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Necessary and sufficient Tauberian conditions under which convergence follows from $A^{r,\delta}$ summability method for double sequences *

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ABSTRACT: Let $x = (x_{mn})$ be a double sequence of real or complex numbers. The $A^{r,\delta}$-transform of a sequence (x_{mn}) is defined by

$$(A^{r,\delta}x)_{mn} = \sigma^{r,\delta}_{mn}(x) = \frac{1}{(m+1)(n+1)} \sum_{j=0}^{m} \sum_{k=0}^{n} (1+r^j)(1+\delta^k)x_{jk}, \qquad 0 < r, \delta < 1$$

The $A^{r,*}$ and $A^{*,\delta}$ transformations are defined respectively by

$$(A^{r,*}x)_{mn} = \sigma^{r,*}_{mn}(x) = \frac{1}{m+1} \sum_{j=0}^{m} (1+r^j)x_{jn}, \quad 0 < r < 1,$$

and

$$(A^{*,\delta}x)_{mn} = \sigma^{*,\delta}_{mn}(x) = \frac{1}{n+1} \sum_{k=0}^{n} (1+\delta^k)x_{mk}, \quad 0 < \delta < 1.$$

We say that (x_{mn}) is ($A^{r,\delta}$,1,1) summable to l if $(\sigma^{r,\delta}_{mn}(x))$ has a finite limit l. It is known that if $\lim_{m,n\to\infty} x_{mn} = l$ and (x_{mn}) is bounded, then the limit $\lim_{m,n\to\infty} \sigma^{r,\delta}_{mn}(x) = l$ exists. But the inverse of this implication is not true in general. Our aim is to obtain necessary and sufficient conditions for ($A^{r,\delta}$,1,1) summability method under which the inverse of this implication holds. Following Tauberian theorems for $(A^{r,\delta}, 1, 1)$ summability method, we also introduce $A^{r,*}$ and $A^{*,\delta}$ transformations of double sequences and obtain Tauberian theorems for the $(A^{r,*}, 1, 0)$ and $(A^{*,\delta}, 0, 1)$ summability methods.

Key Words: $(A^{r,\delta}, 1, 1)$, $(A^{r,*}, 1, 0)$ and $(A^{*,\delta}, 0, 1)$ summability methods, Pringsheim's convergence, slow decrease and slow oscillation in different senses, Tauberian conditions and theorems.

Contents

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1. Introduction

Let $x = (x_{mn})$ be a double sequence of real or complex numbers. The $A^{r,\delta}$-transform of a sequence (x_{mn}) is defined by

$$(A^{r,\delta}x)_{mn} = \sigma^{r,\delta}_{mn}(x) = \frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}\sum_{k=0}^{n}(1+r^j)(1+\delta^k)x_{jk},$$

where $0 < r, \delta < 1$. If

$$\lim_{m,n\to\infty} \sigma^{r,\delta}_{mn}(x) = l, \tag{1.1}$$

then we say that (x_{mn}) is summable to l by ($A^{r,\delta}$,1,1) summability method.

The $A^{r,*}$ and $A^{*,\delta}$ transformations are defined respectively by

$$(A^{r,*}x)_{mn} = \sigma^{r,*}_{mn}(x) = \frac{1}{m+1}\sum_{j=0}^{m}(1+r^j)x_{jn}, \quad 0 < r < 1$$

and

$$(A^{*,\delta}x)_{mn} = \sigma^{*,\delta}_{mn}(x) = \frac{1}{n+1}\sum_{k=0}^{n}(1+\delta^k)x_{mk}, \quad 0 < \delta < 1.$$

A double sequence (x_{mn}) is said to be convergent in Pringsheim's sense (or P-convergent) to l if for every $\epsilon > 0$ there exists a positive integer $n_0 = n_0(\epsilon)$ such that $|x_{mn} - l| < \epsilon$ whenever $m, n \geq n_0$ (see [4]). The number l is called the Pringsheim limit of x and we denote by $P - \lim_{m,n\to\infty} x_{mn} = l$.

If

$$\lim_{m,n\to\infty} \sigma^{r,*}_{mn}(x) = l \tag{1.2}$$

or

$$\lim_{m,n\to\infty} \sigma^{*,\delta}_{mn}(x) = l, \tag{1.3}$$

then we say that (x_{mn}) is summable to l by ($A^{r,*}$,1,0) or ($A^{*,\delta}$,0,1) summability method, respectively. It is easy to check that if the limit

$$\lim_{m,n\to\infty} x_{mn} = l \tag{1.4}$$

exists and (x_{mn}) is bounded, then we also have (1.1). However, the converse implication is not true in general. If we define the sequence (x_{mn}) by $x_{mn} = (-1)^{mn}\left((1+r^m)(1+\delta^n)\right)^{-1}$ for all nonnegative integers m and n, then it is easy to see that

$$(A^{r,\delta}x)_{mn} = \sigma^{r,\delta}_{mn}(x) = \frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}\sum_{k=0}^{n}(-1)^{jk}.$$

Hence, we have $\sigma^{r,\delta}_{mn}(x) \to 0$ as $m, n \to \infty$. This shows that (x_{mn}) is ($A^{r,\delta}$,1,1) summable to zero, but it is not convergent. We notice that (1.1) implies (1.4) under certain condition, which is called a Tauberian condition, imposed on the sequence (x_{mn}). Any theorem which states that convergence of sequences follows from its ($A^{r,\delta}$,1,1) and some Tauberian condition(s) is said to be a Tauberian theorem for the ($A^{r,\delta}$,1,1) summability method.

