) = \max\{ & G(S(x), T(y), U(z)), G(f(x), S(x), S(x)), G(g(y), T(y), T(y)), G(h(z), U(z), U(z)), \\ & \frac{1}{3}(G(S(x), g(y), h(z)) + G(f(x), T(y), h(z)) + G(f(x), g(y), U(z)))\} \end{aligned}$$

for all$x, y, z \in X$ with $x \preceq y \preceq z$ and $\alpha \in \mathsf{F}$. Then f, g, h, S, T and U have a common fixed point. Moreover, the common fixed point of f, g, h, S, T and U is unique if and only if the set of common fixed point of f, g, h, S, T and U is well ordered.

Proof. Let x_0 be an arbitrary point in X. Since $f(X) \subseteq U(X)$ then there exists $x_1 \in X$ such that $f(x_0) = U(x_1)$, since $g(X) \subseteq T(X)$ then there exists $x_2 \in X$ such that $g(x_1) = T(x_2)$, since $h(X) \subseteq S(X)$ then there exists $x_3 \in X$ such that $h(x_2) = S(x_3)$. on continuing this process, We can construct sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that $y_{3n-1} = f(x_{3n-2}) = U(x_{3n-1}), y_{3n} = g(x_{3n-1}) = T(x_{2n}), y_{3n+1} = h(x_{3n}) = S(x_{2n+1})$ for all $n = 1, 2, 3, \cdots$ From conditions (3.3.3) and (3.3.4) we have $x_{3n-2} \leq f(x_{3n-2}) = U(x_{3n-1}) \leq f(U(x_{3n-1})) \leq x_{3n-1}, x_{3n-1} \leq g(x_{3n-1}) = T(x_{2n}) \leq T(g(x_{3n})) \leq x_{3n}$ and $x_{3n} \leq h(x_{3n}) = S(x_{3n+1}) \leq S(h(x_{3n+1})) \leq x_{3n+1}$. Thus, for all $n \geq 1$ we obtain $x_1 \leq x_2 \leq .... \leq x_n \leq x_{n+1}$. that is a non decreasing sequence. Now we prove that y_n is a $G-$ Cauchy sequence in X. For this let us consider that $G(y_{3n}, y_{3n+1}, y_{3n+2}) > 0$ for every n. If not then $y_{3n} = y_{3n+1} = y_{3n+2}$, for some n, therefore using (3.3.7) we have

$$G(y_{3n+1}, y_{3n+2}, y_{3n+3}) = G(f(x_{3n}), g(x_{3n+1}), h(x_{3n+2})) \leq \alpha(M(x_{3n}, x_{3n+1}, x_{3n+2}))M(x_{3n}, x_{3n+1}, x_{3n+2})$$

where

$$\begin{aligned}
M(x_{3n}, x_{3n+1}, x_{3n+2}) =& max\{G(S(x_{3n}), T(x_{3n+1}), U(x_{3n+2})), G(f(x_{3n}), S(x_{3n}), S(x_{3n})),\\
& G(g(x_{3n+1}), T(x_{3n+1}), T(x_{3n+1})), G(h(x_{3n+2}), U(x_{3n+2}), U(x_{3n+2})),\\
& \frac{1}{3}(G(S(x_{3n}), g(x_{3n+1}), h(x_{3n+2})) + G(T(x_{3n+1}), f(x_{3n}), h(x_{3n+2}))\\
& + G(U(x_{3n+2}), f(x_{3n}), g(x_{3n+1})))\}\\
=& max\{G(y_{3n}, y_{3n+1}, y_{3n+2}), G(y_{3n+1}, y_{3n}, y_{3n}), G(y_{3n+2}, y_{3n+1}, y_{3n+1}),\\
& G(y_{3n+3}, y_{3n+2}, y_{3n+2}), \frac{1}{3}(G(y_{3n}, y_{3n+2}, y_{3n+3}) + G(y_{3n+1}, y_{3n}, y_{3n+3})\\
& + G(y_{3n+2}, y_{3n+1}, y_{3n+2}))\}\\
=& max\{0, 0, 0, G(y_{3n+3}, y_{3n+2}, y_{3n+2}), \frac{1}{3}(G(y_{3n}, y_{3n+1}, y_{3n+3})\\
& + G(y_{3n}, y_{3n+1}, y_{3n+3}) + 0)\}\\
=& max\{0, 0, 0, G(y_{3n}, y_{3n+1}, y_{3n+3}), \frac{2}{3}(G(y_{3n}, y_{3n+1}, y_{3n+3})\}\\
=& G(y_{3n}, y_{3n+1}, y_{3n+3}) = G(y_{3n+1}, y_{3n+2}, y_{3n+3})
\end{aligned}$$

hence

$$\begin{aligned}
G(y_{3n+1}, y_{3n+2}, y_{3n+3}) =& G(f(x_{3n}), g(x_{3n+1}), h(x_{3n+2}))\\
\leq& \alpha(G(y_{3n+1}, y_{3n+2}, y_{3n+3}))G(y_{3n+1}, y_{3n+2}, y_{3n+3})
\end{aligned}$$

using $0 \leq \alpha < 1$, we get

$$G(y_{3n+1}, y_{3n+2}, y_{3n+3}) < G(y_{3n+1}, y_{3n+2}, y_{3n+3})$$

which is a contradiction. Hence we must have $y_{3n+1} = y_{3n+2} = y_{3n+3}$. Similarly, we obtain $y_{3n+2} = y_{3n+3} = y_{3n+4}$ and so on. Thus $\{y_n\}$ be a constant sequence and $\{y_{3n}\}$ is the common fixed point of $f, g, h, S, T$and U. Now we suppose $G(y_{3n}, y_{3n+1}, y_{3n+2}) > 0$ for every n, since $x = x_{3n}$, $y = x_{3n+1}$ and $z = x_{3n+2}$ are comparable elements so using (3.3.7), we have

