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h-open sets in Topological Spaces

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ABSTRACT: In this paper, we introduce a new class of open sets in a topological space (X, τ) called h-open sets. Also, introduce and study topological properties of h-interior, h-closure, h-limit points, h-derived, h-interior points, h-border, h-frontier and h-exterior by using the concept of h-open sets. Moreover, introduce the notion of h-continuous functions, h-open functions, h-irresolute functions, h-totally continuous functions, h-contra-continuous functions, h-homeomorphism and investigate some properties of these functions and study some properties, remarks related to them.

Key Words: h-open sets, h-interior, h-closure, h-limit points, h-border, h-frontier, h-exterior, h-continuous functions, h-open functions, h-irresolute functions, h-homeomorphism, h-totally continuous functions, h-contra-continuous functions.

Contents

1. Introduction and Preliminaries

The concept of open sets is now well-known important notions in topology and its applications. For a subset A of a topological space (X, τ), the closure of A, the interior of A with respect to τ are denoted by Cl(A) and Int(A) respectively. The complement of A is denoted by A^c. A subset A of a topological space (X, τ) is said to be clopen set, if A is open and closed. This work consists of two sections. In section one, we will introduce and study a new class of open sets which is called h-open set and introduce the notions of h-interior, h-closure, h-limit points, h-derived, h-interior points, h-border, h-frontier and h-exterior by using the concept of h-open sets, and study their topological properties. In section two, we will present the notion of h-continuous functions, h-open functions, h-irresolute functions, h-totally continuous functions, h-contra-continuous functions, h-homeomorphism and investigate some properties of these functions and study some properties, remarks related to them.

2. h-open sets

In this section, we introduce a new class of open sets which is called h-open set and introduce the notions of h-interior, h-closure, h-limit points, h-derived, h-interior points, h-border, h-frontier and h-exterior by using the concept of h-open sets, and study their topological properties.

Definition 2.1. *A subset A of the topological space (X, τ) is called h-open set if for every non-empty set U in X, $U \neq X$ and $U \in \tau$, $A \subseteq Int(A \cup U)$. The complement of the h-open set is called h-closed. We denote the family of all h-open sets of a topological space (X, τ) by τ^h.*

Example 2.2. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$.
Then $\tau^h = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$.*

Example 2.3. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.
Then $\tau^h = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.*

Remark 2.4. *From Example.2.1, and Example.2.2. Note that $\tau \subseteq \tau^h$.*

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Theorem 2.5. *Every open set in any topological space (X, τ) is h-open set.*

Proof. Let (X, τ) be any topological space and let $A \subseteq X$ be any open set. Therefore, $A = Int(A) \subseteq Int(A \cup U)$, for every non-empty set $U \neq X$ and $U \in \tau$. Thus, A is h-open set. □

Remark 2.6. *The converse of the Theorem.2.1, need, not be true as shown in the following example.*

Example 2.7. *In Example.2.1, $\{b\}, \{c\}, \{b, c\}, \{b, c, d\}$ are h-open sets but not open sets.*

Theorem 2.8. *Let (X, τ) be a topological space and let A, B be two h-open sets. Then*

1. *$A \cap B$ is h-open set.*
2. *$A \cup B$ is h-open set.*

Proof. 1) Let A and B be two h-open sets. Then from Definition.2.1, $A \subseteq Int(A \cup U)$ and $B \subseteq Int(B \cup U)$, for every non-empty set $U \neq X$, $U \in \tau$. Then $A \cup B \subseteq Int(A \cup U) \cup Int(B \cup U) \subseteq Int((A \cup U) \cup (B \cup U)) = Int((A \cup B) \cup U)$. Therefore, $A \cup B$ is h-open set.
2) Let A and B be two h-open sets. Then from Definition.2.1, $A \subseteq Int(A \cup U)$ and $B \subseteq Int(B \cup U)$, for every non-empty set $U \neq X$, $U \in \tau$. Then $A \cap B \subseteq Int(A \cup U) \cap Int(B \cup U) = Int((A \cup U) \cap (B \cup U)) = Int(((A \cup U) \cap B) \cup ((A \cup U) \cap U)) \subseteq Int((A \cap B) \cup U)$. Therefore, $A \cap B$ is h-open set. □

