



Existence, Error estimation, Rate of convergence, Ulam-Hyers stability, Well-posedness and Limit Shadowing Property Related to a Fixed Point Problem

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ABSTRACT: In this paper we consider a fixed point problem where the mapping is supposed to satisfy a generalized contractive inequality involving rational terms. We first prove the existence of a fixed point of such mappings. Then we show that the fixed point is unique under some additional assumptions. We investigate four aspects of the problem, namely, error estimation and rate of convergence of the fixed point iteration, Ulam-Hyers stability, well-posedness and limit shadowing property. In the existence theorem we use an admissibility condition. Two illustrations are given. The research is in the line with developing fixed point approaches relevant to applied mathematics.

Key Words: Metric space, Fixed point, Error correction, Rate of convergence, Ulam-Hyers stability, Well-posedness, Limit shadowing property.

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1. Introduction

Our primary purpose in this paper is to establish a fixed point theorem for a mapping which satisfies a contractive inequality involving rational terms and also some other conditions which are conceptual extension of admissibility conditions. The latter has been used quite extensively in the recent developments of fixed point theory [1,8,10,19,20,23]. It is shown that the fixed point is unique if some additional conditions are imposed. We investigate some aspects of the fixed point problem considered here. We make an error estimation of the fixed point iteration which we construct in this paper. We also investigate the rate of convergence of the iteration process. Such considerations have appeared in the fixed point theory through works like [2,7,14,25].

Next we investigate the Ulam-Hyers stability of the fixed point problem. It is a type of stability which was initiated by a mathematical question by Ulam [24] and subsequent partial answers by Hyers [9] and Rassias [16]. The investigation of such stability has been performed in various contexts of mathematics like functional equations [5,6], isometries [12,17], etc.

Finally we investigate the well-posedness and limit shadowing property of the problem. These are two related properties of the fixed point problem. The study of well-posedness has appeared in several recent works related to fixed point theory as for instances in [4,10,11,13,15,22].

The relevance of the present study lies in the theoretical development of fixed point methodologies applicable to different domains of applied mathematics like differential equations, functional equations etc. [3,21].

2. Mathematical background

For the purpose of the following three definitions we formally state the following fixed point problem to which they are related.

Problem (P): Let (X, d) be a metric space and $F : X \rightarrow X$ be a mapping. We consider the problem of finding a fixed point of F , that is, the problem of finding $x \in X$ such that

$$x = Fx. \tag{2.1}$$

Definition 2.1 ([10,18]). The problem (P) is called Ulam-Hyers stable if there exists $c > 0$ such that, for any $\epsilon > 0$ and $y \in X$ with $d(y, Fy) \leq \epsilon$ there exists a solution $x^* \in X$ of $x = Fx$ such that $d(y, x^*) \leq c\epsilon$.

Definition 2.2 ([22]). The problem (P) is called generalized Ulam-Hyers stable if there exists a function $\phi : [0, \infty) \rightarrow [0, \infty)$, which is monotone increasing, continuous at 0 with $\phi(0) = 0$, such that for each $\epsilon > 0$ and for each solution $u^* \in X$ of the inequality $d(x, Fx) \leq \epsilon$ there exists a solution $x^* \in X$ of $x = Fx$ such that $d(u^*, x^*) \leq \phi(\epsilon)$.

Remark 2.3. If $\phi : [0, \infty) \rightarrow [0, \infty)$ is defined as $\phi(t) = ct$ for $t \geq 0$, where $c > 0$ is a constant, then Definition 2.2 reduces to Definition 2.1.

Definition 2.4 ([10]). The problem (P) is called well-posed if (i) F has a unique fixed point $x^* \in X$, (ii) $d(x_n, x^*) \rightarrow 0$ as $n \rightarrow \infty$, whenever $\{x_n\}$ is a sequence in X with $d(x_n, Fx_n) \rightarrow 0$ as $n \rightarrow \infty$.

Definition 2.5 ([22]). The problem (P) has the limit shadowing property in X if, for any sequence $\{x_n\} \in X$ for which $d(x_n, Fx_n) \rightarrow 0$ as $n \rightarrow \infty$, it follows that there exists $z \in X$ such that $d(x_n, F^n z) \rightarrow 0$ as $n \rightarrow \infty$.