As an extension of a classical Tauberian theorem for Cesàro summability method [2], Móricz [3] derived necessary and sufficient Tauberian conditions for Cesàro summability method. Following Móricz [3], Talo and Başar [6] obtained necessary and sufficent Tauberian conditions for the A^r summability method which was introduced by Başar [1]. In this paper, we extend their results in [6] to ($A^{r,\delta}$,1,1) summability method for double sequences and introduce necessary and sufficient conditions for summability ($A^{r,\delta}$,1,1) under which the existence of the limit (1.4) follows from that of (1.1). Following Tauberian theorems for $(A^{r,\delta}, 1, 1)$ summability method, we also introduce $A^{r,*}$ and $A^{*,\delta}$ transformations of double sequences and obtain Tauberian theorems for the $(A^{r,*}, 1, 0)$ and $(A^{*,\delta}, 0, 1)$ summability methods.

2. Tauberian theorems for $(A^{r,\delta},1,1)$ summability method

2.1. An auxiliary result

We prove the following lemma which is needed in the sequel. We denote the integer part of the product λ and n by $\lambda_n := [\lambda n]$.

Lemma 2.1. *If a sequence (x_{mn}) is $(A^{r,\delta}$,1,1) summable to a finite number l, then*

$$\lim_{m,n\to\infty} \frac{1}{(\lambda_m - m)(\lambda_n - n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} (1+r^j)(1+\delta^k)x_{jk} = l \tag{2.1}$$

for every $\lambda > 1$ and

$$\lim_{m,n\to\infty} \frac{1}{(m - \lambda_m)(n - \lambda_n)} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} (1+r^j)(1+\delta^k)x_{jk} = l \tag{2.2}$$

for every $0 < \lambda < 1$.

Proof: Case $\lambda > 1$. By definition, we have

$$\begin{aligned}
\tau_{mn}^{r,\delta} := \frac{1}{(\lambda_m - m)(\lambda_n - n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} (1+r^j)(1+\delta^k)x_{jk} \\
= \frac{1}{(\lambda_m - m)(\lambda_n - n)} \sum_{j=0}^{\lambda_m} \sum_{k=0}^{\lambda_n} (1+r^j)(1+\delta^k)x_{jk} \\
- \frac{1}{(\lambda_m - m)(\lambda_n - n)} \sum_{j=0}^{\lambda_m} \sum_{k=0}^{n} (1+r^j)(1+\delta^k)x_{jk} \\
- \frac{1}{(\lambda_m - m)(\lambda_n - n)} \sum_{j=0}^{m} \sum_{k=0}^{\lambda_n} (1+r^j)(1+\delta^k)x_{jk} \\
+ \frac{1}{(\lambda_m - m)(\lambda_n - n)} \sum_{j=0}^{m} \sum_{k=0}^{n} (1+r^j)(1+\delta^k)x_{jk} \\
= \frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m - m)(\lambda_n - n)} \sigma_{\lambda_m,\lambda_n}^{r,\delta} - \frac{(\lambda_m+1)(n+1)}{(\lambda_m - m)(\lambda_n - n)} \sigma_{\lambda_m,n}^{r,\delta} \\
- \frac{(m+1)(\lambda_n+1)}{(\lambda_m - m)(\lambda_n - n)} \sigma_{m,\lambda_n}^{r,\delta} + \frac{(m+1)(n+1)}{(\lambda_m - m)(\lambda_n - n)} \sigma_{mn}^{r,\delta} \\
= \frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m - m)(\lambda_n - n)} \sigma_{\lambda_m,\lambda_n}^{r,\delta} - \left(-\frac{\lambda_m+1}{\lambda_m - m} \sigma_{\lambda_m,n}^{r,\delta} + \frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m - m)(\lambda_n - n)} \sigma_{\lambda_m,n}^{r,\delta} \right) \\
- \left(-\frac{\lambda_n+1}{\lambda_n - n} \sigma_{m,\lambda_n}^{r,\delta} + \frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m - m)(\lambda_n - n)} \sigma_{m,\lambda_n}^{r,\delta} \right) \\
+ \left(-\frac{\lambda_m+1}{\lambda_m - m} \sigma_{mn}^{r,\delta} - \frac{\lambda_n+1}{\lambda_n - n} \sigma_{mn}^{r,\delta} + \frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m - m)(\lambda_n - n)} \sigma_{mn}^{r,\delta} + \sigma_{mn}^{r,\delta} \right).
\end{aligned}$$

Arranging the terms on the last identity gives

$$\begin{aligned}
\tau_{mn}^{r,\delta} &= \sigma_{mn}^{r,\delta} + \frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m - m)(\lambda_n - n)} (\sigma_{\lambda_m,\lambda_n}^{r,\delta} - \sigma_{\lambda_m,n}^{r,\delta} - \sigma_{m,\lambda_n}^{r,\delta} + \sigma_{mn}^{r,\delta}) \\
&+ \frac{\lambda_m+1}{\lambda_m - m} (\sigma_{\lambda_m,n}^{r,\delta} - \sigma_{mn}^{r,\delta}) + \frac{\lambda_n+1}{\lambda_n - n} (\sigma_{m,\lambda_n}^{r,\delta} - \sigma_{mn}^{r,\delta}). \qquad (2.3)
\end{aligned}$$

It is clear that for all $\lambda > 1$ and large enough m and n,

$$\frac{\lambda^2}{2(\lambda-1)^2} \leq \frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m-m)(\lambda_n-n)} \leq \frac{3\lambda^2}{2(\lambda-1)^2} \tag{2.4}$$

and

$$\frac{\lambda}{2(\lambda-1)} \leq \frac{\lambda_m+1}{\lambda_m-m} \leq \frac{3\lambda}{2(\lambda-1)}. \tag{2.5}$$

It follows from (1.1), (2.4) and (2.5) that $\lim_{m,n\to\infty} \tau^{r,\delta}_{mn} = l$.