Definition 2.9. *Let (X, τ) be a topological space and let $A \subseteq X$. The h-interior of A is defined as the union of all h-open sets in X and is denoted by $Int_h(A)$. It is clear that $Int_h(A)$ is h-open set, for any subset A of X.*

Proposition 2.10. *Let (X, τ) be a topological space and let $A \subseteq B \subseteq X$. Then*

1. *$Int_h(A) \subseteq Int_h(B)$.*
2. *$Int_h(A) \subseteq A$.*
3. *A is h-open if and only if $A = Int_h(A)$.*

Definition 2.11. *Let (X, τ) be a topological space and let $A \subseteq X$. The h-closure of A is defined as the intersection of all h-closed sets in X containing A, and is denoted by $Cl_h(A)$. It is clear that $Cl_h(A)$ is h-closed set for any subset A of X.*

Proposition 2.12. *Let (X, τ) be a topological space and let $A \subseteq B \subseteq X$. Then*

1. *$Cl_h(A) \subseteq Cl_h(B)$.*
2. *$A \subseteq Cl_h(A)$.*
3. *A is h-closed if and only if $A = Cl_h(A)$.*

Definition 2.13. *Let (X, τ) be a topological space and let $A \subseteq X$. A point $x \in X$ is said to be h-limit point of A if it satisfies the following assertion:*

$$(\forall G \in \tau^h)(x \in G \Rightarrow G \cap (A \backslash \{x\}) \neq \emptyset).$$

The set of all h-limit points of A is called the h-derived set of A and is denoted by $D_h(A)$.
Note that for a subset A of X, a point $x \in X$ is not a h-limit point of A if and only if there exists a h-open set G in X such that $x \in G$ and $G \cap (A \backslash \{x\}) = \emptyset$ or, equivalently, $x \in G$ and $G \cap A = \emptyset$ or $G \cap A = \{x\}$ or, equivalently, $x \in G$ and $G \cap A \subseteq \{x\}$.

Theorem 2.14. *Let (X, τ) be a topological space and let A be a subset of X. Then the following are equivalent*

1. $(\forall G \in \tau^h)(x \in G \Rightarrow A \cap G \neq \emptyset)$.

2. $x \in Cl_h(A)$.

Proof. (1) ⇒(2) If $x \notin Cl_h(A)$, then there exists a h-closed set F such that $A \subseteq F$and $x \notin F$. Hence $G = X - F$ is a h-open set such that $x \in G$ and $G \cap A = \emptyset$. This is a contradiction, and hence (2) is valid.

(2) ⇒(1) Straightforward. □

Theorem 2.15. *Let (X, τ) be a topological space and let $A \subseteq B \subseteq X$. Then*

1. $Cl_h(A) = A \cup D_h(A)$.
2. *A is h-closed if and only if* $D_h(A) \subseteq A$.
3. $D_h(A) \subseteq D_h(B)$.
4. $D_h(A) \subseteq D(A)$.
5. $Cl_h(A) \subseteq Cl(A)$.

Proof. 1) Let $x \notin Cl_h(A)$. Then there exists a h-closed set F in X such that $A \subseteq F$ and $x \notin F$. Hence $G = X - F$ is a h-open set such that $x \in G$ and $G \cap A = \emptyset$. Therefore $x \notin A$ and $x \notin D_h(A)$, then $x \notin A \cup D_h(A)$. Thus $A \cup D_h(A) \subseteq Cl_h(A)$. On the other hand, $x \notin A \cup D_h(A)$ implies that there exists a h-open set G in X such that $x \in G$ and $G \cap A = \emptyset$. Hence $F = X - G$ is a h-closed set in X such that $A \subseteq F$ and $x \notin F$. Hence $x \notin Cl_h(A)$. Thus $Cl_h(A) \subseteq A \cup D_h(A)$. Therefore $Cl_h(A) = A \cup D_h(A)$. For (2), (3), (4) and (5) the proof is easy. □

Example 2.16. *Let $X = \{a, b, c\}$ with topology, $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$. Then we have the followings*

1. $\tau \subseteq \tau^h = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$.
2. *If* $A = \{a, c\}$, *then* $D(A) = \{c\}$ *and* $D_h(A) = \emptyset$.
3. *If* $B = \{a, b\}$, *then* $D(B) = \{b, c\}$ *and* $D_h(B) = \{c\}$.