Definition 2.6. Let X be a nonempty set and $\alpha : X \times X \rightarrow [0, \infty)$. A mapping $F : X \rightarrow X$ is said to be α -dominated if $\alpha(x, Fx) \geq 1$, for $x \in X$.

The above definition is illustrated though the following example.

Example 2.7. Let $X = [0, 1]$ be equipped with usual metric. Let $F : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, \infty)$ be respectively defined as follows:

$$Fx = \frac{\sin^2 x}{16}, \text{ for } x \in X \text{ and } \alpha(x, y) = \begin{cases} e^{x+y}, & \text{if } 0 \leq x \leq 1, 0 \leq y \leq \frac{1}{8}, \\ 0, & \text{otherwise.} \end{cases}$$

As $Fx \in [0, \frac{1}{16}]$, for all $x \in [0, 1]$, it follows that $\alpha(x, Fx) \geq 1$, for all $x \in [0, 1]$, that is, F is a α -dominated mapping.

Definition 2.8 ([20]). A function $\alpha : X \times X \rightarrow [0, \infty)$, where X is a nonempty set, is said to have triangular property if for $x, y, z \in X$, $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1 \implies \alpha(x, z) \geq 1$.

Definition 2.9 ([19]). Let (X, d) be a metric space and $\alpha : X \times X \rightarrow [0, \infty)$. Then X is said to have regular property with respect to α (or α -regular property) if for every sequence $\{x_n\}$ in X converging to $x \in X$, $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \implies \alpha(x_n, x) \geq 1$, for all n .

Remark 2.10. For the metric space X and the mapping α as in Example 2.7, it can be easily verified that α has triangular property and X is regular with respect to α .

Let (X, d) be a metric space and $\alpha : X \times X \rightarrow [0, \infty)$ be a mapping. We designate the following properties by (A1), (A2) and (A3):

- (A1) X has regular property with respect to α ;
- (A2) α has triangular property;
- (A3) for every $x, x^* \in X$, there exists a $u \in X$ such that $\alpha(x, u) \geq 1$ and $\alpha(x^*, u) \geq 1$.

3. Main results

In this section we establish a fixed point result. We discuss its uniqueness under some additional assumptions. We deduce some corollaries of the main result and illustrate it with an example.

Theorem 3.1. *Let (X, d) be a complete metric space, $F : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, \infty)$. Suppose that F is α -dominated and there exists $k \in (0, 1)$ such that for $x, y \in X$ with $\alpha(x, y) \geq 1$,*

$$d(Fx, Fy) \leq k \max \left\{ d(x, y), \frac{d(x, Fx) + d(y, Fy)}{2}, \frac{d(x, Fy) + d(y, Fx)}{2}, \frac{d(x, Fx) d(y, Fy)}{1 + d(x, y)}, \frac{d(y, Fx) d(x, Fy)}{1 + d(x, y)} \right\}. \quad (3.1)$$

Also, suppose that the property (A1) holds. Then F has a fixed point in X .

Proof. Let $x_0 \in X$ be arbitrary. We construct a sequence $\{x_n\}$ in X such that

$$x_{n+1} = Fx_n, \text{ for all } n \geq 0. \quad (3.2)$$

As F is α -dominated, we have

$$\alpha(x_n, Fx_n) = \alpha(x_n, x_{n+1}) \geq 1, \text{ for all } n \geq 0. \quad (3.3)$$

Let

$$r_n = d(x_n, x_{n+1}), \text{ for all } n \geq 0. \quad (3.4)$$

By (3.1), (3.2), (3.3) and (3.4), we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(Fx_n, Fx_{n+1}) \\ &\leq k \max \left\{ d(x_n, x_{n+1}), \frac{d(x_n, Fx_n) + d(x_{n+1}, Fx_{n+1})}{2}, \frac{d(x_n, Fx_{n+1}) + d(x_{n+1}, Fx_n)}{2}, \frac{d(x_n, Fx_n) d(x_{n+1}, Fx_{n+1})}{1 + d(x_n, x_{n+1})}, \frac{d(x_{n+1}, Fx_n) d(x_n, Fx_{n+1})}{1 + d(x_n, x_{n+1})} \right\} \\ &= k \max \left\{ d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}, \frac{d(x_n, x_{n+2})}{2}, \frac{d(x_n, x_{n+1}) d(x_{n+1}, x_{n+2})}{1 + d(x_n, x_{n+1})}, 0 \right\} \\ &\leq k \max \left\{ d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}, \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}, d(x_{n+1}, x_{n+2}) \right\} \\ &= k \max \left\{ r_n, \frac{r_n + r_{n+1}}{2}, \frac{r_n + r_{n+1}}{2}, r_{n+1} \right\} \\ &= k \max \left\{ r_n, r_{n+1} \right\}, \quad \left[\text{as } \frac{r_n + r_{n+1}}{2} \leq \max \{r_n, r_{n+1}\} \right]. \end{aligned}$$