$$\begin{aligned}
G(y_{3n+1}, y_{3n+2}, y_{3n+3}) &= G(f(x_{3n}), g(x_{3n+1}), h(x_{3n+2}))\\
&\leq \alpha(M(x_{3n}, x_{3n+1}, x_{3n+2}))M(x_{3n}, x_{3n+1}, x_{3n+2}) \qquad (3.23)
\end{aligned}$$

where

$$\begin{aligned} M(x_{3n}, x_{3n+1}, x_{3n+2}) = & \max\{G(S(x_{3n}), T(x_{3n+1}), U(x_{3n+2})), G(f(x_{3n}), S(x_{3n}), S(x_{3n})), \\ & G(g(x_{3n+1}), T(x_{3n+1}), T(x_{3n+1})), G(h(x_{3n+2}), U(x_{3n+2}), U(x_{3n+2})), \\ & \frac{1}{3}(G(S(x_{3n}), g(x_{3n+1}), h(x_{3n+2})) + G(T(x_{3n+1}), f(x_{3n}), h(x_{3n+2})) \\ & + G(U(x_{3n+2}), f(x_{3n}), g(x_{3n+1}))\} \\ = & \max\{G(y_{3n}, y_{3n+1}, y_{3n+2}), G(y_{3n+1}, y_{3n}, y_{3n}), G(y_{3n+2}, y_{3n+1}, y_{3n+1}), \\ & G(y_{3n+3}, y_{3n+2}, y_{3n+2}), \frac{1}{3}(G(y_{3n}, y_{3n+2}, y_{3n+3}) \\ & + G(y_{3n+1}, y_{3n+1}, y_{3n+3}) + G(y_{3n+2}, y_{3n+1}, y_{3n+2}))\} \\ = & \max\{G(y_{3n}, y_{3n+1}, y_{3n+2}), G(y_{3n+1}, y_{3n+2}, y_{3n+3}), \\ & \frac{1}{3}(G(y_{3n}, y_{3n+1}, y_{3n+2}) + 0 + 0)\} \\ = & \max\{G(y_{3n}, y_{3n+1}, y_{3n+2}), G(y_{3n+1}, y_{3n+2}, y_{3n+3})\} \end{aligned}$$

Now $M(x_{3n}, x_{3n+1}, x_{3n+2})$ is either $G(x_{3n}, x_{3n+1}, x_{3n+2})$ or $G(y_{3n+1}, y_{3n+2}, y_{3n+3})$.
If $M(x_{3n}, x_{3n+1}, x_{3n+2}) = G(y_{3n+1}, y_{3n+2}, y_{3n+3})$ then from (3.3.7), we have

$$\begin{aligned} G(y_{3n+1}, y_{3n+2}, y_{3n+3}) = G(f(x_{3n}), g(x_{3n+1}), h(x_{3n+2})) \leq \alpha(G(y_{3n+1}, y_{3n+2}, y_{3n+3})) \\ G(y_{3n+1}, y_{3n+2}, y_{3n+3}) \end{aligned}$$

using $0 \leq \alpha < 1$, we get $G(y_{3n+1}, y_{3n+2}, y_{3n+3}) < G(y_{3n+1}, y_{3n+2}, y_{3n+3})$ which is a contradiction. hence $M(x_{3n}, x_{3n+1}, x_{3n+2}) = G(y_{3n}, y_{3n+1}, y_{3n+2})$ and from (3.3.7), we have

$$G(y_{3n+1}, y_{3n+2}, y_{3n+3}) = G(f(x_{3n}), g(x_{3n+1}), h(x_{3n+2})) \leq \alpha(G(y_{3n}, y_{3n+1}, y_{3n+2}))G(y_{3n}, y_{3n+1}, y_{3n+2}) \tag{3.24}$$

using $0 \leq \alpha < 1$, we get $G(y_{3n+1}, y_{3n+2}, y_{3n+3}) \leq G(y_{3n}, y_{3n+1}, y_{3n+2})$. Similarly put $x = x_{3n-1}, y = x_{3n}$ and $z = x_{3n+1}$ in (3.3.7), we have $G(y_{3n}, y_{3n+1}, y_{3n+2}) \leq G(y_{3n-1}, y_{3n}, y_{3n+1})$.
Hence for any n, $G(y_{3n}, y_{3n+1}, y_{3n+2}) \leq G(y_{3n-1}, y_{3n}, y_{3n+1}) \leq ...... \leq G(y_2, y_1, y_0)$ implies that the sequence $\{G(y_{n+2}, y_{n+1}, y_n)\}$ is monotonically non increasing sequence. Hence there exist $r \geq 0$ such that

$$\lim_{n\to\infty} G(y_{n+2}, y_{n+1}, y_n) = r \tag{3.25}$$

using (3.24), we have $\frac{G(y_{3n+1}, y_{3n+2}, y_{3n+3})}{G(y_{3n}, y_{3n+1}, y_{3n+2})} \leq \alpha(G(y_{3n}, y_{3n+1}, y_{3n+2})) < 1$, letting $n \to \infty$ and using (3.25), we have $\lim_{n\to\infty} \alpha(G(y_{3n}, y_{3n+1}, y_{3n+2})) = 1$, since $\alpha \in F$ yields that $r = 0$. Consequently

$$\lim_{n\to\infty} \alpha(G(y_{n+2}, y_{n+1}, y_n)) = 0 \tag{3.26}$$

Now we claim that $\{y_{3n}\}$ is a $G-$Cauchy sequence. suppose on the contrary that $\{y_{3n}\}$ is not a $G-$Cauchy sequence then for any $\epsilon > 0$ and there exist an integers $3m_k$ and $3n_k$ with $3m_k > 3n_K > k$ for all $k > 0$ such that $G(y_{3m_k}, y_{3n_k}, y_{3n_k}) \geq \epsilon$ and

$$G(y_{2m_k-2}, y_{2n_k-1}, y_{2p_k}) < \epsilon \tag{3.27}$$

from triangle inequality, we have

$$\begin{aligned} \epsilon \leq G(y_{3m_k}, y_{3n_k}, y_{3n_k}) = G(y_{3m_k}, y_{3n_k-2}, y_{3n_k-2}) + G(y_{3n_k-2}, y_{3n_k-1}, y_{3n_k-1}) + G(y_{3n_k-1}, y_{3n_k}, y_{3n_k}) \\ \leq \epsilon + G(y_{3n_k-2}, y_{3n_k-1}, y_{3n_k-1}) + G(y_{3n_k-1}, y_{3n_k}, y_{3n_k}). \end{aligned}$$