ii-) Case $0 < \lambda < 1$. By definition, we have, as in the case $\lambda > 1$,

$$\begin{aligned} \tau^{r,\delta}_{mn} &= \sigma^{r,\delta}_{mn} + \frac{(\lambda_m+1)(\lambda_n+1)}{(m-\lambda_m)(n-\lambda_n)}(\sigma^{r,\delta}_{\lambda_m,\lambda_n} - \sigma^{r,\delta}_{\lambda_m,n} - \sigma^{r,\delta}_{m,\lambda_n} + \sigma^{r,\delta}_{mn}) \\ &+ \frac{\lambda_m+1}{m-\lambda_m}(\sigma^{r,\delta}_{\lambda_m,n} - \sigma^{r,\delta}_{\lambda_m,\lambda_n}) + \frac{\lambda_n+1}{n-\lambda_n}(\sigma^{r,\delta}_{m,\lambda_n} - \sigma^{r,\delta}_{\lambda_m,\lambda_n}). \end{aligned} \tag{2.6}$$

It is clear that for all $0 < \lambda < 1$ and large enough m and n,

$$\frac{\lambda^2}{2(1-\lambda)^2} \leq \frac{(\lambda_m+1)(\lambda_n+1)}{(m-\lambda_m)(n-\lambda_n)} \leq \frac{3\lambda^2}{2(1-\lambda)^2} \tag{2.7}$$

and

$$\frac{\lambda}{2(1-\lambda)} \leq \frac{\lambda_m+1}{m-\lambda_m} \leq \frac{3\lambda}{2(1-\lambda)}. \tag{2.8}$$

It follows from (1.1), (2.7) and (2.8) that $\lim_{m,n\to\infty} \tau^{r,\delta}_{mn} = l$. □

2.2. Main results

First, we consider double sequences of real numbers and prove the following one-sided Tauberian theorem.

Theorem 2.2. *If (x_{mn}) is a sequence of real numbers which is $(A^{r,\delta},1,1)$ summable to a finite limit l, then (1.4) holds if and only if the following two conditions are satisfied:*

$$\sup_{\lambda>1} \liminf_{m,n\to\infty} \frac{1}{(\lambda_m-m)(\lambda_n-n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} \Big((1+r^j)(1+\delta^k)x_{jk} - x_{mn}\Big) \geq 0 \tag{2.9}$$

and

$$\sup_{0<\lambda<1} \liminf_{m,n\to\infty} \frac{1}{(m-\lambda_m)(n-\lambda_n)} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} \Big(x_{mn} - (1+r^j)(1+\delta^k)x_{jk}\Big) \geq 0. \tag{2.10}$$

A sequence (x_{mn}) of real numbers is said to be slowly decreasing in sense (1,1) if

$$\lim_{\lambda\to1^+} \liminf_{m,n\to\infty} \min_{\substack{m<j\leq\lambda_m \\ n<k\leq\lambda_n}} (x_{jk} - x_{mn}) \geq 0. \tag{2.11}$$

Note that condition (2.11) can be equivalently reformulated as follows:

$$\lim_{\lambda\to1^-} \liminf_{m,n\to\infty} \min_{\substack{\lambda_m<j\leq m \\ \lambda_n<k\leq n}} (x_{mn} - x_{jk}) \geq 0. \tag{2.12}$$

Note that the concept of slow decreasing was introduced by Schmidt [5] for the sequences of real numbers.

Corollary 2.3. *Let (1.1), (1.2) and (1.3) be satisfied. If a sequence (x_{mn}) of real numbers is slowly decreasing in sense (1,1), then (1.4) is satisfied.*

Remark 2.4. *If conditions (1.1) and (1.4) or equivalently, conditions (1.1), (2.9) and (2.10) are satisfied, then we necessarily have*

$$\lim_{m,n\to\infty}\frac{1}{(\lambda_m-m)(\lambda_n-n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}\Big((1+r^j)(1+\delta^k)x_{jk}-x_{mn}\Big)=0 \tag{2.13}$$

for every $\lambda>1$ and

$$\lim_{m,n\to\infty}\frac{1}{(m-\lambda_m)(n-\lambda_n)}\sum_{j=\lambda_m+1}^{m}\sum_{k=\lambda_n+1}^{n}\Big(x_{mn}-(1+r^j)(1+\delta^k)x_{jk}\Big)=0 \tag{2.14}$$

for every $0<\lambda<1$.

Remark 2.5. *Theorem 2.2 remains true if conditions (2.9) and (2.10) are replaced by their symmetric counterparts:*

$$\inf_{\lambda>1}\limsup_{m,n\to\infty}\frac{1}{(\lambda_m-m)(\lambda_n-n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}\Big((1+r^j)(1+\delta^k)x_{jk}-x_{mn}\Big)\le 0 \tag{2.15}$$

and

$$\inf_{0<\lambda<1}\limsup_{m,n\to\infty}\frac{1}{(m-\lambda_m)(n-\lambda_n)}\sum_{j=\lambda_m+1}^{m}\sum_{k=\lambda_n+1}^{n}\Big(x_{mn}-(1+r^j)(1+\delta^k)x_{jk}\Big)\le 0. \tag{2.16}$$

Second, we consider double sequences of complex numbers and prove the following two-sided Tauberian theorem.

Theorem 2.6. *If (x_{mn}) is a double sequence of complex numbers which is ($A^{r,\delta}$,1,1) summable to a finite limit l, then (x_{mn}) converges to the same limit if and only if one of the following two conditions is satisfied:*

$$\inf_{\lambda>1}\limsup_{m,n\to\infty}\left|\frac{1}{(\lambda_m-m)(\lambda_n-n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}\Big((1+r^j)(1+\delta^k)x_{jk}-x_{mn}\Big)\right|=0 \tag{2.17}$$

or

$$\inf_{0<\lambda<1}\limsup_{m,n\to\infty}\left|\frac{1}{(m-\lambda_m)(n-\lambda_n)}\sum_{j=\lambda_m+1}^{m}\sum_{k=\lambda_n+1}^{n}\Big(x_{mn}-(1+r^j)(1+\delta^k)x_{jk}\Big)\right|=0. \tag{2.18}$$

We recall that a sequence (x_{mn}) of complex numbers is said to be slowly oscillating in sense (1,1) if

$$\lim_{\lambda\to1^+}\limsup_{m,n\to\infty}\max_{\substack{m<j\le\lambda_m\\ n<k\le\lambda_n}}|x_{jk}-x_{mn}|=0. \tag{2.19}$$

The concept of slow oscillation was introduced by Hardy [2] for the sequences of complex numbers. An equivalent reformulation of (2.19) can be given as follows:

$$\lim_{\lambda\to1^-}\limsup_{m,n\to\infty}\max_{\substack{\lambda_m<j\le m\\ \lambda_n<k\le n}}|x_{mn}-x_{jk}|=0. \tag{2.20}$$

We have the following corollary of Theorem 2.6.

