Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 ON LINE
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–10.
ISSN-0037-8712 IN PRESS
doi:10.5269/bspm.51025

A New Version of Energy and Elastica for Curves with Extended Darboux Frame

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ABSTRACT: In this work, we research geometrical interpretation involved with the energy by ED-frame field of first and second kind on an orientable hypersurface in E^4. We explore the geometric properties of some graphics by way of energy. We apply totally diverse discussion and approach to illustrate bending energy functional for ED-frame field of first and second kind. Moreover, we have an original and satisfactorily association among energy of the curve on orientable hypersurface in E^4.

Key Words: Energy, ED-frame field, hypersurface, vector fields, bending energy.

Contents

1. Introduction

Materials having the feature of deformable structure such as cloth, flexible metals, rubber, paper are the main subject and examples for the elasticity theory. However, elastica can be considered from a different perspective that enlightens broad range of physical and mathematical studies. Studies concerned about the elastica firstly focus on the research of mechanical equilibrium, the study of variational problems, and the solution of the elliptic integral, [1]- [7].

Related principles designed for energy in curvature centered energy is regarded to be at its first phases in development. A few of the legendary fields and landmark research for this theory may be discovered in mathematical physics, membrane chemistry, computer aided geometric design and geometric modeling, shell engineering, biology and thin plate [8]- [9]. One of the well-known functional and related works is bending energy functional, which appeared firstly Bernoulli-Euler elastica formulation for energy.

We organize the manuscript by starting to state fundamental definitions and proposition for ED-frame fields and energy. Then we recall interpretation of geometrical meaning of the energy for unit vector fields. Based on these relations we compute the energy of curves defined on an orientable hypersurface in E^4. Finally, we give examples about particle's energy for different cases by computing their values and drawing their graphs.

2. ED-frame fields

By way of design and style, this is model to kind of a moving frame with regards to a particle. In the quick stages of regular differential geometry, the Frenet-Serret frame was applied to create a curve in location. After that, Frenet-Serret frame is established by way of subsequent equations for a presented framework [10]- [11],

$$\begin{bmatrix} \nabla_{\mathbf{t}}\mathbf{t} \\ \nabla_{\mathbf{t}}\mathbf{n} \\ \nabla_{\mathbf{t}}\mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix},$$

2010 *Mathematics Subject Classification:* 53C41, 53A10.
Submitted November 18, 2019. Published May 02, 2021

where $\kappa = \|\mathbf{t}\|$ and τ are the curvature and torsion of γ, respectively.

Let M be an orientable hypersurface oriented by the unit normal vector field $\mathbf{N}$ in E^4 and β be a Frenet curve of class C^n (n≥4) with arc-length parameter s lying on M. We denote the unit tangent vector field of the curve by T, and denote the hypersurface unit normal vector field restricted to the curve by $\mathbf{N}$, i.e.

$$\mathbf{T}(s) = \beta'(s), \ \mathbf{N}(s) = \mathbf{N}(\beta(s)).$$

We can construct the extended Darboux frame field along the Frenet curve as follows [12]:

Case 1. If the set $\{\mathbf{N}, \mathbf{T}, \beta''\}$ is linearly independent, then using the Gram-Schmidt orthonormalization method gives the orthonormal set $\{\mathbf{N}, \mathbf{T}, \mathbf{E}\}$, where

$$E = \frac{\beta'' - <\beta'', \mathbf{N}> \mathbf{N}}{\left\|\beta'' - <\beta'', \mathbf{N}> \mathbf{N}\right\|}.$$

Case 2. If the set $\{\mathbf{N}, \mathbf{T}, \beta''\}$ is linearly dependent, i.e. if β'' is in the direction of the normal vector $\mathbf{N}$, applying the Gram-Schmidt orthonormalization method to $\{\mathbf{N}, \mathbf{T}, \beta'''\}$ yields the orthonormal set $\{\mathbf{N}, \mathbf{T}, \mathbf{E}\}$, where

$$\mathbf{E} = \frac{\beta''' - <\beta''', \mathbf{N}> \mathbf{N} - <\beta''', \mathbf{T}> \mathbf{T}}{\left\|\beta''' - <\beta''', \mathbf{N}> \mathbf{N} - <\beta''', \mathbf{T}> \mathbf{T}\right\|}.$$