Letting $k \to \infty$ and using (3.26), we have

$$\lim_{k\to\infty} G(y_{3m_k}, y_{3n_k}, y_{3n_k}) = \epsilon \tag{3.28}$$

Now for all $k > 0$ from (3.26) and (3.27), we have

$$\begin{aligned}\epsilon &\le G(y_{3m_k}, y_{3n_k}, y_{3n_k})\\ &\le G(y_{3m_k}, y_{3m_k-1}, y_{3m_k-1}) + G(y_{3m_k-1}, y_{3n_k}, y_{3n_k})\end{aligned}$$

implies that $\epsilon \le \lim_{k\to\infty} G(y_{3m_k-1}, y_{3n_k}, y_{3n_k})$ on the other hand from (3.26)and (3.28), we have $G(y_{3m_k-1}, y_{3n_k}, y_{3n_k}) \le G(y_{3m_k-1}, y_{3m_k}, y_{3m_k}) + G(y_{3m_k}, y_{3n_k}, y_{3n_k})$ implies that $\lim_{k\to\infty} G(y_{3m_k-1}, y_{3n_k}, y_{3n_k}) \le \epsilon$. Hence

$$\lim_{k\to\infty} G(y_{3m_k-1}, y_{3n_k}, y_{3n_k}) = \epsilon. \tag{3.29}$$

Similarly for all k, from (3.26) and (3.27), we have $G(y_{3m_k}, y_{3n_k}, y_{3n_k}) \le G(y_{3m_k}, y_{3n_k+1}, y_{3n_k+1}) + G(y_{3n_k+1}, y_{3n_k}, y_{3n_k})$ implies that $\epsilon \le \lim_{k\to\infty} G(y_{3m_k}, y_{3n_k+1}, y_{3n_k+1})$ on the other hand from (3.26)and (3.28), we have $G(y_{3m_k}, y_{3n_k+1}, y_{3n_k+1}) \le G(y_{3n_k}, y_{3n_k}, y_{3n_k+1}) + G(y_{3n_k}, y_{3n_k+1}, y_{3m_k})$ implies that $\lim_{k\to\infty} G(y_{3m_k}, y_{3n_k+1}, y_{3n_k+1}) \le \epsilon$. Hence

$$\lim_{k\to\infty} G(y_{3m_k}, y_{3n_k+1}, y_{3n_k+1}) = \epsilon \tag{3.30}$$

$$\begin{aligned}G(y_{3n_k+1}, y_{3n_k+1}, y_{3m_k}) =& G(f(x_{3n_k}), g(x_{3n_k}), h(x_{3m_k-1}))\\ &\le \alpha(M(x_{3n_k+1}, x_{3n_k}, x_{3m_k-1}))M(x_{3n_k}, x_{3n_k}, x_{3m_k-1})\end{aligned} \tag{3.31}$$

where

$$\begin{aligned}M(x_{3n_k}, x_{3n_k}, x_{3m_k-1}) =& max\{G(Sx_{3n_k}, Tx_{3n_k}, Ux_{3m_k-1})), G(f(x_{3n_k}), S(x_{3n_k}), S(x_{3n_k})),\\ & G(g(x_{3n_k}), T(x_{3n_k}), T(x_{3n_k})), G(h(x_{3m_k-1}), U(x_{3m_k-1}), U(x_{3m_k-1})),\\ & \frac{1}{3}(G(S(x_{3n_k}), g(x_{3n_k}), h(x_{3m_k-1})) + G(T(x_{3n_k}), f(x_{3n_k+1}), h(x_{3m_k-1}))+\\ & G(U(x_{3m_k-1}), f(x_{3n_k}), g(x_{3n_k})))\}\\ =& max\{G(y_{3n_k}, y_{3n_k}, y_{3m_k-1}), G(y_{3n_k+1}, y_{3n_k}, y_{3n_k}),\\ & G(y_{3n_k+1}, y_{3n_k}, y_{3n_k}), G(y_{3m_k}, y_{3m_k-1}, y_{3m_k-1}),\\ & \frac{1}{3}(G(y_{3n_k}, y_{3n_k+1}, y_{3m_k}) + G(y_{3n_k}, y_{3n_k+1}, y_{3m_k})+\\ & G(y_{3m_k-1}, y_{3n_k+1}, y_{3n_k+1})\}\end{aligned}$$

letting $k \to \infty$ and using (3.26), (3.29) and (3.30), we have $\lim_{k\to\infty} M(x_{3n_k}, x_{3n_k}, x_{3m_k-1}) = max\{\epsilon, 0, 0, 0, \frac{1}{3}(\epsilon+\epsilon+\epsilon)\} = \epsilon$.
Therefore from (3.31) $\frac{G(f(x_{3n_k}), g(x_{3n_k}), h(x_{3m_k-1}))}{M(x_{3n_k}, x_{3n_k}, x_{3m_k-1})} < \alpha(M(x_{3n_k}, x_{3n_k}, x_{3m_k-1})) < 1$.
Using fact that $\epsilon = \lim_{k\to\infty} G(f(x_{3n_k}), g(x_{3n_k}), h(x_{3m_k-1})) = \lim_{k\to\infty} M(x_{3n_k}, x_{3n_k}, x_{3m_k-1})$, we get $\lim_{k\to\infty} \alpha(M(x_{3n_k}, x_{3n_k}, x_{3m_k-1})) = 1$ since $\alpha \in F$, hence $M(x_{3n_k}, x_{3n_k}, x_{3m_k-1}) = 0$ which is a contradiction. Hence y_{3n} is a $G-$ Cauchy sequence by completeness of X there exist a point u in X such that $\{y_{3n}\}$ and its subsequences $\{y_{3n+1}\}$, $\{y_{3n+2}\}$ and $\{y_{3n+3}\}$ are also converges to u . That is