Corollary 2.7. *Let (1.1), (1.2) and (1.3) be satisfied. If a sequence (x_{mn}) of complex numbers is slowly oscillating in sense (1,1), then (1.4) is satisfied.*

2.3. Proofs

Proof of Theorem 2.2
Necessity. Assume that (1.1) and (1.4) are satisfied. Then Lemma 2.1 yields (2.9) in case $\lambda > 1$ and (2.10) in case $0 < \lambda < 1$.
Sufficiency. Assume that (1.1), (2.9) and (2.10) are satisfied. First we consider the case $\lambda > 1$. Let $\epsilon > 0$ be given. By (2.9), there exists some $\lambda > 1$ such that

$$\liminf_{m,n\to\infty} \frac{1}{(\lambda_m - m)(\lambda_n - n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} \Big((1+r^j)(1+\delta^k)x_{jk} - x_{mn}\Big) \geq -\epsilon. \tag{2.21}$$

By (2.3), we have

$$\begin{aligned} x_{mn} - \sigma^{r,\delta}_{mn} &= \frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m - m)(\lambda_n - n)}(\sigma^{r,\delta}_{mn} - \sigma^{r,\delta}_{\lambda_m,n} - \sigma^{r,\delta}_{m,\lambda_n} + \sigma^{r,\delta}_{\lambda_m,\lambda_n}) \\ &+ \frac{\lambda_m+1}{\lambda_m - m}(\sigma^{r,\delta}_{\lambda_m,n} - \sigma^{r,\delta}_{mn}) + \frac{\lambda_n+1}{\lambda_n - n}(\sigma^{r,\delta}_{m,\lambda_n} - \sigma^{r,\delta}_{mn}) \\ &- \frac{1}{(\lambda_m - m)(\lambda_n - n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} \Big((1+r^j)(1+\delta^k)x_{jk} - x_{mn}\Big). \end{aligned} \tag{2.22}$$

By (1.1), (2.4) and (2.5), the first three terms on the right hand-side of (2.22) vanish as $m, n \to \infty$.
It follows from (2.22) that

$$\begin{aligned} &\limsup_{m,n\to\infty}(x_{mn} - \sigma^{r,\delta}_{mn}) \\ &\qquad \leq -\liminf_{m,n\to\infty} \frac{1}{(\lambda_m - m)(\lambda_n - n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} \Big((1+r^j)(1+\delta^k)x_{jk} - x_{mn}\Big) \\ &\qquad \leq \epsilon. \end{aligned}$$

Consequently, we have

$$\limsup_{m,n\to\infty} x_{mn} \leq l + \epsilon. \tag{2.23}$$

Second, we consider the case $0 < \lambda < 1$. Let $\epsilon > 0$ be given. By (2.10), there exists some $0 < \lambda < 1$ such that

$$\liminf_{m,n\to\infty} \frac{1}{(m - \lambda_m)(n - \lambda_n)} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} \Big(x_{mn} - (1+r^j)(1+\delta^k)x_{jk}\Big) \geq -\epsilon. \tag{2.24}$$

By (2.6), we have

$$\begin{aligned} x_{mn} - \sigma^{r,\delta}_{mn} &= \frac{(\lambda_m+1)(\lambda_n+1)}{(m - \lambda_m)(n - \lambda_n)}(\sigma^{r,\delta}_{\lambda_m,\lambda_n} - \sigma^{r,\delta}_{\lambda_m,n} - \sigma^{r,\delta}_{m,\lambda_n} + \sigma^{r,\delta}_{mn}) \\ &+ \frac{\lambda_m+1}{m - \lambda_m}(\sigma^{r,\delta}_{\lambda_m,n} - \sigma^{r,\delta}_{\lambda_m,\lambda_n}) + \frac{\lambda_n+1}{n - \lambda_n}(\sigma^{r,\delta}_{m,\lambda_n} - \sigma^{r,\delta}_{\lambda_m,\lambda_n}) \\ &+ \frac{1}{(m - \lambda_m)(n - \lambda_n)} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} \Big(x_{mn} - (1+r^j)(1+\delta^k)x_{jk}\Big). \end{aligned} \tag{2.25}$$

By (1.1), (2.7) and (2.8), the first three terms on the right hand-side of (2.25) vanish as $m, n \to \infty$.

It follows from (2.25) that

$$\liminf_{m,n\to\infty}(x_{mn}-\sigma^{r,\delta}_{mn}) \geq \liminf_{m,n\to\infty}\frac{1}{(m-\lambda_m)(n-\lambda_n)}\sum_{j=\lambda_m+1}^{m}\sum_{k=\lambda_n+1}^{n}\Big(x_{mn}-(1+r^j)(1+\delta^k)x_{jk}\Big) \geq -\epsilon.$$

Consequently, we have

$$\liminf_{m,n\to\infty} x_{mn} \geq l-\epsilon. \tag{2.26}$$

Combining (2.23) and (2.26) yields

$$l-\epsilon \leq \liminf_{m,n\to\infty} x_{mn} \leq \limsup_{m,n\to\infty} x_{mn} \leq l+\epsilon.$$