Theorem 2.17. *Let τ_1 and τ_2 be topologies on X such that $\tau_1^h \subseteq \tau_2^h$. For any subset A of X, every h-limit point of A with respect to τ_2 is a h-limit point of A with respect to τ_1.*

Proof. Let x be a h-limit point of A with respect to τ_2. Then $G \cap (A \backslash \{x\}) \neq \emptyset$ for every $G \in \tau_2^h$ such that $x \in G$. But $\tau_1^h \subseteq \tau_2^h$ so, in particular, $G \cap (A \backslash \{x\}) \neq \emptyset$ for every $G \in \tau_1^h$ such that $x \in G$. Hence x is a h-limit point of A with respect to τ_1. □

Remark 2.18. *The converse of the Theorem.2.5, need not be true as shown in the following example.*

Example 2.19. *$X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}\}$ and $\tau_2 = \{\emptyset, X, \{a\}, \{a, b\}$. Then $\tau_1^h = \{\emptyset, X, \{a\}, \{b, c\}$ and $\tau_2^h = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}$. Not that $\tau_1^h \subseteq \tau_2^h$ and b is a h-limit point of $A = \{a, b\}$ with respect to τ_1, but it is not a h-limit point of A with respect to τ_2.*

Theorem 2.20. *If τ is the indiscrete (resp. discrete) topology on a set X, then τ^h is indiscrete (resp. discrete) topology on X.*

Proof. Straightforward. □

Lemma 2.21. *If A is a subset of a discrete topological space (X, τ), then $D_h(A) = \emptyset$.*

Proof. Let $x \in X$. Recall that every subset of X is open, and so h-open. In particular, the singleton set $G = \{x\}$ is h-open. But $x \in G$ and $G \cap A = \{x\} \cap A \subseteq \{x\}$. Hence x is not a h-limit point of A, and so $D_h(A) = \emptyset$. □

Theorem 2.22. *Let (X, τ) be a topological space and let A, B subsets of X. If A is h-closed, then $Cl_h(A \cap B) \subseteq A \cap Cl_h(B)$.*

Proof. If A is h-closed, then $Cl_h(A) = A$ and so $Cl_h(A \cap B) \subset Cl_h(A) \cap Cl_h(B) \subseteq A \cap Cl_h(B)$. □

Lemma 2.23. *Let (X, τ) be a topological space and let A subset of X. Then A is h-open if and only if there exists an open set U in X such that $A \subseteq U \subseteq Cl(A)$.*

Proof. Straightforward. □

Lemma 2.24. *The intersection of an open set and h-open set is h-open set.*

Proof. Let A be an open set in X and B a h-open set in X. Then there exists an open set U in X such that $B \subseteq U \subseteq Cl(B)$. It follows that $A \cap B \subseteq A \cap U \subseteq A \cap Cl(B) \subseteq Cl(A \cap B)$. Now since $A \cap U$ is open, it follows from Lemma.2.1 that $A \cap B$ is h-open. □

Definition 2.25. *Let (X, τ) be a topological space and let $A \subseteq X$. Then $b_h(A) = A \backslash Int_h(A)$ is called the h-border of A, and the set $Fr_h(A) = Cl_h(A) \backslash Int_h(A)$ is called the h-frontier of A.*
Note that if A is a h-closed subset of X, then $b_h(A) = Fr_h(A)$.