$$\lim_{n\to\infty} y_{3n+1} = \lim_{n\to\infty} U(x_{3n+1}) = \lim_{n\to\infty} f(x_{3n}) = u \tag{3.32}$$

$$\lim_{n\to\infty} y_{3n+2} = \lim_{n\to\infty} T(x_{2n+2}) = \lim_{n\to\infty} g(x_{3n+1}) = u \tag{3.33}$$

$$\lim_{n\to\infty} y_{3n+3} = \lim_{n\to\infty} S(x_{2n+3}) = \lim_{n\to\infty} h(x_{3n+2}) = u. \tag{3.34}$$

Suppose that S is continuous and by compatibility of (f, S), we have

$$\lim_{n\to\infty} f(S(x_{3n+2})) = \lim_{n\to\infty} S(f(x_{3n+2})) = S(u). \tag{3.35}$$

Again since $x_{3n+1} \leq gx_{3n+1} = Tx_{3n+2}$, using (3.3.7), we have

$$\begin{aligned} G(f(x_{3n+1}), g(x_{3n+1}), h(x_{3n+1})) =& G(f(S(x_{3n+2})), g(x_{3n+1}), h(x_{3n+1})) \\ & \leq \alpha(M(S(x_{3n+2}), x_{3n+1}, x_{3n+1}))M(S(x_{3n+2}), x_{3n+1}, x_{3n+1}) \end{aligned} \tag{3.36}$$

where

$$\begin{aligned} M(S(x_{3n+2}), x_{3n+1}, x_{3n+1}) =& max\{G(S(Sx_{3n+2}), T(x_{3n+1}), U(x_{3n+1})), \\ & G(f(S(x_{3n+2})), S(S(x_{3n+2})), S(S(x_{3n+2}))), \\ & G(g(x_{3n+1}), T(x_{3n+1}), T(x_{3n+1})), \\ & G(h(x_{3n+2}), U(x_{3n+1}), U(x_{3n+1})), \\ & \frac{1}{3}(G(S(S(x_{3n+2})), g(x_{3n+1}), h(x_{3n+1})) \\ & + G(T(x_{3n+1}), f(S(x_{3n+2})), h(x_{3n+1})) \\ & + G(U(x_{3n+1}), f(S(x_{3n+2})), g(x_{3n+1})))\} \end{aligned}$$

letting $n \to \infty$ and using (3.32), (3.33), (3.34) and (3.35), we have

$$\begin{aligned} \lim_{n\to\infty} M(S(x_{3n+2}), x_{3n+1}, x_{3n+1}) =& max\{G(S(u), u, u), G(S(u), S(u), S(u))G(u, u, u), G(u, u, u), \\ & \frac{1}{3}(G(S(u), u, u) + G(u, S(u), u) + G(u, S(u), u))\} \\ =& maxG(S(u), u, u), 0, 0, 0, G(S(u), u, u) \\ =& G(S(u), u, u). \end{aligned}$$

Therefore from (3.36) as $n \to \infty$, we have
$G(S(u), u, u) \leq \alpha(G(S(u), u, u))G(S(u), u, u) < G(S(u), u, u)$ yields that

$$S(u) = u. \tag{3.37}$$

Since $x_{3n+1} \leq gx_{3n+1}$ and $gx_{3n+1} \to u$ as $n \to \infty$,$x_{3n+1} \leq u$. From (3.3.7), we have

$$\begin{aligned} G(f(x_{3n+1}), g(x_{3n+1}), h(x_{3n+1})) =& G(f(u), g(x_{3n+1}), h(x_{3n+1})) \\ & \leq \alpha(M(u, x_{3n+1}, x_{3n+1}))M(u, x_{3n+1}, x_{3n+1}) \end{aligned} \tag{3.38}$$

where

$$\begin{aligned} M(u, x_{3n+1}, x_{3n+1}) =& max\{G(S(u), T(x_{3n+1}), U(x_{3n+1})), G(f(u), S(u), S(u)), \\ & G(g(x_{3n+1}), T(x_{3n+1}), T(x_{3n+1})), G(h(x_{3n+1}), U(x_{3n+1}), U(x_{3n+1})), \\ & \frac{1}{3}(G(S(u), g(x_{3n+1}), h(x_{3n+1})) + G(T(x_{3n+1}), f(u), h(x_{3n+1})) \\ & + G(U(x_{3n+1}), f(u), g(x_{3n+1})))\} \end{aligned}$$

letting $n \to \infty$ and using (3.32), (3.33), (3.34) and (3.37), we have

$$\begin{aligned} \lim_{n\to\infty} M(u, x_{3n+1}, x_{3n+1}) =& max\{G(u, u, u), G(f(u), u, u), G(u, u, u), \\ & G(u, u, u), \frac{1}{3}(G(u, u, u) + G(u, f(u), f(u)) + G(u, f(u), u)) \\ =& max\{0, G(f(u), u, u), 0, 0, \frac{2}{3}G(f(u), u, u)\} \\ =& G(f(u), u, u). \end{aligned}$$

Therefore from (3.38) as $n \to \infty$, we have
$G(f(u), u, u) \leq \alpha(G(f(u), u, u))G(f(u), u, u) < G(f(u), u, u)$. yields that

$$f(u) = u. \tag{3.39}$$

Suppose that T is continuous and by compatibility of (g, T), we have

$$\lim_{n\to\infty} g(T(x_{3n+3})) = \lim_{n\to\infty} T(g(x_{3n+3})) = T(u). \tag{3.40}$$