Being ϵ arbitrary small, hence (1.4) follows. □

Proof of Corollary 2.3. For $\lambda>1$, we have the following inequality:

$$\frac{1}{(\lambda_m-m)(\lambda_n-n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}\Big((1+r^j)(1+\delta^k)x_{jk}-x_{mn}\Big) \geq \min_{\substack{m<j\leq\lambda_m\\ n<k\leq\lambda_n}}\Big(\delta^k x_{jk}+r^j x_{jk}+r^j\delta^k x_{jk}\Big)+\min_{\substack{m<j\leq\lambda_m\\ n<k\leq\lambda_n}}(x_{jk}-x_{mn})\,.$$

It follows from the definition of $\sigma^{r,\delta}_{mn}(x)$ that

$$\frac{x_{mn}}{mn}=\frac{(m+1)(n+1)\sigma^{r,\delta}_{mn}-m(n+1)\sigma^{r,\delta}_{m-1,n}-(m+1)n\sigma^{r,\delta}_{m,n-1}+mn\sigma^{r,\delta}_{m-1,n-1}}{mn(1+r^m)(1+\delta^n)}.$$

Since (x_{mn}) is $(A^{r,\delta},1,1)$ summable to l, then we have $x_{mn}/mn\to 0$, as $m,n\to\infty$. Therefore, $r^m\delta^n x_{mn}\to 0$ as $m,n\to\infty$. By the definition of $\sigma^{r,*}_{mn}(x)$ and $\sigma^{*,\delta}_{mn}(x)$, we have

$$\frac{x_{mn}}{m}=\frac{(m+1)\sigma^{r}_{mn}-m\sigma^{r}_{m-1,n}}{m(1+r^m)}$$

and

$$\frac{x_{mn}}{n}=\frac{(n+1)\sigma^{\delta}_{mn}-n\sigma^{\delta}_{m,n-1}}{n(1+\delta^n)},$$

respectively. Since (x_{mn}) is $(A^{r,*},1,0)$ and $(A^{*,\delta},0,1)$ summable to l, then we have $x_{mn}/m\to 0$ and $x_{mn}/n\to 0$ as $m,n\to\infty$, respectively. Hence, $r^m x_{mn}\to 0$ and $\delta^n x_{mn}\to 0$ as $m,n\to\infty$. So, condition (2.11) clearly implies condition (2.9). Similarly, (2.12) implies (2.10). By Theorem 2.2, we have (1.4). □

Proof of Theorem 2.6
Necessity. The proof is similar to the proof of the necessity part of Theorem 2.2.
Sufficiency. Assume that (1.1) and one of conditions (2.17) and (2.18) is satisfied. Let any $\epsilon>0$ be given. By (2.17), there exists some $\lambda>1$ such that

$$\limsup_{m,n\to\infty}\left|\frac{1}{(\lambda_m-m)(\lambda_n-n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}\Big((1+r^j)(1+\delta^k)x_{jk}-x_{mn}\Big)\right|<\epsilon. \tag{2.27}$$

By (1.1), (2.4) and (2.5), we have

$$\limsup_{m,n\to\infty}|x_{mn}-\sigma^{r,\delta}_{mn}| \leq \limsup_{m,n\to\infty}\left|\frac{1}{(\lambda_m-m)(\lambda_n-n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}\Big((1+r^j)(1+\delta^k)x_{jk}-x_{mn}\Big)\right| \leq \epsilon. \quad (2.28)$$

Let any $\epsilon>0$ be given. By (2.18), there exists some $0<\lambda<1$ such that

$$\limsup_{m,n\to\infty}\left|\frac{1}{(m-\lambda_m)(n-\lambda_n)}\sum_{j=\lambda_m+1}^{m}\sum_{k=\lambda_n+1}^{n}\Big(x_{mn}-(1+r^j)(1+\delta^k)x_{jk}\Big)\right|<\epsilon.$$

By (1.1), (2.7) and (2.8), we have

$$\limsup_{m,n\to\infty}|x_{mn}-\sigma^{r,\delta}_{mn}| \leq \limsup_{m,n\to\infty}\left|\frac{1}{(m-\lambda_m)(n-\lambda_n)}\sum_{j=\lambda_m+1}^{m}\sum_{k=\lambda_n+1}^{n}\Big(x_{mn}-(1+r^j)(1+\delta^k)x_{jk}\Big)\right| \leq \epsilon \quad (2.29)$$

By (2.28) or (2.29), in either case we obtain

$$\limsup_{m,n\to\infty}|x_{mn}-\sigma^{r,\delta}_{mn}|=0 \quad (2.30)$$

whence it follows that

$$\lim_{m,n\to\infty}|x_{mn}-\sigma^{r,\delta}_{mn}|=0. \quad (2.31)$$

Now we conclude (1.4) from (1.1) and (2.31). □

Proof of Corollary 2.7. For $\lambda>1$, we have the following inequality:

$$\left|\frac{1}{(\lambda_m-m)(\lambda_n-n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}\Big((1+r^j)(1+\delta^k)x_{jk}-x_{mn}\Big)\right| \leq \max_{\substack{m<j\leq\lambda_m\\ n<k\leq\lambda_n}}|\delta^k x_{jk}+r^j x_{jk}+r^j\delta^k x_{jk}|+\max_{\substack{m<j\leq\lambda_m\\ n<k\leq\lambda_n}}|x_{jk}-x_{mn}|$$

As in the proof of Corollary 2.3, we have $x_{mn}/mn\to 0$ as $m,n\to\infty$, $x_{mn}/m\to 0$, as $m,n\to\infty$ and $x_{mn}/n\to 0$, as $m,n\to\infty$. Hence, $r^m\delta^n x_{mn}\to 0$, $r^m x_{mn}\to 0$ and $\delta^n x_{mn}\to 0$ as $m,n\to\infty$. Hence, condition (2.19) clearly implies the condition (2.17). Similarly, (2.20) implies (2.18). By Theorem 2.6, we have (1.4). □

3. Tauberian theorems for $(A^{r,*},1,0)$ summability method

3.1. An auxiliary result

Lemma 3.1. *If a sequence (x_{mn}) is $(A^{r,*},1,0)$ summable to a finite limit l, then*

$$\lim_{m,n\to\infty}\frac{1}{\lambda_m-m}\sum_{j=m+1}^{\lambda_m}(1+r^j)x_{jn}=l \quad (3.1)$$

for every $\lambda > 1$ *and*

$$\lim_{m,n\to\infty} \frac{1}{m-\lambda_m} \sum_{j=\lambda_m+1}^{m} (1+r^j)x_{jn} = l \tag{3.2}$$

for every $0 < \lambda < 1$.