In each case, if we define $\mathbf{D} = \mathbf{N} \otimes \mathbf{T} \otimes \mathbf{E}$, we have four unit vector fields $\mathbf{T}, \mathbf{E}, \mathbf{D}$, and $\mathbf{N}$, which are mutually orthogonal at each point of β. Thus, we have a new orthonormal frame field $\{\mathbf{T}, \mathbf{E}, \mathbf{D}, \mathbf{N}\}$ along the curve β instead of its Frenet frame field. It is obvious that $\mathbf{E}$(s) and $\mathbf{D}$(s) are also tangent to the hypersurface M for all s. Thus, the set $\{\mathbf{T}, \mathbf{E}, \mathbf{D}\}$ spans the tangent hyperplane of the hypersurface at the point β(s) . We call these new frame fields "extended Darboux frame field of first kind" or in short "ED-frame field of first kind" in case 1, and "extended Darboux frame field of second kind" or in short "ED-frame field of second kind" in case 2, respectively.

Therefore, we obtain the differential equations of ED-frame fields:

Case 1:

$$\begin{aligned}
\mathbf{T}' &= \kappa_g \mathbf{E} + \kappa_n \mathbf{N} \\
\mathbf{E}' &= -\kappa_g \mathbf{T} + \tilde{\kappa}_g \mathbf{D} + \tau_g \mathbf{N} \\
\mathbf{D}' &= -\tilde{\kappa}_g \mathbf{E} + \tilde{\tau}_g \mathbf{N} \\
\mathbf{N}' &= -\kappa_n \mathbf{T} - \tau_g \mathbf{E} - \tilde{\tau}_g \mathbf{D}.
\end{aligned}$$

Case 2:

$$\begin{aligned}
\mathbf{T}' &= \kappa_n \mathbf{N} \\
\mathbf{E}' &= \tilde{\kappa}_g \mathbf{D} + \tau_g \mathbf{N} \\
\mathbf{D}' &= -\tilde{\kappa}_g \mathbf{E} \\
\mathbf{N}' &= -\kappa_n \mathbf{T} - \tau_g \mathbf{E}.
\end{aligned}$$

3. Energy with Sasaki metric

For two Riemannian manifolds (M, ρ) and $\left(N, \tilde{h}\right)$ the energy of a differentiable map $f : (M, \rho) \to \left(N, \tilde{h}\right)$ can be defined as

$$\varepsilon nergy\,(f) = \frac{1}{2} \int_M \sum_{a=1}^{n} \tilde{h}\left(df\left(e_a\right), df\left(e_a\right)\right) v,$$

where $\{e_a\}$ is a local basis of the tangent space and v is the canonical volume form in M [13]-[14]. Let T^1M be the unit tangent bundle endowed with the restriction of the Sasaki metric on TM. Then the energy of a unit vector field X is defined to be the section's energy of $X : M \rightarrow T^1M$. For the bundle projection $\omega : T^1M \rightarrow M$, vertical/horizontal splitting induced by the Levi-Civita connection can be stated as $T\left(T^1M\right) = \mathcal{V} \oplus \mathcal{H}$. Further, we write $TM = \mathcal{F} \oplus \mathcal{G}$ where $\mathcal{F}$ shows the line bundle generated by X and $\mathcal{G}$ is the orthogonal complement [13].