Example 2.26. *Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, X, \{b\}, \{b, c\}\}$, $\tau^h = \{\emptyset, X, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$. If $A = \{a, b\}$, then $Int_h(A) = \{b\}$, $b_h(A) = \{a\}$ and so $Cl_h(A) = \{a, b\}$, $Fr_h(A) = \{a\}$. If we take $A = \{b, c\}$, then $Int_h(A) = \{b, c\}$, $b_h(A) = \emptyset$ and so $Cl_h(A) = X$, $Fr_h(A) = \{a\}$.*

Theorem 2.27. *Let (X, τ) be a topological space and let $A \subseteq X$. Then*

1. $A = Int_h(A) \cup b_h(A)$.
2. $Int_h(A) \cap b_h(A) = \emptyset$.
3. *A is a h-open set if and only if* $b_h(A) = \emptyset$.
4. $b_h(Int_h(A)) = \emptyset$.
5. $Int_h(b_h(A)) = \emptyset$.
6. $b_h(b_h(A)) = b_h(A)$.
7. $b_h(A) = A \cap Cl_h(X \backslash A)$.
8. $b_h(A) = A \cap D_h(X \backslash A)$.

Proof. (1) and (2). Straightforward.

(3) Since $Int_h(A) \subseteq A$, it follows from Proposition.2.1(3) that A is h-open $\Leftrightarrow A = Int_h(A) \Leftrightarrow b_h(A) = A \backslash Int_h(A) = \emptyset$.

(4) Since $Int_h(A)$ is h-open, it follows from (3) that $b_h(Int_h(A)) = \emptyset$.

(5) If $x \in Int_h(b_h(A))$, then $x \in b_h(A) \subseteq A$ and $x \in Int_h(A)$. Since $Int_h(b_h(A)) \subseteq Int_h(A)$. Thus $x \in b_h(A) \cap Int_h(A) = \emptyset$, which is a contradiction. Hence $Int_h(b_h(A)) = \emptyset$.

(6) Using (5), we get $b_h(b_h(A)) = b_h(A) \backslash Int_h(b_h(A)) = b_h(A)$.

(7) $b_h(A) = A \backslash Int_h(A) = A \backslash (X \backslash Cl_h(X \backslash A)) = A \cap Cl_h(X \backslash A)$.

(8) Applying (7) and Theorem.2.4 (1), we have $b_h(A) = A \cap Cl_h(X \backslash A) = A \cap ((X \backslash A) \cup D_h(X \backslash A)) = A \cap D_h(X \backslash A)$. □

Lemma 2.28. *Let (X,τ) be a topological space and let $A \subseteq X$. Then A a h-closed if and only if $Fr_h(A) \subseteq A$.*

Proof. Assume that A is h-closed. Then $Fr_h(A) = Cl_h(A)\backslash Int_h(A) = A\backslash Int_h(A) \subseteq A$. Conversely suppose that $Fr_h(A) \subseteq A$ Then $Cl_h(A)\backslash Int_h(A) \subseteq A$ and so $Cl_h(A) \subseteq A$ Since $Int_h(A) \subseteq A$. Noticing that $A \subseteq Cl_h(A)$, we have $A = Cl_h(A)$. □

Definition 2.29. *Let (X,τ) be a topological space and let $A \subseteq X$. Then $Ext_h(A) = Int_h(X\backslash A)$ is called the h-exterior of A.*

Example 2.30. *Let $X = \{a,b,c\}$ with topology $\tau = \{\emptyset, X, \{a\}, \{a,b\}\}$, $\tau^h = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{b,c\}\}$. If $A = \{a,c\}$, then we have $Ext_h(A) = \{b\}$.*

Theorem 2.31. *Let (X,τ) be a topological space and let $A \subseteq B \subseteq X$. Then*

1. *$Ext_h(A)$ is h-open.*
2. *$Ext_h(A) = X\backslash Cl_h(A)$.*
3. *If $A \subseteq B$, then $Ext_h(B) \subseteq Ext_h(A)$.*
4. *$Ext_h(A \cup B) \subseteq Ext_h(A) \cap Ext_h(B)$.*
5. *$Ext_h(A \cap B) \supseteq Ext_h(A) \cup Ext_h(B)$.*
6. *$Ext_h(X) = \emptyset, Ext_h(\emptyset) = X$.*
7. *$Ext_h(A) = Ext_h(X\backslash Ext_h(A))$.*
8. *$X = Int_h(A) \cup Ext_h(A) \cup Fr_h(A)$.*

Proof. (1) and (2) straightforward.