Proof of Lemma 3.1. i-) Case $\lambda > 1$. By definition of $\sigma^{r,*}_{mn}$, we have

$$\begin{aligned}
\sigma^{r,*}_{\lambda_m,n} - \sigma^{r,*}_{mn} &= \frac{1}{\lambda_m+1}\sum_{j=0}^{\lambda_m}(1+r^j)x_{jn} - \frac{1}{m+1}\sum_{j=0}^{m}(1+r^j)x_{jn} \\
&= \frac{1}{\lambda_m+1}\sum_{j=0}^{m}(1+r^j)x_{jn} + \frac{1}{\lambda_m+1}\sum_{j=m+1}^{\lambda_m}(1+r^j)x_{jn} - \frac{1}{m+1}\sum_{j=0}^{m}(1+r^j)x_{jn} \\
&= \left(\frac{1}{\lambda_m+1} - \frac{1}{m+1}\right)\sum_{j=0}^{m}(1+r^j)x_{jn} + \frac{1}{\lambda_m+1}\sum_{j=m+1}^{\lambda_m}(1+r^j)x_{jn} \\
&= \frac{m-\lambda_m}{(m+1)(\lambda_m+1)}\sum_{j=0}^{m}(1+r^j)x_{jn} + \frac{1}{\lambda_m+1}\sum_{j=m+1}^{\lambda_m}(1+r^j)x_{jn} \\
&= \frac{m-\lambda_m}{\lambda_m+1}\sigma^{r}_{mn} + \frac{1}{\lambda_m+1}\sum_{j=m+1}^{\lambda_m}(1+r^j)x_{jn}.
\end{aligned}$$

Multiplying both sides by $\dfrac{\lambda_m+1}{\lambda_m-m}$, we have

$$\frac{\lambda_m+1}{\lambda_m-m}(\sigma^{r,*}_{\lambda_m,n} - \sigma^{r,*}_{mn}) = -\sigma^{r}_{mn} + \frac{1}{\lambda_m-m}\sum_{j=m+1}^{\lambda_m}(1+r^j)x_{jn}. \tag{3.3}$$

It follows from (3.3) that

$$\frac{1}{\lambda_m-m}\sum_{j=m+1}^{\lambda_m}(1+r^j)x_{jn} = \sigma^{r,*}_{mn} + \frac{\lambda_m+1}{\lambda_m-m}(\sigma^{r,*}_{\lambda_m,n} - \sigma^{r,*}_{mn}). \tag{3.4}$$

Taking (1.2) and (2.5) into account, we have (3.1) from (3.4).
ii-) Case $0 < \lambda < 1$. By definition, we have

$$\frac{1}{m-\lambda_m}\sum_{j=\lambda_m+1}^{m}(1+r^j)x_{jn} = \sigma^{r,*}_{mn} + \frac{\lambda_m+1}{m-\lambda_m}(\sigma^{r,*}_{mn} - \sigma^{r,*}_{\lambda_m,n}). \tag{3.5}$$

Taking (1.3) and (2.8) into account, we have (3.2) from (3.5). □

3.2. Main results

Theorem 3.2. *If* (x_{mn}) *is a sequence of real numbers which is* $(A^{r,*},1,0)$ *summable to a finite limit* l, *then (1.4) holds if and only if*

$$\sup_{\lambda>1}\liminf_{m,n\to\infty}\frac{1}{\lambda_m-m}\sum_{j=m+1}^{\lambda_m}\big((1+r^j)x_{jn} - x_{mn}\big) \geq 0 \tag{3.6}$$

and

$$\sup_{0<\lambda<1}\liminf_{m,n\to\infty}\frac{1}{m-\lambda_m}\sum_{j=\lambda_m+1}^{m}\left(x_{mn}-(1+r^j)x_{jn}\right)\geq 0, \tag{3.7}$$

in which case we necessarily have

$$\lim_{m,n\to\infty}\frac{1}{\lambda_m-m}\sum_{j=m+1}^{\lambda_m}\left((1+r^j)x_{jn}-x_{mn}\right)=0$$

for all $\lambda>1$ *and*

$$\lim_{m,n\to\infty}\frac{1}{m-\lambda_m}\sum_{j=\lambda_m+1}^{m}\left(x_{mn}-(1+r^j)x_{jn}\right)=0$$

for all $0<\lambda<1$.

A sequence (x_{mn}) of real numbers is said to be slowly decreasing in sense $(1,0)$ if

$$\lim_{\lambda\to1^+}\liminf_{m,n\to\infty}\min_{m<j\leq\lambda_m}(x_{jn}-x_{mn})\geq 0 \tag{3.8}$$

Note that condition (3.8) can be equivalently reformulated as follows:

$$\lim_{\lambda\to1^-}\liminf_{m,n\to\infty}\min_{\lambda_m<j\leq m}(x_{mn}-x_{jn})\geq 0 \tag{3.9}$$

Corollary 3.3. *Let (1.2) be satisfied. If a sequence (x_{mn}) of real numbers is slowly decreasing in sense $(1,0)$, then (1.4) is satisfied.*