Therefore,

$$d(x_{n+1}, x_{n+2}) \leq k \max \{r_n, r_{n+1}\}. \quad (3.5)$$

Suppose that $0 \leq r_n < r_{n+1}$. From (3.4) and (3.5), we have

$$r_{n+1} = d(x_{n+1}, x_{n+2}) \leq k r_{n+1},$$

which is a contradiction. Therefore, $r_{n+1} \leq r_n$, for all $n \geq 0$. Then from (3.5), we have

$$d(x_{n+1}, x_{n+2}) = r_{n+1} \leq k r_n = k d(x_n, x_{n+1}), \text{ for all } n \geq 0. \quad (3.6)$$

By repeated application of (3.6), we have

$$d(x_{n+1}, x_{n+2}) \leq k d(x_n, x_{n+1}) \leq k^2 d(x_{n-1}, x_n) \leq \dots \leq k^{n+1} d(x_0, x_1). \quad (3.7)$$

With the help of (3.7), we have

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) \leq \sum_{n=1}^{\infty} k^n d(x_0, x_1) = \frac{1}{1-k} d(x_0, x_1) < \infty,$$

which implies that $\{x_n\}$ is a Cauchy sequence in X . As X is complete, there exists $x \in X$ such that

$$\lim_{n \rightarrow \infty} x_n = x. \quad (3.8)$$

By (3.3), (3.8) and the assumption (A1), we have $\alpha(x_n, x) \geq 1$, for all $n \geq 0$. Using (3.2), we have

$$\begin{aligned} d(x_{n+1}, Fx) &= d(Fx_n, Fx) \\ &\leq \max \left\{ d(x_n, x), \frac{d(x_n, Fx_n) + d(x, Fx)}{2}, \frac{d(x_n, Fx) + d(x, Fx_n)}{2}, \right. \\ &\quad \left. \frac{d(x_n, Fx_n) d(x, Fx)}{1 + d(x_n, x)}, \frac{d(x, Fx_n) d(x_n, Fx)}{1 + d(x_n, x)} \right\} \\ &= k \max \left\{ d(x_n, x), \frac{d(x_n, x_{n+1}) + d(x, Fx)}{2}, \frac{d(x_n, Fx) + d(x, x_{n+1})}{2}, \right. \\ &\quad \left. \frac{d(x_n, x_{n+1}) d(x, Fx)}{1 + d(x_n, x)}, \frac{d(x, x_{n+1}) d(x_n, Fx)}{1 + d(x_n, x)} \right\}. \end{aligned} \quad (3.9)$$

Taking limit as $n \rightarrow \infty$ in (3.9) and using (3.8), we have

$$\begin{aligned} d(x, Fx) &\leq k \max \left\{ 0, \frac{d(x, Fx)}{2}, \frac{d(x, Fx)}{2}, 0, 0 \right\} \\ &= k \frac{d(x, Fx)}{2}, \end{aligned}$$

which implies that $d(x, Fx) = 0$, that is, $x = Fx$, that is, x is a fixed point of F .

Theorem 3.2. *In addition to the hypothesis of Theorem 3.1, suppose that (A2) and (A3) hold. Then F has a unique fixed point.*

Proof. By Theorem 3.1, the set of fixed points of F is nonempty. If possible, let x and x^* be two fixed points of F . Then $x = Fx$ and $x^* = Fx^*$. Our aim is to show that $x = x^*$. By the assumption (A3), there exists $u \in X$ such that $\alpha(x, u) \geq 1$ and $\alpha(x^*, u) \geq 1$. Put $u_0 = u$. Then $\alpha(x, u_0) \geq 1$. Let $u_1 = Fu_0$. Similarly, as in the proof of Theorem 3.1, we inductively define a sequence $\{u_n\}$ such that