Again since $x_{3n+2} \leq hx_{3n+2} = Sx_{3n+3}$, using (3.3.7), we have

$$G(f(x_{3n+2}), g(T(x_{3n+3})), h(x_{3n+2})) \leq \alpha(M(x_{3n+2}, Tx_{3n+3}, x_{3n+2}))M(x_{3n+2}, Tx_{3n+3}, x_{3n+2}) \tag{3.41}$$

where

$$\begin{aligned} M(x_{3n+2}, Tx_{3n+3}, x_{3n+2}) =& max\{G(S(x_{3n+2}), TT(x_{3n+3}), U(x_{3n+2})), \\ & G(f(x_{3n+2}), S(x_{3n+2}), S(x_{3n+2})), G(g(T(x_{3n+3})), T(T(x_{3n+3})), T(T(x_{3n+3}))), \\ & G(h(x_{3n+2}), U(x_{3n+2}), U(x_{3n+2})), \frac{1}{3}(G(S(x_{3n+2}), g(T(x_{3n+3})), h(x_{3n+2}))+ \\ & G(T(T(x_{3n+3})), f(x_{3n+2}), h(x_{3n+2})) + G(U(x_{3n+2}), f(x_{3n+2}), g(T(x_{3n+3}))))\} \end{aligned}$$

letting $n \to \infty$ and using (3.32), (3.33), (3.34) and (3.35), we have

$$\begin{aligned} \lim_{n\to\infty} M(x_{3n+2}, T(x_{3n+3}), x_{3n+2}) =& max\{G(u, T(u), u), G(u, u, u), G(T(u), T(u), T(u)), G(u, u, u), \\ & \frac{1}{3}(G(u, T(u), u) + G(T(u), u, u) + G(u, u, T(u)))\} \\ =& maxG(T(u), u, u), 0, 0, 0, G(T(u), u, u) \\ =& G(T(u), u, u). \end{aligned}$$

Therefore from (3.41) as $n \to \infty$, we have
$G(u, T(u), u) \leq \alpha(G(T(u), u, u))G(T(u), u, u) < G(T(u), u, u)$. yields that

$$T(u) = u. \tag{3.42}$$

Since $x_{3n+2} \leq h(x_{3n+2})$ and $h(x_{3n+2}) \to u$ as $n \to \infty$,$x_{3n+2} \leq u$. From (3.3.7), we have

$$G(f(u), g(u), h(x_{3n+2})) \leq \alpha(M(u, u, x_{3n+2}))M(u, u, x_{3n+2}) \tag{3.43}$$

where

$$\begin{aligned} M(u, u, x_{3n+2}) =& max\{G(S(u), T(u), U(x_{3n+2})), G(f(u), S(u), S(u)), G(g(u), T(u), T(u)), \\ & G(h(x_{3n+2}), U(x_{3n+2}), U(x_{3n+2})), \frac{1}{3}(G(S(u), g(u), h(x_{3n+2}))+ \\ & G(T(u), f(u), h(x_{3n+2})) + G(U(x_{3n+2}), f(u), g(u)))\} \end{aligned}$$

letting $n \to \infty$ and using (3.32), (3.33), (3.34) and (3.37), we have

$$\begin{aligned} \lim_{n\to\infty} M(u, u, x_{3n+2}) =& max\{G(u, u, u), G(u, u, u), G(g(u), u, u), G(u, u, u), \\ & \frac{1}{3}(G(u, g(u), u) + G(u, u, u) + G(u, g(u), u))\} \\ & = max\{0, G(g(u), u, u), 0, 0, \frac{2}{3}G(g(u), u, u)\} \\ & = G(g(u), u, u). \end{aligned}$$

Therefore from (3.43) as $n \to \infty$, we have
$G(f(u), gu, hx_{3n+2}) \leq \alpha(G(g(u), u, u))G(g(u), u, u) < G(g(u), u, u)$. yields that

$$g(u) = u. \tag{3.44}$$

Since $f(X) \subseteq U(X)$ then there exists a point $v \in X$ such that $u = f(u) = U(v)$.
Suppose that $h(v) \neq U(v)$. Since $u \leq f(u) = U(v) \leq f(U(v)) \leq v$ implies that $u \leq v$ and using (3.3.7), we have

$$G(U(v), h(v), h(v)) = G(f(u), h(v), h(v)) \leq \alpha(M(u, v, v))M(u, v, v) \tag{3.45}$$

where

$$\begin{aligned} M(u, v, v) =& max\{G(S(u), U(v), U(v)), G(f(u), S(u), S(u)), G(h(v), U(v), U(v)), G(h(v), U(v), U(v)), \\ & \frac{1}{3}(G(S(u), h(v), h(v)) + G(T(v), f(u), h(v)) + G(U(v), U(v), h(v)))\} \\ =& max\{G(u, u, u), G(u, u, u), G(h(v), U(v), U(v)), G(h(v), U(v), U(v)), \\ & \frac{1}{3}(G(U(v), h(v), h(v)) + G(U(v), U(v), h(v)) + G(U(v), U(v), h(v)))\} \\ =& G(U(v), h(v), h(v)). \end{aligned}$$

Therefore from (3.45), we have $G(U(v), h(v), h(v)) < G(U(v), h(v), h(v))$ yields that $Uv = hv$. Now by the weakly compatibility of the pair (h, U), $h(u) = h(f(u)) = h(U(v)) = U(h(v)) = U(f(u)) = U(u)$. That is u is a coincidence point of h and U. Next since $x_{3n+2} \leq h(x_{3n+2})$ and $h(x_{3n+2}) \to u$ as $n \to \infty$ implies $x_{3n+2} \leq u$. From (3.3.7), we have

$$G(f(u), g(u), h(x_{3n+2})) \leq \alpha(M(u, u, x_{3n+2}))M(u, u, x_{3n+2}) \tag{3.46}$$

where

$$\begin{aligned} M(u, u, x_{3n+2}) =& max\{G(S(u), T(u), U(x_{3n+2})), G(f(u), S(u), S(u)), G(g(u), T(u), T(u)), \\ & G(h(x_{3n+2}), U(x_{3n+2}), U(x_{3n+2})), \frac{1}{3}(G(S(u), g(u), h(x_{3n+2})) \\ & + G(T(u), f(u), h(x_{3n+2})) + G(U(x_{3n+2}), f(u), g(u)))\} \end{aligned}$$

letting $n \to \infty$ and using (3.32), (3.33), and (3.34), we have

$$\begin{aligned} \lim_{n\to\infty} M(u, u, x_{3n+2}) =& max\{G(u, u, u), G(u, u, u), G(u, u, u), G(hu, u, u), \\ & \frac{1}{3}(G(u, u, h(u)) + G(g(u, u, h(u)) + G(u, u, u))\} \\ =& max\{G(h(u), u, u), 0, 0, 0, G(h(u), u, u)\} \\ =& G(h(u), u, u). \end{aligned}$$