Theorem 3.4. *If (x_{mn}) is a sequence of complex numbers which is $(A^{r,*},1,0)$ summable to l, then (x_{mn}) converges to the same limit if and only if one of the following two conditions is satisfied:*

$$\inf_{\lambda>1}\limsup_{m,n\to\infty}\left|\frac{1}{\lambda_m-m}\sum_{j=m+1}^{\lambda_m}\left((1+r^j)x_{jn}-x_{mn}\right)\right|=0 \tag{3.10}$$

or

$$\inf_{0<\lambda<1}\limsup_{m,n\to\infty}\left|\frac{1}{m-\lambda_m}\sum_{j=\lambda_m+1}^{m}\left(x_{mn}-(1+r^j)x_{jn}\right)\right|=0. \tag{3.11}$$

We recall that a sequence (x_{mn}) of complex numbers is said to be slowly oscillating in sense (1,0) if

$$\lim_{\lambda\to1^+}\limsup_{m,n\to\infty}\max_{m<j\leq\lambda_m}|x_{jn}-x_{mn}|=0. \tag{3.12}$$

An equivalent reformulation of (3.12) can be given as follows:

$$\lim_{\lambda\to1^-}\limsup_{m,n\to\infty}\max_{\lambda_m<j\leq m}|x_{mn}-x_{jn}|=0. \tag{3.13}$$

Corollary 3.5. *Let (1.2) be satisfied. If a sequence (x_{mn}) of complex numbers is slowly oscillating in sense (1,0), then (1.4) is satisfied.*

3.3. Proofs

Proof of Theorem 3.2.
Necessity. Assume that (1.1) and (1.2) are satisfied. Then Lemma 3.1 yields (3.6) in case $\lambda > 1$ and (3.7) in case $0 < \lambda < 1$.
Sufficiency. Assume that (1.1), (3.6) and (3.7) are satisfied. First, we consider the case $\lambda > 1$. Let $\epsilon > 0$ be given. By (3.6), there exists some $\lambda > 1$ such that

$$\liminf_{m,n\to\infty} \frac{1}{\lambda_m - m} \sum_{j=m+1}^{\lambda_m} \left((1+r^j)x_{jn} - x_{mn}\right) \geq -\epsilon. \tag{3.14}$$

It follows from (3.4) that

$$x_{mn} - \sigma^{r,*}_{mn} = \frac{\lambda_m + 1}{\lambda_m - m}(\sigma^{r,*}_{\lambda_m,n} - \sigma^{r,*}_{mn}) - \frac{1}{\lambda_m - m} \sum_{j=m+1}^{\lambda_m} \left((1+r^j)x_{jn} - x_{mn}\right). \tag{3.15}$$

By (1.2) and (2.5), the first term on the right hand-side of (3.15) vanishes as $m, n \to \infty$. It follows from (3.15) that

$$\begin{aligned} \limsup_{m,n\to\infty}(x_{mn} - \sigma^{r,*}_{mn}) &\leq -\liminf_{m,n\to\infty} \frac{1}{\lambda_m - m} \sum_{j=m+1}^{\lambda_m} \left((1+r^j)x_{jn} - x_{mn}\right) \\ &\leq \epsilon. \end{aligned}$$

Consequently, we have

$$\limsup_{m,n\to\infty} x_{mn} \leq l + \epsilon. \tag{3.16}$$

Second, we consider the case $0 < \lambda < 1$. Let $\epsilon > 0$ be given. By (3.7), there exists some $0 < \lambda < 1$ such that

$$\liminf_{m,n\to\infty} \frac{1}{m - \lambda_m} \sum_{j=\lambda_m+1}^{m} \left(x_{mn} - (1+r^j)x_{jn}\right) \geq -\epsilon. \tag{3.17}$$

It follows from (3.5) that

$$x_{mn} - \sigma^{r,*}_{mn} = \frac{\lambda_m + 1}{m - \lambda_m}(\sigma^{r,*}_{mn} - \sigma^{r,*}_{\lambda_m,n}) + \frac{1}{m - \lambda_m} \sum_{j=\lambda_m+1}^{m} \left(x_{mn} - (1+r^j)x_{jn}\right). \tag{3.18}$$

Using a similar argument as above, we conclude by (3.18) that

$$\begin{aligned} \liminf_{m,n\to\infty}(x_{mn} - \sigma^{r,*}_{mn}) &\geq \liminf_{m,n\to\infty} \frac{1}{m - \lambda_m} \sum_{j=\lambda_m+1}^{m} \left(x_{mn} - (1+r^j)x_{jn}\right) \\ &\geq -\epsilon. \end{aligned}$$

Consequently, we have

$$\liminf_{m,n\to\infty} x_{mn} \geq l - \epsilon. \tag{3.19}$$

Combining (3.16) and (3.19) yields

$$l - \epsilon \leq \liminf_{m,n\to\infty} x_{mn} \leq \limsup_{m,n\to\infty} x_{mn} \leq l + \epsilon.$$

Being ϵ arbitrary small, hence (1.4) follows. □

Proof of Corollary 3.3. For $\lambda > 1$, we have the following inequality:

$$\frac{1}{\lambda_m - m}\sum_{j=m+1}^{\lambda_m}\left((1+r^j)x_{jn} - x_{mn}\right) \geq \min_{m<j\leq\lambda_m}(r^j x_{jk}) + \min_{m<j\leq\lambda_m}(x_{jn} - x_{mn}).$$

We have

$$\frac{x_{mn}}{m} = \frac{(m+1)\sigma^{r,*}_{mn} - m\sigma^{r,*}_{m-1,n}}{m(1+r^m)}$$

Since (x_{mn}) is $(A^{r,*}, 1, 0)$ summable to l, then we have $x_{mn}/m \to 0$ as $m, n \to \infty$. Therefore, $r^m x_{mn} \to 0$ as $m, n \to \infty$. So condition (3.8) clearly implies the condition (3.6). Similarly, (3.9) implies (3.7). By Theorem 3.2, we have (1.4). □