Let $Q : T\left(T^1M\right) \rightarrow T^1M$ be the connection map. Then the following two conditions hold:
i) $\omega \circ Q = \omega \circ d\omega$ and $\omega \circ Q = \omega \circ \tilde{\omega}$, where $\tilde{\omega} : T\left(T^1M\right) \rightarrow T^1M$ is the tangent bundle projection;
ii) for $\varrho \in T_xM$ and a section$\xi : M \rightarrow T^1M$; we have

$$Q\left(d\xi\left(\varrho\right)\right) = \nabla_{\varrho}\xi,$$

where ∇ is the Levi-Civita covariant derivative [13].

Also, for $\varsigma_1, \varsigma_2 \in T_{\xi}\left(T^1M\right)$, we define

$$\rho_S\left(\varsigma_1, \varsigma_2\right) = \rho\left(d\omega\left(\varsigma_1\right), d\omega\left(\varsigma_2\right)\right) + \rho\left(Q\left(\varsigma_1\right), Q\left(\varsigma_2\right)\right).$$

This yields a Riemannian metric on TM: As we know ρ_S is called the Sasaki metric that also makes the projection $\omega : T^1M \rightarrow M$ a Riemannian submersion.

4. Results and discussion

In the theory of relativity, all the energy moving through an object contributes to the body's total mass that measures how much it can resist to acceleration. Each kinetic and potential energy makes a highly proportional contribution to the mass. In this study not only we compute the energy of surface curves but we also investigate its close correlation with bending energy of elastica which is a variational problem proposed firstly by Daniel Bernoulli to Leonard Euler in 1744. Euler elastica bending energy formula for a space curve in the 3-dimensional Frenet curvature along the curve is known as

$$H_B = \frac{1}{2}\int_{\beta}\left|\nabla_{\mathbf{T}}\mathbf{T}\right| ds.$$

♠ **Case 1:** *Energy of ED-frame field of first kind with Sasaki metric are given by*

$$\begin{aligned}
\varepsilon nergy\left(\mathbf{T}\right) &= \frac{1}{2}\int_{\beta}(1+\kappa_g^2+\kappa_n^2)ds,\\
\varepsilon nergy\left(\mathbf{E}\right) &= \frac{1}{2}\int_{\beta}(1+\kappa_g^2+\tilde{\kappa}_g^2+\tau_g^2)ds,\\
\varepsilon nergy\left(\mathbf{D}\right) &= \frac{1}{2}\int_{\beta}(1+\tilde{\kappa}_g^2+\tilde{\tau}_g^2)ds,\\
\varepsilon nergy\left(\mathbf{N}\right) &= \frac{1}{2}\int_{\beta}(1+\kappa_n^2+\tau_g^2+\tilde{\tau}_g^2)ds.
\end{aligned}$$

Putting

$$\mathbf{X} = \phi_1\mathbf{T}+\phi_2\mathbf{E}+\phi_3\mathbf{D}+\phi_4\mathbf{N}.$$

From ED-frame field of first kind, we obtain

$$\begin{aligned}
\nabla_{\mathbf{T}}\mathbf{X} &= (\phi_1' - \phi_2\kappa_g - \phi_4\kappa_n)\mathbf{T} + (\phi_2' + \phi_1\kappa_g - \tilde{\kappa}_g\phi_3 - \phi_4\tau_g)\mathbf{E}\\
&\quad +(\phi_3' + \phi_2\tilde{\kappa}_g - \phi_4\tilde{\tau}_g)\mathbf{D} + (\phi_4' + \phi_1\kappa_n + \phi_2\tau_g + \phi_3\tilde{\tau}_g)\mathbf{N}.
\end{aligned}$$

Since

$$\begin{aligned}
\varepsilon nergy\left(\mathbf{X}\right) &= \frac{1}{2}\int_{\beta}(1+(\phi_1' - \phi_2\kappa_g - \phi_4\kappa_n)^2+(\phi_2' + \phi_1\kappa_g - \tilde{\kappa}_g\phi_3 - \phi_4\tau_g)^2\\
&\quad +(\phi_3' + \phi_2\tilde{\kappa}_g - \phi_4\tilde{\tau}_g)^2+(\phi_4' + \phi_1\kappa_n + \phi_2\tau_g + \phi_3\tilde{\tau}_g)^2)ds.
\end{aligned}$$