(3) Assume that $A \subseteq B$. Then $Ext_h(B) = Int_h(X\backslash B) \subseteq Int_h(X\backslash A) = Ext_h(A)$.

(4) $Ext_h(A \cup B) = Int_h(X\backslash(A \cup B)) = Int_h((X\backslash A) \cap (X\backslash B)) \subseteq Int_h(X\backslash A) \cap Int_h(X\backslash B) = Ext_h(A) \cap Ext_h(B)$.

(5) $Ext_h(A \cap B) = Int_h(X\backslash(A \cap B)) = Int_h((X\backslash A) \cup (X\backslash B)) \supseteq Int_h(X\backslash A) \cup Int_h(X\backslash B) = Ext_h(A) \cup Ext_h(B)$.

(6) Straightforward.

(7) $Ext_h(X\backslash Ext_h(A)) = Ext_h(X\backslash Int_h(X\backslash A)) = Int_h(X\backslash A) = Ext_h(A)$.

(8)Straightforward. □

Definition 2.32. *A function $f : (X,\tau) \longrightarrow (Y,\sigma)$ is said to be*

1. *totally-continuous if $f^{-1}(U)$ is clopen set in X, for every open set U in Y.*
2. *contra-continuous if $f^{-1}(U)$ is closed set in X, for every open set U in Y.*

3. h-continuous functions and h-homeomorphism

In this section, we introduce new classes of functions called h-continuous functions, h-open functions, h-irresolute functions, h-totally continuous functions, h-contra-continuous functions, h-homeomorphism and study some properties of these functions.

Definition 3.1. *A function $f:(X,\tau)\to(Y,\sigma)$ is said to be h-continuous, if $f^{-1}(U)$ is h-open set in X for every open set U in Y.*

Example 3.2. *Let $X = Y = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a,c\}\}$, $\tau^h = \{\emptyset, X, \{a\}, \{c\}, \{a,c\}\}$ and $\sigma = \{\emptyset, Y, \{a,c\}\}$. Clearly, the identity function $f:(X,\tau)\to(Y,\sigma)$ is h-continuous.*

Theorem 3.3. *Every continuous function is h-continuous.*

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ be continuous function and U be any open subset in Y. Since, f is continuous, then $f^{-1}(U)$ is open set in X. Since, every open set is h-open set by Theorem.2.1, then $f^{-1}(U)$ is h-open set in X. Therefore, f is h-continuous. □

Remark 3.4. *The converse of the Theorem 3.1, need, not be true as shown in the following example.*

Example 3.5. *Let $X = \{a,b,c\}$ and $Y = \{1,2,3\}$, $\tau = \{\emptyset, X, \{b\}\}$, $\tau^h = \{\emptyset, X, \{b\}, \{a,c\}\}$, $\sigma = \{\emptyset, Y, \{1\}, \{2,3\}\}$. A function $f:(X,\tau)\to(Y,\sigma)$ is defined by $f(\{a\}) = \{2\}, f(\{b\}) = \{1\}, f(\{c\}) = \{3\}$. Clearly, f is a h-continuous, but f is not continuous.*

Theorem 3.6. *If $f:(X,\tau)\to(Y,\sigma)$ is h-continuous and $g:(Y,\sigma)\to(Z,\eta)$ is continuous, then $g\circ f:(X,\tau)\to(Z,\eta)$ is h-continuous.*

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ be h-continuous and $g:(Y,\sigma)\to(Z,\eta)$ be continuous . Let U be an open set in Z. Since, g is continuous, then $g^{-1}(U)$ is an open set in Y. Since, f is h-continuous, then $f^{-1}((g^{-1}(U)) = (g\circ f)^{-1}(U)$ is h-open set in X. Therefore, $g\circ f:(X,\tau)\to(Z,\eta)$ is h-continuous. □

Definition 3.7. *A function $f:(X,\tau)\to(Y,\sigma)$ is said to be h-open, if $f(U)$ is h-open set in Y for every open set U in X.*

Example 3.8. *Let $X = Y = \{a,b,c\}$, $\tau = \{\emptyset, X, \{b,c\}\}$, $\sigma = \{\emptyset, Y, \{a\}\}$ and $\sigma^h = \{\emptyset, Y, \{a\}, \{b,c\}\}$. Clearly, the identity function $f:(X,\tau)\to(Y,\sigma)$ is h-open.*