Therefore from (3.3.31) as $n \to \infty$, we have
$G(fu, g(u), h(u)) \leq \alpha(G(h(u), u, u))G(h(u), u, u) < G(h(u), u, u)$. Therefore $h(u) = u$. Therefore $f(u) = g(u) = h(u) = S(u) = T(u) = U(u) = u$. Therefore u is a common fixed point of f, g, h, S, T, U.
Suppose that the set of common fixed points of f, g, h, S, T *and* U is well ordered and let u *and* v be any two distinct fixed points of f, g, h, S, T *and* U then from (3.3.7), we have

$$G(f(u), g(v), h(v)) \leq \alpha(M(u, v, v))M(u, v, v) \tag{3.47}$$

where

$$\begin{aligned} M(u, v, v) =& max\{G(S(u), T(v), U(v)), G(f(u), S(u), S(u)), G(g(v), T(v), T(v)), G(h(v), U(v), U(v)), \\ & \frac{1}{3}(G(S(u), g(v), h(v)) + G(T(v), f(u), h(v)) + G(U(v), f(u), g(v)))\} \\ =& max\{G(u, v, v), G(u, u, u), G(v, v, v), G(v, v, v), \frac{1}{3}(G(u, v, v) + G(v, u, v) + G(v, u, v)\} \\ =& G(u, v, v). \end{aligned}$$

From (3.3.32), we have $G(f(u), g(v), h(v)) \leq \alpha(G(u, v, v))G(u, v, v)$ implies $G(u, v, v) < G(u, v, v)$ which is a contradiction. Therefore $u = v$. Therefore common fixed point of f, g, h, S, T *and* U is unique. Conversely, if f, g, h, S, T and U have only one common fixed point then the set of common fixed point of f, g, h, S, T and U being singleton is well ordered. □

In the sequel we give some corollaries of our theoretical results:
If we take $f = g = h$ and $S = T = U$ in the Theorem 3.2, we have the following result.

Corollary 3.4. *Let $(X, \preceq)$ be a partially ordered set and there exists a metric G on X such that (X, G) is a complete $G-$ metric space. Let $f, T : X \to X$ be given mappings satisfying*

(1) $f(X) \subseteq T(X)$,

(2) for every comparable elements $x, y, z \in X$,
$G(f(x), f(y), f(z)) \leq \alpha(G(T(x), T(y), T(z)))G(T(x), T(y), T(z))$, where $\alpha \in \mathcal{F}$

(3) The pairs (T, f) are partially weakly increasing,

(4) f is dominating and weak annihilator maps of T.

(5) there exist a non decreasing sequence $\{x_n\}$ with $x_n \preceq y_n$ for all n and $y_n \to u$ implies $x_n \preceq u$.

(6) (f, T) is compatible, weakly compatible and f or T is continuous map.

Then f, T have a common fixed point. Moreover, f, T have one common fixed point if and only if the set of common fixed point of f, T is well ordered.

If we take $f = g = h$ and $S = T = U$ in the Theorem 3.3, we have the following result.

Corollary 3.5. *Let $(X, \preceq)$ be a partially ordered set and there exists a metric G on X such that (X, G) is a complete $G-$ metric space. Let $f, T : X \to X$ be given mappings satisfying*

(1) $f(X) \subseteq T(X)$,

(2) for every comparable elements $x, y, z \in X$,
$G(f(x), f(y), f(z)) \leq \alpha(G(T(x), T(y), T(z)))G(T(x), T(y), T(z))$, where $\alpha \in \mathcal{F}$

(3) The pairs (T, f) are partially weakly increasing,

(4) f is dominating and weak annihilator maps of T.

(5) there exist a non decreasing sequence $\{x_n\}$ with $x_n \preceq y_n$ for all n and $y_n \to u$ implies $x_n \preceq u$.

(6) (f, T) is compatible, weakly compatible and f or T is continuous map.

(7) $G(f(x), f(y), f(z)) \leq \alpha(M(x, y, z))M(x, y, z)$, where

$$\begin{aligned} M(x,y,z) = &\max\{G(S(x),T(y),U(z)), G(f(x),S(x),S(x)), G(g(y),T(y),T(y)), G(h(z),U(z),U(z)), \\ &\frac{1}{3}(G(S(x),g(y),h(z)) + G(T(y),f(x),h(z)) + G(U(z),f(x),g(y)))\} \end{aligned}$$

for all $x, y, z \in X$ with $x \preceq y \preceq z$ and $\alpha \in \mathsf{F}$ Then f, T have a common fixed point. Moreover, f, T have one common fixed point if and only if the set of common fixed point of f, T is well ordered.