$$u_{n+1} = Fu_n, \text{ for all } n \geq 0. \quad (3.10)$$

As F is α -dominated, we have

$$\alpha(u_n, u_{n+1}) \geq 1, \text{ for all } n \geq 0. \quad (3.11)$$

Arguing similarly as in proof of Theorem 3.1, we prove that $\{u_n\}$ is a Cauchy sequence in X and there exists $p \in X$ such that

$$\lim_{n \rightarrow \infty} u_n = p. \quad (3.12)$$

We claim that

$$\alpha(x, u_n) \geq 1, \text{ for all } n \geq 0. \quad (3.13)$$

In fact, we shall use mathematical induction. As $\alpha(x, u_0) \geq 1$ and $\alpha(u_0, u_1) \geq 1$, by the assumption (A2), we have $\alpha(x, u_1) \geq 1$. Therefore, our claim is true for $n = 1$. We assume that $\alpha(x, u_m) \geq 1$ holds for some $m > 1$. Now by (3.11), $\alpha(u_m, u_{m+1}) \geq 1$. Then applying the assumption (A2), we have $\alpha(x, u_{m+1}) \geq 1$ and this proves our claim.

By (3.1) and (3.13) we have, for all $n \geq 0$

$$\begin{aligned}
d(x, u_{n+1}) &= d(Fx, Fu_n) \\
&\leq k \max \left\{ d(x, u_n), \frac{d(x, Fx) + d(u_n, Fu_n)}{2}, \frac{d(x, Fu_n) + d(u_n, Fx)}{2}, \right. \\
&\quad \left. \frac{d(x, Fx) d(u_n, Fu_n)}{1 + d(x, u_n)}, \frac{d(x, Fu_n) d(u_n, Fx)}{1 + d(x, u_n)} \right\} \\
&= k \max \left\{ d(x, u_n), \frac{d(u_n, u_{n+1})}{2}, \frac{d(x, u_{n+1}) + d(u_n, x)}{2}, \right. \\
&\quad \left. 0, \frac{d(x, u_{n+1}) d(u_n, x)}{1 + d(x, u_n)} \right\}. \tag{3.14}
\end{aligned}$$

Taking limit as $n \rightarrow \infty$ in (3.14) and using (3.12), we have

$$\begin{aligned}
d(x, p) &\leq k \max \left\{ d(x, p), 0, \frac{d(x, p) + d(p, x)}{2}, 0, \frac{d(x, p) d(p, x)}{1 + d(x, p)} \right\} \\
&\leq k \max \left\{ d(x, p), d(x, p), d(x, p) \right\} = k d(x, p), \tag{3.15}
\end{aligned}$$

which is a contradiction unless $d(x, p) = 0$, that is, $x = p$. Similarly, we can prove that $x^* = p$. Hence we have $x = x^*$, that is, the fixed point of F is unique.

We present the following illustrative examples in support of Theorems 3.1 .

Example 3.3. Using the metric space X , mappings α and F as in Example 2.7, we see that α has triangular property and X is regular with respect to α (see Remark 2.10) and F is a α -dominated mapping. Take $k = \frac{1}{4}$.

Let $x, y \in X$ with $\alpha(x, y) \geq 1$. Then $x \in [0, 1]$ and $y \in [0, \frac{1}{8}]$. Therefore, it is required to verify the inequality in Theorem 3.1 for $x \in [0, 1]$ and $y \in [0, \frac{1}{8}]$. Now, $d(x, y) = |x - y|$ and

$$\begin{aligned}
d(Fx, Fy) &= \left| \frac{\sin^2 x}{16} - \frac{\sin^2 y}{16} \right| = \frac{1}{16} |\sin(x - y) \sin(x + y)| \leq \frac{1}{16} |\sin(x - y)| \leq \frac{|x - y|}{16} \\
&= \frac{1}{4} \frac{|x - y|}{4} = \frac{1}{4} \frac{d(x, y)}{4} \\
&\leq \frac{1}{4} \max \left\{ d(x, y), \frac{d(x, Fx) + d(y, Fy)}{2}, \frac{d(x, Fy) + d(y, Fx)}{2}, \right. \\
&\quad \left. \frac{d(x, Fx) d(y, Fy)}{1 + d(x, y)}, \frac{d(y, Fx) d(x, Fy)}{1 + d(x, y)} \right\}.
\end{aligned}$$

Then it follows that the inequality in Theorem 3.1 is satisfied for all $x, y \in X$ with $\alpha(x, y) \geq 1$. Hence all the conditions of Theorem 3.2 are satisfied and 0 is the unique fixed point of F .