Regularly, we have

$$d(\omega)\circ d(\mathbf{e}^{\mu}_{(0)})=d(\omega\circ\mathbf{e}^{\mu}_{(0)})=d(id_C)=id_{TC}.$$

Moreoever it is clear that

$$Q(\mathbf{T}(\mathbf{X}))=\triangledown_{\mathbf{T}}\mathbf{X}.$$

Since

$$\begin{aligned}Q(\mathbf{T}(\mathbf{X})) &= (\phi_1'-\phi_2\kappa_g-\phi_4\kappa_n)\mathbf{T}+(\phi_2'+\phi_1\kappa_g-\tilde{\kappa}_g\phi_3-\phi_4\tau_g)\mathbf{E}\\ &\quad+(\phi_3'+\phi_2\tilde{\kappa}_g-\phi_4\tilde{\tau}_g)\mathbf{D}+(\phi_4'+\phi_1\kappa_n+\phi_2\tau_g+\phi_3\tilde{\tau}_g)\mathbf{N}.\end{aligned}$$

Thus, we find from Sasaki metric

$$\begin{aligned}\rho_S\left(d\mathbf{T}(\mathbf{X}),d\mathbf{T}(\mathbf{X})\right) &= \rho\left(\mathbf{T},\mathbf{T}\right)+\rho\left(\nabla_{\mathbf{T}}\mathbf{X},\nabla_{\mathbf{T}}\mathbf{X}\right)\\ &= 1+(\phi_1'-\phi_2\kappa_g-\phi_4\kappa_n)^2+(\phi_2'+\phi_1\kappa_g-\tilde{\kappa}_g\phi_3-\phi_4\tau_g)^2\\ &\quad+(\phi_3'+\phi_2\tilde{\kappa}_g-\phi_4\tilde{\tau}_g)^2+(\phi_4'+\phi_1\kappa_n+\phi_2\tau_g+\phi_3\tilde{\tau}_g)^2.\end{aligned}$$

From elastica, we easily have

$$\varepsilon nergy\left(\mathbf{T}\right)-H_B=\frac{1}{2}s.$$

By using derivative formula, we get

$$\begin{aligned}\nabla^2_{\mathbf{T}}\mathbf{X} &= ((\phi_1'-\phi_2\kappa_g-\phi_4\kappa_n)'-\kappa_g(\phi_2'+\phi_1\kappa_g-\tilde{\kappa}_g\phi_3-\phi_4\tau_g)\\ &\quad-\kappa_n(\phi_4'+\phi_1\kappa_n+\phi_2\tau_g+\phi_3\tilde{\tau}_g))\mathbf{T}+((\phi_2'+\phi_1\kappa_g-\tilde{\kappa}_g\phi_3-\phi_4\tau_g)'\\ &\quad+(\phi_1'-\phi_2\kappa_g-\phi_4\kappa_n)\kappa_g-\tilde{\kappa}_g(\phi_3'+\phi_2\tilde{\kappa}_g-\phi_4\tilde{\tau}_g)-\tau_g(\phi_4'+\phi_1\kappa_n\\ &\quad+\phi_2\tau_g+\phi_3\tilde{\tau}_g))\mathbf{E}+((\phi_3'+\phi_2\tilde{\kappa}_g-\phi_4\tilde{\tau}_g)'+\tilde{\kappa}_g(\phi_2'+\phi_1\kappa_g-\tilde{\kappa}_g\phi_3-\phi_4\tau_g)\\ &\quad-\tilde{\tau}_g(\phi_4'+\phi_1\kappa_n+\phi_2\tau_g+\phi_3\tilde{\tau}_g))\mathbf{D}+((\phi_4'+\phi_1\kappa_n+\phi_2\tau_g+\phi_3\tilde{\tau}_g)'\\ &\quad+\kappa_n(\phi_1'-\phi_2\kappa_g-\phi_4\kappa_n)+\tau_g(\phi_2'+\phi_1\kappa_g-\tilde{\kappa}_g\phi_3-\phi_4\tau_g)+\tilde{\tau}_g(\phi_3'+\phi_2\tilde{\kappa}_g-\phi_4\tilde{\tau}_g))\mathbf{N}.\end{aligned}$$

Using above equation we easily have following condition.