The following example support our theoretical results:

Example 3.6. *Let $X = [0, \infty)$ be endowed with the usual $G-$ metric $G(x,y,z) = |x-y|+|y-z|+|z-x|$, and $\leq$ be the usual ordering on $\mathbb{R}$. We define a new ordering $\preceq$ on X such that $x \preceq y$ if and only if $y \leq x$, for all $x, y \in X$. Define $f, g, h, T, S, U : X \to X$ as $f(x) = \ln(1+\frac{x}{6})$, $g(x) = \ln(1+\frac{x}{12})$, $h(x) = \ln(1+\frac{x}{18})$, $U(x) = e^{6x}-1$, $T(x) = e^{12x}-1$ and $S(x) = e^{18x}-1$. Here $f(X) \subseteq U(X)$, $g(X) \subseteq T(X)$, $h(X) \subseteq S(X)$. Now for each $x \in X$ we have $1+\frac{x}{6} \leq e^x$, $1+\frac{x}{12} \leq e^x$, $1+\frac{x}{18} \leq e^x$, so $f(x) = \ln(1+\frac{x}{6}) \leq x$, $g(x) = \ln(1+\frac{x}{12}) \leq x$, $h(x) = \ln(1+\frac{x}{18}) \leq x$. It implies that $x \preceq f(x)$, $x \preceq g(x)$, $x \preceq h(x)$. So f, g, h are dominating maps. Also for each $x \in X$, we have $f(U(x)) = \ln(1+\frac{T(x)}{6}) = \ln(1+\frac{e^{6x}-1}{6}) = \ln(\frac{5+e^{6x}}{6}) = \ln(e^{3x}.\frac{5e^{-3x}+e^{3x}}{6}) = 3x+\ln(\frac{5e^{-3x}+e^{3x}}{6}) \geq x$. This implies that $f(U(x)) \leq x$. Also, $g(T(x)) = \ln(1+\frac{S(x)}{12}) = \ln(1+\frac{e^{12x}-1}{12}) = \ln(\frac{11+e^{12x}}{12}) = \ln(e^{6x}.\frac{11e^{-6x}+e^{6x}}{12}) = 6x+\ln(\frac{11e^{-6x}+e^{6x}}{12}) \geq x$. This implies that $g(T(x)) \leq x$. Further, $h(S(x)) = \ln(1+\frac{Ux}{18}) = \ln(1+\frac{e^{18x}-1}{18}) = \ln(\frac{17+e^{18x}}{18}) = \ln(e^{9x}.\frac{17e^{-9x}+e^{9x}}{18}) = 9x+\ln(\frac{17e^{-9x}+e^{9x}}{18}) \geq x$. This implies that $h(S(x)) \leq x$. Thus f, g, h are weak annihilators of U, T and S respectively. Since $f(U(x)) \leq x, x \leq f(x)$, $f(U(x)) \leq f(x)$. Hence (U, f) is partially weakly increasing. Also since, $g(T(x)) \leq x, x \leq g(x)$, $g(T(x)) \leq g(x)$ and hence (T, g) is partially weakly increasing. Similarly $h(S(x)) \leq x, x \leq h(x)$, $h(S(x)) \leq h(x)$ and (S, h) is partially weakly increasing. Now there exist a non-decreasing sequence $\{x_n\} = \frac{1}{n}$ in X such that $\frac{1}{n} \to 0$, $f(x_n) = \ln(1+\frac{x_n}{6}) = \ln(1+\frac{1}{6n}) \to 0$ and $S(x_n) = e^{18x_n}-1 = e^{\frac{18}{n}}-1 \to 0$ as $n \to \infty$. Also $f(S(x_n)) = \ln(1+\frac{S(x_n)}{6}) \to 0$ and $S(f(x_n)) = e^{18f(x_n)}-1 \to 0$. Therefore $\lim_{n\to\infty} G(f(S(x_n)), S(f(x_n)), S(f(x_n))) = 0$, that is the pair (f, S) is compatible and continuous maps. Similarly the pair (g, T) is also compatible and continuous maps. 0 is the coincidence point of the pair (h, U) and we have $h(U(0)) = h(0) = 0 = U(0) = U(h(0))$. Therefore the pair (h, U) is weakly compatible. Now, we define $\alpha(t) = \frac{1}{1+t}$ if $t \in (0, \infty)$ and $\alpha(t) = 0$ if $t = 0$ then for $t_n = \frac{1}{n}$, $\lim_{n\to\infty} \alpha(t_n) = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} \to 1$. It implies that $\lim_{n\to\infty} t_n = \frac{1}{n} \to 0$. Thus $\alpha \in F$. For $x, y, z \in X$, we have*

$$
\begin{aligned}
&G(f(x), g(y), h(z)) = |f(x)-g(y)| + |g(y)-h(z)| + |h(z)-f(x)| \\
&= |\ln(1+\frac{x}{6}) - \ln(1+\frac{y}{12})| + |\ln(1+\frac{y}{12}) - \ln(1+\frac{z}{18})| + |\ln(1+\frac{z}{18}) - \ln(1+\frac{x}{6})| \\
&\leq |1+\frac{x}{6}-1-\frac{y}{12}| + |1+\frac{y}{12}-1-\frac{z}{18}| + |1+\frac{z}{18}-1-\frac{x}{6}| \\
&\leq \frac{1}{12}|2x-y| + \frac{1}{36}|3y-2z| + \frac{1}{18}|z-3x| \\
&= \frac{1}{72}|12x-6y| + \frac{1}{216}|18y-12z| + \frac{1}{108}|6z-18x| \\
&\leq \frac{1}{72}|e^{12x}-e^{6y}| + \frac{1}{216}|e^{18y}-e^{12z}| + \frac{1}{108}|e^{6z}-e^{18x}| \\
&= \frac{1}{72}G(T(x), U(y), U(y)) + \frac{1}{216}G(S(y), T(z), T(z)) + \frac{1}{108}G(U(z), S(x), S(x)) \\
&\leq \frac{1}{36}[\frac{1}{2}G(T(x), U(y), U(y)) + \frac{1}{6}G(S(y), T(z), T(z)) + \frac{1}{3}G(U(z), S(x), S(x))] \\
&\leq \frac{1}{36}M(x,y,z) \leq \alpha(M(x,y,z))M(x,y,z).
\end{aligned}
$$

It holds if $\frac{1}{36} \leq \alpha(M(x,y,z)) < 1$, for all $x, y, z \in X$. Thus all the conditions of Theorem 3.3 are satisfied and 0 is the unique common fixed point f, g, h, S, T and U.