Theorem 3.9. *Every open function is h-open.*

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ be open function and U be any open set in X. Since, f is open, then $f(U)$ is open set in Y. Since, every open set is h-open set by Theorem 2.1, then $f(U)$ is h-open set in Y. Therefore, f is h-open. □

Remark 3.10. *The converse of the Theorem 3.3, need not be true as shown in the following example.*

Example 3.11. *In Example 3.3, the identity function $f:(X,\tau)\to(Y,\sigma)$ is h-open but not open.*

Theorem 3.12. *If $f:(X,\tau)\to(Y,\sigma)$ is open and $g:(Y,\sigma)\to(Z,\eta)$ is h-open, then $g\circ f:(X,\tau)\to(Z,\eta)$ is h-open.*

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ be open and $g:(Y,\sigma)\to(Z,\eta)$ is a h-open. Let U be an open set in X. Since, f is an open, then $f(U)$ is an open set in Y. Since, g is a h-open, then $(g\circ f)(U) = g(f(U))$ is a h-open set in Z. Therefore, $g\circ f:(X,\tau)\to(Z,\eta)$ is h-open. □

Definition 3.13. *A function $f:(X,\tau)\to(Y,\sigma)$ is said to be h-irresolute, if $f^{-1}(U)$ is h-open set in X for every h-open set U in Y.*

Example 3.14. *Let $X = Y = \{a,b,c\}, \tau = \{\emptyset, X, \{b\}, \{b,c\}\}$, $\tau^h = \{\emptyset, X, \{b\}, \{c\}, \{a,c\}, \{b,c\}\}$, $\sigma = \{\emptyset, Y, \{b\}\}$ and $\sigma^h = \{\emptyset, Y, \{b\}, \{a,c\}\}$. Clearly, the identity function $f:(X,\tau)\to(Y,\sigma)$ is h-irresolute.*

Theorem 3.15. *Every continuous function is h-irresolute.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a continuous function and U be any h-open set in Y. Since, f is a continuous, then Then $f^{-1}(U)$ is open set in X. Hence, h-open set in X by Theorem 2.1. Therefore, f is h-irresolute. □

Remark 3.16. *The converse of the Theorem 3.5, need not be true as shown in the following example.*

Example 3.17. *Let* $X = Y = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{a, c\}\}$, $\tau^h = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$, $\sigma = \{\emptyset, Y, \{a\}, \{c\}, \{a, c\}\}$ *and* $\sigma^h = \{\emptyset, Y, \{a\}, \{c\}, \{a, c\}\}$. *Clearly, the identity function* $f : (X, \tau) \to (Y, \sigma)$ *is h-irresolute, but* f *is not continuous function.*

Theorem 3.18. *Every h-irresolute function is h-continuous.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be h-irresolute function and U be any open set in Y. Since, every open set is h-open set by Theorem 2.1. Since, f is h-irresolute, then $f^{-1}(U)$ is h-open set in X. Therefore f is h-continuous. □

Remark 3.19. *The converse of the Theorem 3.6, need not be true as shown in the following example.*

Example 3.20. *Let* $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$, $\tau^h = \{\emptyset, X, \{a\}, \{b, c\}\}$, $\sigma = \{\emptyset, Y, \{b, c\}\}$ *and* $\sigma^h = \{\emptyset, Y, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. *Clearly, the identity function* $f : (X, \tau) \to (Y, \sigma)$ *is h-continuous, but* f *is not h-irresolute.*

Theorem 3.21. *The composition of two h-irresolute function is also h-irresolute.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be any two h-irresolute. Let U be any h-open in Z. Since, g is h-irresolute, then $g^{-1}(U)$ is h-open set in Y. Since, f is h-irresolute, then $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is h-open in X. Therefore, $g \circ f : (X, \tau) \to (Z, \eta)$ is h-irresolute. □