4. Error estimation and rate of convergence

We now study the rate at which the iteration method of finding the fixed point of the problem (P) converges if the initial approximation to the fixed point is sufficiently close to the desired fixed point.

Definition 4.1. The problem (P) is said to be of order r or has the rate of convergence r if (i) F has a unique fixed point x , (ii) r is the positive real number for which there exists a finite constant $C > 0$ such that $R_{k+1} \leq C R_k^r$, where $R_k = d(x, x_k)$ is the error in k^{th} iterate. The constant C is called the asymptotic error.

When $r = 1$ we say that the problem (P) is linearly convergent.

Theorem 4.2. *Let (X, d) be a complete metric space, $F : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, \infty)$. Suppose that F satisfies all the assumptions of Theorem 3.2. Then $R_{n+1} \leq \frac{k^{n+1}}{(1-k)} d(x_1, x_0)$.*

Proof. By Theorem 3.2, F has a unique fixed point $x \in X$. Let $x_0 \in X$ be the initial approximation of x and $x_1 = Fx_0$. Similarly, as in the proof of Theorem 3.1, we define a sequence $\{x_n\}$ such that $x_{n+1} = Fx_n$, for all $n \geq 0$. Then arguing similarly as in proof of Theorem 3.1, we can show that

(a) (3.3) and (3.7) are satisfied;

(b) $\{x_n\}$ is a Cauchy sequence in X and converges to a fixed point of F in X .

As, we consider that x is the unique fixed point of F , we have $\lim_{n \rightarrow \infty} x_n = x$. By (3.3), (3.8) and the assumption (A1), we have $\alpha(x_n, x) \geq 1$ for all $n \geq 0$. Then we have

$$\begin{aligned}
R_{n+1} &= d(x, x_{n+1}) = d(Fx, Fx_n) = d(Fx_n, Fx) \\
&\leq k \max \left\{ d(x_n, x), \frac{d(x_n, Fx_n) + d(x, Fx)}{2}, \frac{d(x_n, Fx) + d(x, Fx_n)}{2}, \right. \\
&\quad \left. \frac{d(x_n, Fx_n) d(x, Fx)}{1 + d(x_n, x)}, \frac{d(x, Fx_n) d(x_n, Fx)}{1 + d(x_n, x)} \right\} \\
&\leq k \max \left\{ d(x_n, x), \frac{d(x_n, x_{n+1})}{2}, \frac{d(x_n, x) + d(x, x_{n+1})}{2}, 0, \frac{d(x, x_{n+1}) d(x_n, x)}{1 + d(x_n, x)} \right\} \\
&\leq k \max \left\{ d(x_n, x), \frac{d(x_n, x) + d(x, x_{n+1})}{2}, \right. \\
&\quad \left. 0, \frac{d(x, x_{n+1}) d(x_n, x)}{1 + d(x_n, x)} \right\} \\
&= k \max \left\{ R_n, \frac{R_n + R_{n+1}}{2}, \frac{R_n + R_{n+1}}{2}, \frac{R_{n+1} R_n}{1 + R_n} \right\} \\
&\leq k \max \left\{ R_n, \frac{R_n + R_{n+1}}{2}, \frac{R_n + R_{n+1}}{2}, R_{n+1} \right\} \\
&= k \max \left\{ R_n, R_{n+1} \right\}, \quad [\text{as } \frac{R_n + R_{n+1}}{2} \leq \max \{ R_n, R_{n+1} \}]. \tag{4.1}
\end{aligned}$$

Suppose that $R_{n+1} > R_n \geq 0$. Then we have $R_{n+1} \leq k R_{n+1}$. As $0 < k < 1$ and $R_{n+1} > 0$, it leads to a contradiction. Therefore, $R_{n+1} \leq R_n$. Hence it follows from (4.1) that

$$R_{n+1} \leq k R_n \leq k [d(x_{n+1}, x_n) + R_{n+1}], \tag{4.2}$$

that is,

$$R_{n+1} \leq \frac{k}{(1-k)} d(x_{n+1}, x_n), \tag{4.3}$$

which, by (3.7), implies that

$$R_{n+1} \leq \frac{k^{n+1}}{(1-k)} d(x_1, x_0).$$

Remark 4.3. In general, the speed of the iteration depends on the value of k ; the smaller is the value of k , the faster would be the convergence.