References

1. M. Abbas, T. Nazir, and S. Radenović, *Common fixed point of generalized weakly contractive maps in partially ordered G-metric spaces,* Appl. Math. Comput. 218 (18), 9883-9395, (2012).

2. M. Abbas, Azhar Hussain, Branislav Popović and Stojan Radenović, *Istratescu-Susuki-Ćirí-type fixed points results in the framewoek of G-metric spaces,* J. Nonlinear Sci. Appl. 9, 6077-6095, (2016).

3. I. Altun and H. Simsek, *Some fixed point theorems on ordered metric spaces and application*, Fixed point theory appl. 17, (2010).

4. H. Aydi, *Coincidence and common fixed point results for contraction type maps in partially ordered metric spaces* , arXiv preprint arXiv:1102.5493. (2011).

5. S. Chandok, *Some fixed point theorems for (α, β)-admissible geraghty type contractive mappings and related results,* Math. Sci. 9, 127–135, (2015).

6. B. C. Dhage, *Generalized metric spaces and mappings with fixed point*, Bull. Calcutta Math. Soc. 84(4), 329-336, (1994).

7. M. E. Gordji, H. Baghani and G. H. Kim, *A fixed point theorem for contraction type maps in ordered metric spaces and application to ordinary differential equations*, Discrete Dynamics Nature Soc. (2012).

8. M.A. Geraghty, *On contractive mappings*, Proc. Amer. Math. Soc. 40(2), 604-608, (1973).

9. Hamid Faraji, Dragana Savić and S. Radenović, *Fixed point theorems for Geraghty contraction type mappings in b-metric spaces and applications*, Aximoms 8(34), 2019.

10. G. Jungck, *Compatible mappings and common fixed points(2)*, Internat. J. Math. and Math. Sci. 11(2), 285-288, (1988).

11. G. Jungck and B.E. Rhoades, *Fixed point for set valued functions without continuity*, Internat. J. of Pure and Applied Mathematics. 29(3), 227-238, (1998).

12. R. Kannan, *On certain sets and fixed point theorems*, Roum. Math. Pure Appl. 14, 51-54, (1969).

13. R. Kannan, *Some results on fixed points- II*, American Math. Monthly, 76, 406, (1969).

14. Ljiljana Gajić, Zoran Kadelburg, Stojan Radenović, *Gp-metric spaces symmetric and asymmetric*, Scientific Publications of the State University of Novi Pazar, Ser. A: Appl. Math. Inform. and Mech. 9(1), 37-46, (2017).

15. Ljubomir Ćirić, *Some Recent Results in Metrical Fixed Point Theory,* University of Belgrade, Beograd, Serbia, (2003).

16. Mujahid Abbas, Talat Nazir, Stojan Radenović, *Some periodic point results in generalized metric space,* Applied Mathematics and Computation, 217 4094-4099, (2010).

17. Z. Mustafa and B. Sims, *A new approach to generalized metric spaces*, Journal of Nonlinear and Convex Analysis, 7(2), 289-297, (2006).

18. B.E. Rhoades, *A fixed point theorem for generalized metric spaces*,Internat. J. Math. and Math. Sci. 19(3), 457-460, (1996).

19. G. S. M. Reddy, *A common fixed point theorem on complete G-metric spaces*, Internat. J. Pure Appl. Math. 118(2), 195-202, (2018).

20. S. Sessa *On a weakly commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math. 32(46), 149-153, (1982).

21. R.K. Sharma, V. Raich and C.S. Chauhan, *On generalization of Banach contraction principle in partially ordered metric spaces*, Global J. Pure Appl. Math. 13(6), 2213-2234, (2017).

22. Stojan Radenović, *Remarks on some recent coupled coincidence point results in symmetric G-metric spaces,* Journal of Operators, (2013).

23. Zeid I. Al-Muhiameed and M. Bousselsal, *Common fixed point theorem for four maps in partially ordered metric spaces*, Int. J. Math. Anal. 7(22), 1051-1058, (2013).

24. G. S. M. Reddy, *Generalization of contraction principle on G-metric spaces*, Global J. Pure Appl. Math. 14(9), 1177-1283, (2018).

25. G. S. M. Reddy, *Fixed point theorems of contractions of G-metric Spaces and property 'P' in G-Metric spaces*, Global J. Pure Appl. Math. 14(6), 885-896, (2018).

26. G. S. M. Reddy, *Fixed Point Theorems for (ε, λ)-Uniformly Locally Generalized Contractions*, Global J. Pure Appl. Math. 14(9), 1177-1183, (2018).

27. Ravi P. Agarwal, Zoran Kadelburg, Stojan Radenović, *On coupled fixed point results in asymmetric G-metric spaces,* Journal of Inequalities and Applications, 528, (2013).

28. Ravi P. Agarwal, Erdal Karapinar, Donal O'Oregan, Antonio Francisco Roldan-Lopez-de-Hierro, *Fixed Point Theory in Metric Type Spaces,* Springer International Publishing Switzerland, (2015).

29. Vesna Todorčević, *Harmonic Quasiconformal Mappings and Hyperbolic Type Metrics,* Springer Nature Switzerland AG, (2019).

30. V. Srinivas Chary, G. Sudhaamsh Mohan Reddy, *Fixed Point Results for Almost Z_G-contraction via simulation functions in G-metric spaces,* International Journal of Control and Automation, 12, 6, 608-615, (2019).

31. V. Srinivas Chary, G. Sudhaamsh Mohan Reddy, D. Srinivasa Chary, Hüseyin Işik and Aydi Hassen, *Some fixed point theorems for modified JS-G-contractions and an application to integral equation,* Journal of Applied Mathematics and Informatics, 38, 5-6, 507-518, (2020).

32. V. Srinivas Chary, G. Sudhaamsh Mohan Reddy, D. Srinivasa Chary, Hüseyin Işik and Aydi Hassen, *Some fixed point theorems on α-β-G-complete G-metric spaces,* Carpathian Mathematical Publications, 13, 1, 58-67, (2021).

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