Proof of Theorem 3.4.
Necessity. The proof is similar to the proof of the necessity part of Theorem 2.6.
Sufficiency. Assume that (1.2) and one of conditions (3.10) and (3.11) is satisfied. Let $\epsilon > 0$ be given. By (3.10), there exists some $\lambda > 1$ such that

$$\limsup_{m,n\to\infty}\left|\frac{1}{\lambda_m - m}\sum_{j=m+1}^{\lambda_m}\left((1+r^j)x_{jn} - x_{mn}\right)\right| < \epsilon. \tag{3.20}$$

By (1.2), (2.4) and (2.5), we have

$$\limsup_{m,n\to\infty}|x_{mn} - \sigma^{r,*}_{mn}| \leq \limsup_{m,n\to\infty}\left|\frac{1}{\lambda_m - m}\sum_{j=m+1}^{\lambda_m}\left((1+r^j)x_{jn} - x_{mn}\right)\right| \leq \epsilon. \tag{3.21}$$

Let any $\epsilon > 0$ be given. By (3.11, there exists some $0 < \lambda < 1$ such that

$$\limsup_{m,n\to\infty}\left|\frac{1}{m - \lambda_m}\sum_{j=\lambda_m+1}^{m}\left(x_{mn} - (1+r^j)x_{jn}\right)\right| < \epsilon.$$

By (1.2), (2.7) and (2.8), we have

$$\limsup_{m,n\to\infty}|x_{mn} - \sigma^{r,*}_{mn}| \leq \limsup_{m,n\to\infty}\left|\frac{1}{m - \lambda_m}\sum_{j=\lambda_m+1}^{m}\left(x_{mn} - (1+r^j)x_{jn}\right)\right| \leq \epsilon. \tag{3.22}$$

By (3.21) or (3.22), in either case we obtain

$$\limsup_{m,n\to\infty}|x_{mn} - \sigma^{r,*}_{mn}| = 0$$

whence it follows that

$$\lim_{m,n\to\infty}|x_{mn} - \sigma^{r,*}_{mn}| = 0. \tag{3.23}$$

Now, we conclude (1.4) from (1.2) and (3.23).

4. Tauberian theorems for $(A^{*,\delta}, 0, 1)$ summability method

The symmetric counterparts of Theorem 3.2 and Corollary 3.3 are also valid, when we consider summability $(A^{*,\delta}, 0, 1)$ instead of $(A^{r,*}, 1, 0)$.

Lemma 4.1. *If a sequence (x_{mn}) is $(A^{*,\delta}, 0, 1)$ summable to a finite limit l, then*

$$\lim_{m,n\to\infty} \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} (1+\delta^k)x_{mk} = l \tag{4.1}$$

for every $\lambda > 1$ and

$$\lim_{m,n\to\infty} \frac{1}{n - \lambda_n} \sum_{k=\lambda_n+1}^{n} (1+\delta^k)x_{mk} = l \tag{4.2}$$

for every $0 < \lambda < 1$.

Theorem 4.2. *If (x_{mn}) is a sequence of real numbers which is $(A^{*,\delta}, 0, 1)$ summable to a finite limit l, then (1.4) holds if and only if*

$$\sup_{\lambda>1} \liminf_{m,n\to\infty} \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} \Big((1+\delta^k)x_{mk} - x_{mn}\Big) \geq 0 \tag{4.3}$$

and

$$\sup_{0<\lambda<1} \liminf_{m,n\to\infty} \frac{1}{n - \lambda_n} \sum_{k=\lambda_n+1}^{n} \Big(x_{mn} - (1+\delta^k)x_{mk}\Big) \geq 0. \tag{4.4}$$

A sequence (x_{mn}) of real numbers is said to be slowly decreasing in sense $(0, 1)$ if

$$\lim_{\lambda\to 1^+} \liminf_{m,n\to\infty} \min_{n<k\leq\lambda_n} (x_{mk} - x_{mn}) \geq 0. \tag{4.5}$$

Note that condition (4.5) can be equivalently reformulated as follows:

$$\lim_{\lambda\to 1^-} \liminf_{m,n\to\infty} \min_{\lambda_n<k\leq n} (x_{mn} - x_{mk}) \geq 0 \tag{4.6}$$

Corollary 4.3. *Let (1.3) be satisfied. If a sequence (x_{mn}) of real numbers is slowly decreasing in sense $(0, 1)$, then (1.4) is satisfied.*

Theorem 4.4. *If (x_{mn}) is a sequence of complex numbers which is $(A^{*,\delta}, 0, 1)$ summable to l, then (x_{mn}) converges to the same limit if and only if one of the following two conditions is satisfied:*

$$\inf_{\lambda>1} \limsup_{m,n\to\infty} \left| \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} \Big((1+\delta^k)x_{mk} - x_{mn}\Big) \right| = 0 \tag{4.7}$$

or

$$\inf_{0<\lambda<1} \limsup_{m,n\to\infty} \left| \frac{1}{n - \lambda_n} \sum_{k=\lambda_n+1}^{n} \Big(x_{mn} - (1+\delta^k)x_{mk}\Big) \right| = 0. \tag{4.8}$$

Theorem 4.2 and Theorem 4.4 can be proved by the similar techniques as in the proofs of Theorem 3.2 and Theorem 3.4. So we omit them.

We recall that a sequence (x_{mn}) of complex numbers is said to be slowly oscillating in sense (0,1) if

$$\lim_{\lambda\to 1^+} \limsup_{m,n\to\infty} \max_{n<k\leq\lambda_n} |x_{mk} - x_{mn}| = 0. \tag{4.9}$$

An equivalent reformulation of (4.9) can be given as follows:

$$\lim_{\lambda\to 1^-} \limsup_{m,n\to\infty} \max_{\lambda_n<k\leq n} |x_{mn} - x_{mk}| = 0. \tag{4.10}$$

Corollary 4.5. *Let (1.3) be satisfied. If a sequence (x_{mn}) of complex numbers is slowly oscillating in sense $(0, 1)$, then (1.4) is satisfied.*

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