Remark 4.4. Above theorem shows that if $0 < k < 1$ the error in taking the point x_n instead of x does not exceed $\frac{k^n}{1-k} d(x_1, x_0)$. This error can be made less than a preassigned real number $\varepsilon > 0$, if

$n \geq \left\lceil \frac{\log\left(\frac{d(x_1, x_0)}{\varepsilon(1-k)}\right)}{\log\left(\frac{1}{k}\right)} \right\rceil + 1$, where $[y]$ denotes the greatest integer function. This gives the number of iterations n needed to bring the point x_n within ε distance of the actual fixed point.

Remark 4.5. By Theorem 3.2, F has unique fixed point x . The inequality (4.2), that is, $R_{n+1} \leq k R_n$ shows that the fixed point problem (P) is linearly convergent with asymptotic error k .

5. Ulam-Hyers stability

In this section we discuss Ulam-Hyers stability of fixed problem (P) via α - dominated mapping. For this purpose we consider the fixed point problem (P) and the inequality

$$d(x, Fx) \leq \epsilon, \quad \text{where } \epsilon > 0. \quad (5.1)$$

We consider the following assumption which we use in the theorem of this section.

(A4) For any solution x^* of $x = Fx$ and any solution u^* of (5.1), one has $\alpha(x^*, u^*) \geq 1$.

Theorem 5.1. *In addition to the hypothesis of Theorem 3.2, suppose that (A4) holds. Then the fixed point problem (P) is Ulam-Hyers stable.*

Proof. By Theorem 3.2, F has unique a fixed point $x^* \in X$. Therefore, x^* is a solution of $x = Fx$. Let $u^* \in X$ be a solution of (5.1). Then $d(u^*, Fu^*) \leq \epsilon$. By the assumption (A4), we have $\alpha(x^*, u^*) \geq 1$. Using (3.1), we have

$$\begin{aligned} d(x^*, u^*) &= d(Fx^*, u^*) \leq d(Fx^*, Fu^*) + d(Fu^*, u^*) \\ &\leq k \max \left\{ d(x^*, u^*), \frac{d(x^*, Fx^*) + d(u^*, Fu^*)}{2}, \frac{d(x^*, Fu^*) + d(u^*, Fx^*)}{2}, \right. \\ &\quad \left. \frac{d(x^*, Fx^*) d(u^*, Fu^*)}{1 + d(x^*, u^*)}, \frac{d(u^*, Fx^*) d(x^*, Fu^*)}{1 + d(x^*, u^*)} \right\} + d(Fu^*, u^*) \\ &\leq k \max \left\{ d(x^*, u^*), \frac{\epsilon}{2}, \frac{d(x^*, u^*) + d(u^*, Fu^*) + d(u^*, x^*)}{2}, 0, \right. \\ &\quad \left. \frac{d(u^*, x^*) [d(x^*, u^*) + d(u^*, Fu^*)]}{1 + d(x^*, u^*)} \right\} + \epsilon \\ &\leq k \max \left\{ d(x^*, u^*), \frac{\epsilon}{2}, \frac{d(x^*, u^*) + \epsilon + d(u^*, x^*)}{2}, 0, \frac{d(u^*, x^*) [d(x^*, u^*) + \epsilon]}{1 + d(x^*, u^*)} \right\} + \epsilon \\ &\leq k \max \left\{ d(x^*, u^*), \frac{\epsilon}{2}, \frac{2 d(x^*, u^*) + \epsilon}{2}, 0, d(x^*, u^*) + \epsilon \right\} + \epsilon \\ &= k [d(x^*, u^*) + \epsilon] + \epsilon, \end{aligned}$$

which implies that

$$d(x^*, u^*) \leq \frac{(k+1)\epsilon}{1-k}. \quad (5.2)$$

Let us define a function $\phi : [0, \infty) \rightarrow [0, \infty)$ as $\phi(t) = \frac{(k+1)t}{1-k}$. Then by (5.2), we have

$$d(x^*, u^*) < \frac{(k+1)\epsilon}{1-k} = \phi(\epsilon).$$

Since ϕ is monotone increasing, continuous and $\phi(0) = 0$. Therefore, the fixed point problem (P) is Ulam-Hyers stable.