Theorem 3.22. *If* $f : (X, \tau) \to (Y, \sigma)$ *is h-irresolute and* $g : (Y, \sigma) \to (Z, \eta)$ *is h-continuous, then* $gof : (X, \tau) \to (Z, \eta)$ *is h-irresolute.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ is h-irresolute and $g : (Y, \sigma) \to (Z, \eta)$ is h-continuous. Let $U \subset Z$. Since, g is h-continuous and f is h-irresolute, then $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is h-open in X. Therefore, $g \circ f : (X, \tau) \to (Z, \eta)$ is h-irresolute. □

Definition 3.23. *A bijective function* $f : (X, \tau) \to (Y, \sigma)$ *is said to be h-homeomorphism if f is h-continuous and h-open function.*

Theorem 3.24. *If* $f : (X, \tau) \to (Y, \sigma)$ *is homomorphism, then* f *is h-homomorphism.*

Proof. Since, every continuous function is h-continuous by Theorem 3.1. Also, since every open function is h-open by Theorem 3.3. Further, since f is bijective. Therefore, f is h-homomorphism. □

Remark 3.25. *The converse of the Theorem 3.9, need not be true as shown in the following example.*

Example 3.26. *Let* $X = Y = \{a, b, c\}, \tau = \{\emptyset, X, \{a, c\}\}, \tau^h = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}, \sigma = \{\emptyset, Y, \{b, c\}\}$ *and* $\sigma^h = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. *Clearly, the identity function* $f : (X, \tau) \to (Y, \sigma)$ *is h-homomorphism, but it is not homomorphism.*

Definition 3.27. *A function* $f : (X, \tau) \to (Y, \sigma)$ *is said to be h-totally continuous, if* $f^{-1}(U)$ *is clopen set in X for every h-open set U in Y.*

Example 3.28. *Let* $X = Y = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{b, c\}\}, \sigma = \{\emptyset, Y, \{a\}\}$ *and* $\sigma^h = \{\emptyset, Y, \{a\}, \{b, c\}\}$. *Clearly, the identity function* $f : (X, \tau) \to (Y, \sigma)$ *is h-totally continuous function.*

Theorem 3.29. *Every h-totally continuous function is totally continuous.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be h-totally continuous and U be any open set in Y. Since, every open set is h-open set by Theorem 2.1, then U is h-open set in Y. Since, f is h-totally continuous function, then $f^{-1}(U)$ is clopen set in X. Therefore, f is totally continuous. □

Remark 3.30. *The converse of the Theorem 3.10, need not be true as shown in the following example.*

Example 3.31. *Let* $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$, $\sigma = \{\emptyset, Y, \{b, c\}\}$ *and* $\sigma^h = \{\emptyset, Y, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. *Clearly, the identity function* $f : (X, \tau) \to (Y, \sigma)$ *is totally continuous function but it is not h-totally continuous.*

Theorem 3.32. *Every h-totally continuous function is h-irresolute.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be h-totally continuous and U be h-open set in Y. Since, f is h-totally continuous function, then $f^{-1}(U)$ is clopen set in X, which implies $f^{-1}(U)$ open, it follow $f^{-1}(U)$ is h-open set in X. Therefore, f is h-irresolute. □

Remark 3.33. *The converse of the Theorem 3.11, need not be true as shown in the following example.*

Example 3.34. *In Example 3.5, the identity function* $f : (X, \tau) \to (Y, \sigma)$ *is h-irresolute but not h-totally continuous.*

Theorem 3.35. *The composition of two h-totally continuous function is also h-totally continuous.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be any two h-totally continuous. Let U be any h-open in Z. Since, g is h-totally continuous, then $g^{-1}(U)$ is clopen set in Y, which implies $f^{-1}(U)$ open set, it follow $f^{-1}(U)$ is h-open set. Since, f is h-totally continuous, then $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is clopen in X. Therefore, $g \circ f : (X, \tau) \to (Z, \eta)$ is h-totally continuous. □