6. Well-Posedness and Limit shadowing property

In this section we discuss the well-posedness and Limit shadowing property of fixed point problem (P) via α -dominated mapping. For this purpose we use the following assumption.

(A5) If x^* is any solution of $x = Fx$ and $\{x_n\}$ is any sequence in X with $d(x_n, Fx_n) \rightarrow 0$ as $n \rightarrow \infty$, then $\alpha(x_n, x^*) \geq 1$, for all n .

Theorem 6.1. *In addition to the hypothesis of Theorem 3.2, suppose that the assumption (A5) holds. Then the fixed point problem (P) is well-posed. Also, the problem (P) has limit shadowing property.*

Proof. By Theorem 3.2, F has a unique fixed point $x^* \in X$. Then x^* is a solution of $x = Fx$. Let $\{x_n\}$ be a sequence in X with $d(x_n, Fx_n) \rightarrow 0$ as $n \rightarrow \infty$. By the assumption (A5), we have $\alpha(x_n, x^*) \geq 1$, for all n . Using (3.1), we have

$$\begin{aligned}
d(x_n, x^*) &= d(x_n, Fx^*) \leq d(x_n, Fx_n) + d(Fx_n, Fx^*) = d(Fx_n, Fx^*) + d(x_n, Fx_n) \\
&\leq k \max \left\{ d(x_n, x^*), \frac{d(x_n, Fx_n) + d(x^*, Fx^*)}{2}, \frac{d(x_n, Fx^*) + d(x^*, Fx_n)}{2}, \right. \\
&\quad \left. \frac{d(x_n, Fx_n) d(x^*, Fx^*)}{1 + d(x_n, x^*)}, \frac{d(x^*, Fx_n) d(x_n, Fx^*)}{1 + d(x_n, x^*)} \right\} + d(x_n, Fx_n) \\
&\leq k \max \left\{ d(x_n, x^*), \frac{d(x_n, Fx_n)}{2}, \frac{d(x_n, x^*) + d(x^*, Fx_n)}{2}, 0, \frac{d(x^*, Fx_n) d(x_n, x^*)}{1 + d(x_n, x^*)} \right\} \\
&\quad + d(x_n, Fx_n) \\
&\leq k \max \left\{ d(x_n, x^*), \frac{d(x_n, Fx_n)}{2}, \frac{d(x_n, x^*) + d(x^*, x_n) + d(x_n, Fx_n)}{2}, \right. \\
&\quad \left. \frac{[d(x^*, x_n) + d(x_n, Fx_n)] d(x_n, x^*)}{1 + d(x_n, x^*)} \right\} + d(x_n, Fx_n) \\
&= k \max \left\{ d(x_n, x^*), \frac{d(x_n, Fx_n)}{2}, \frac{2 d(x_n, x^*) + d(x_n, Fx_n)}{2}, d(x^*, x_n) + d(x_n, Fx_n) \right\} \\
&\quad + d(x_n, Fx_n) \\
&\leq k \max \left\{ d(x_n, x^*), \frac{d(x_n, Fx_n)}{2}, d(x_n, x^*) + \frac{d(x_n, Fx_n)}{2}, d(x^*, x_n) + d(x_n, Fx_n) \right\} \\
&\quad + d(x_n, Fx_n) \\
&= k [d(x^*, x_n) + d(x_n, Fx_n)] + d(x_n, Fx_n),
\end{aligned}$$

which implies that

$$d(x^*, x_n) \leq \frac{(1+k)}{1-k} d(x_n, Fx_n).$$

Then it follows that $\lim_{n \rightarrow \infty} d(x_n, x^*) = 0$, that is, $x_n \rightarrow x^*$ as $n \rightarrow \infty$. Hence the fixed point problem (P) is well-posed.

As $x^* \in X$ is the unique fixed point of F and $\lim_{n \rightarrow \infty} d(x_n, x^*) = 0$, for any arbitrary sequence $\{x_n\}$ in X with $d(x_n, Fx_n) \rightarrow 0$ as $n \rightarrow \infty$, it follows that $\lim_{n \rightarrow \infty} d(x_n, F^n x^*) = \lim_{n \rightarrow \infty} d(x_n, x^*) = 0$. Hence, the problem (P) has limit shadowing property.

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References

1. Amiri, P., Rezapour, Sh., Shahzad, N., *Fixed points of generalized $\alpha - \psi$ -contractions*, RACSAM 108 (2), 519–526, (2014).
2. Berinde, V., *Error estimates for approximating fixed points of quasi contractions*, Gen. Math. 13(2), 23–34, (2005).

3. Carl, S., Heikkilä, S., *Fixed Point Theory in Ordered Sets and Applications*, Springer-Verlag New York, 2011, DOI 10.1007/978-1-4419-7585-0.
4. Chifu, C., Petruşel, G., *Coupled fixed point results for (φ, G) -contractions of type (b) in b -metric spaces endowed with a graph*, J. Nonlinear Sci. Appl. 10, 671–683, (2017).
5. Ciepliński, K., *Applications of fixed point theorems to the Hyers-Ulam stability of functional equations - a survey*, Ann. Funct. Anal. 3(1), 151–164, (2012).
6. Elqorachi, E., Rassias, T. M., *Generalized Hyers-Ulam stability of trigonometric functional equations*, Σ Mathematics 6, (2018), doi:10.3390/math6050083.
7. Hussain, N., Rafiq, A., Damjanović, B., Lazović, R., *On rate of convergence of various iterative schemes*, Fixed Point Theory Appl. 2011: 45, (2011).
8. Hussain, N., Karapinar, E., Salimi, P., Akbar, F., *α -admissible mappings and related fixed point theorems*, J. Inequal. Appl. 2013 : 114, (2013).
9. Hyers, D. H., *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA 27(4), 222–224, (1941).
10. Kutbi, M. A., Sintunavarat, W., *Ulam-Hyers stability and well-posedness of fixed point problems for $\alpha - \lambda$ -contraction mapping in metric spaces*, Abstr. Appl. Anal. 2014, Article ID 268230, 6 pages, (2014).
11. Lahiri, B. K., Das, P., *Well-posedness and porosity of a certain class of operators*, Demonstratio Math. XXXVIII(1), 169–176, (2005).
12. Murali, R., Antony Raj, A., Deboral, M., *Hyers-Ulam stability of the isometric Cauchy-Jenson mapping in generalized quasi-banach spaces*, Int. J. Adv. Appl. Math. Mech. 3(4), 16–21, (2016).
13. Phiangsungnoen, S., Kumam, P., *Generalized Ulam-Hyers stability and well-posedness for fixed point equation via α -admissibility*, J. Inequal. Appl. 2014: 418, (2014).
14. Phuengrattana, W., Suantai, S., *On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval*, J. Comput. Appl. Math. 235, 3006–3014, (2011).
15. Popa, V., *Well-posedness of fixed point problem in orbitally complete metric spaces*, Stud. Cercet. Stiint., Ser. Mat. 16, 209–214, (2006).
16. Rassias, T. M., *On the stability of the linear mappings in Banach spaces*, Proc. Amer. Math. Soc. 72, 297–300, (1978).
17. Rassias, T. M., *Isometries and approximate isometries*, IJMMS 25:2, 73–91, (2001).
18. Rus, I. A., *Remarks on Ulam stability of the operatorial equations*, Fixed Point Theory 10(2), 305–320, (2009).
19. Samet, B., Vetro, C., Vetro, P., *Fixed point theorem for $\alpha - \psi$ - contractive type mappings*, Nonlinear Anal. 75, 2154–2165, (2012).
20. Samet, B., *Fixed points for $\alpha - \psi$ - contractive mappings with an application to quadratic integral equations*, Electron. J. Differential Equations 2014 (No.152), 1–18, (2014).
21. Sen, M., Saha, D., Agarwal, R. P., *A Darbo fixed point theory approach towards the existence of a functional integral equation in a Banach algebra*, Appl. Math. Comput. 358, 111–118, (2019).
22. Sintunavarat, W., *Generalized Ulam-Hyers stability, well-posedness and limit shadowing of fixed point problems for $\alpha - \beta$ - contraction mapping in metric spaces*, The Scientific World Journal 2014, Article ID 569174, 7 pages, (2014).
23. Salimi, P., Latif, A., Hussain, N., *Modified $\alpha - \psi$ - contractive mappings with applications*, Fixed Point Theory Appl. 2013 : 151, (2013).
24. Ulam, S. M., *Problems in Modern Mathematics*, Wiley, New York (1964).
25. Yildirim, I., Abbas, M., *Convergence rate of implicit iteration process and a data dependence result*, Eur. J. Pure Appl. Math. 11 (1), 189–201, (2018).

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