Theorem 3.36. *If* $f : (X, \tau) \to (Y, \sigma)$ *be h-totally continuous and* $g : (Y, \sigma) \to (Z, \eta)$ *be h-irresolute, then* $g \circ f : (X, \tau) \to (Z, \eta)$ *is h-totally continuous.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be h-totally continuous and $g : (Y, \sigma) \to (Z, \eta)$ be h-irresolute. Let U be h-open set in Z. Since, g is h-irresolute, then $g^{-1}(U)$ is h-open set in Y. Since, f is h-totally continuous, then $f^{-1}((g^{-1}(U)) = (g \circ f)^{-1}(U)$ is clopen set in X. Therefore, $g \circ f : (X, \tau) \to (Z, \eta)$ is h-totally continuous. □

Theorem 3.37. *If* $f : (X, \tau) \to (Y, \sigma)$ *is h-totally continuous and* $g : (Y, \sigma) \to (Z, \eta)$ *is h-continuous, then* $g \circ f : (X, \tau) \to (Z, \eta)$ *is totally continuous.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be h-totally continuous and $g : (Y, \sigma) \to (Z, \eta)$ is h-continuous. Let U be open set in Z. Since, g is h-continuous, then $g^{-1}(U)$ is h-open set in Y. Since, f is h-totally continuous, then $f^{-1}((g^{-1}(U)) = (g \circ f)^{-1}(U)$ is clopen set in X. Therefore, $g \circ f : (X, \tau) \to (Z, \eta)$ is totally continuous. □

Definition 3.38. *A function* $f : (X, \tau) \to (Y, \sigma)$ *is said to be h-contra-continuous if* $f^{-1}(U)$ *is h-closed set in X for every open set U in Y.*

Example 3.39. *Let* $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$, $\sigma = \{\emptyset, Y, \{a\}\}$ *and* $\tau^h = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. *Clearly, the identity function* $f : (X, \tau) \to (Y, \sigma)$ *is a h-contra-continuous.*

Theorem 3.40. *Every contra-continuous function is h-contra-continuous.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be contra-continuous function and U any open set in Y. Since, f is contra-continuous, then $f^{-1}(U)$ is closed sets in X. Since, every closed set is h-closed set, then $f^{-1}(U)$ is h-closed set in X. Therefore, f is h-contra-continuous. □

Remark 3.41. *The converse of the Theorem 3.15, need not be true as shown in the following example.*

Example 3.42. *In Example 3.12, the identity function $f : (X, \tau) \to (Y, \sigma)$ is h-contra-continuous but not contra-continuous.*

Theorem 3.43. *Every totally continuous function is h-contra-continuous.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be totally continuous and U be any open set in Y. Since, f is totally continuous function, then $f^{-1}(U)$ is clopen set in X, and hence closed, it follows h-closed set. Therefore, f is h-contra-continuous. □

Remark 3.44. *The converse of the Theorem 3.16, need not be true as shown in the following example.*

Example 3.45. *In Example 3.12, the identity function $f : (X, \tau) \to (Y, \sigma)$ is h-contra-continuous but not totally continuous.*

References

[1] E. Ekici and M. Caldas, Slightly continuous functions, Bol. Soc. Para Mat., 22(2004), 63 - 74.

[2] J. Donchev, Contra continuous functions and strongly S-closed spaces, Internat. J. Math. Math. Sci., 19 (1996), 303-310.

[3] N. Biswas, Some mappings in topological spaces, Bull. Cal. Math. Soc., 61 (1969), 127-133.

[4] O. Njastad, On some classes of nearly open sets, Pacific J. Math., 15(1965), 961-970.

[5] R. L. Ellis, A non-Archimedean analogue of the Tietze-Urysohn extension theorem, Nederl. Akad. Wetensch. Proc. Ser.A, 70 (1967), 332-333.

[6] R. Staum, The algebra of bounded continuous functions into a nonarchimedean field, Pacific J. Math., 50 (1974), 169-185.

[7] S. N. Maheshwari and R. Prasad, On s-normal spaces, Bull. Math. Soc. Sci. Math. R. S. Roumanie (N. S.), 22(1978), 27-29.

[8] J. Tong, On decomposition of continuity in topological spaces, Acta Math. Hungar. 54, no. 12(1989), 51-55.

[9] T. Noiri, Super continuity and some strong forms of continuity, Indian J. Pure Appl. Math., 15(1984), 241-250.

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