$\circledast$ **X** *have fixed energy iff*

$$\begin{gathered}((\phi_1'-\phi_2\kappa_g-\phi_4\kappa_n)'-\kappa_g(\phi_2'+\phi_1\kappa_g-\tilde{\kappa}_g\phi_3-\phi_4\tau_g)-\kappa_n(\phi_4'+\phi_1\kappa_n+\phi_2\tau_g\\ +\phi_3\tilde{\tau}_g))(\phi_1'-\phi_2\kappa_g-\phi_4\kappa_n)+((\phi_2'+\phi_1\kappa_g-\tilde{\kappa}_g\phi_3-\phi_4\tau_g)'+(\phi_1'-\phi_2\kappa_g\\ -\phi_4\kappa_n)\kappa_g-\tilde{\kappa}_g(\phi_3'+\phi_2\tilde{\kappa}_g-\phi_4\tilde{\tau}_g)-\tau_g(\phi_4'+\phi_1\kappa_n+\phi_2\tau_g+\phi_3\tilde{\tau}_g))(\phi_2'\\ +\phi_1\kappa_g-\tilde{\kappa}_g\phi_3-\phi_4\tau_g)+((\phi_3'+\phi_2\tilde{\kappa}_g-\phi_4\tilde{\tau}_g)'+\tilde{\kappa}_g(\phi_2'+\phi_1\kappa_g-\tilde{\kappa}_g\phi_3-\phi_4\tau_g)\\ -\tilde{\tau}_g(\phi_4'+\phi_1\kappa_n+\phi_2\tau_g+\phi_3\tilde{\tau}_g))(\phi_3'+\phi_2\tilde{\kappa}_g-\phi_4\tilde{\tau}_g)+((\phi_4'+\phi_1\kappa_n+\phi_2\tau_g+\phi_3\tilde{\tau}_g)'\\ +\kappa_n(\phi_1'-\phi_2\kappa_g-\phi_4\kappa_n)+\tau_g(\phi_2'+\phi_1\kappa_g-\tilde{\kappa}_g\phi_3-\phi_4\tau_g)+\tilde{\tau}_g(\phi_3'+\phi_2\tilde{\kappa}_g\\ -\phi_4\tilde{\tau}_g))(\phi_4'+\phi_1\kappa_n+\phi_2\tau_g+\phi_3\tilde{\tau}_g)=0.\end{gathered}$$

With respect to ED-frame field of first kind, angle of Frenet vectors can be respectively given by

$$\begin{aligned}\mathcal{A}\left(\mathbf{T}\right) &= \int_0^s(\kappa_g^2+\kappa_n^2)^{\frac{1}{2}}ds,\\ \mathcal{A}\left(\mathbf{E}\right) &= \int_0^s(\kappa_g^2+\tilde{\kappa}_g^2+\tau_g^2)^{\frac{1}{2}}ds,\\ \mathcal{A}\left(\mathbf{D}\right) &= \int_0^s(\tilde{\kappa}_g^2+\tilde{\tau}_g^2)^{\frac{1}{2}}ds,\\ \mathcal{A}\left(\mathbf{N}\right) &= \int_0^s(\kappa_n^2+\tau_g^2+\tilde{\tau}_g^2)^{\frac{1}{2}}ds.\end{aligned}$$

Generalization angle of any field $\mathbf{X}$ is given by

$$\begin{aligned}\mathcal{A}(\mathbf{X}) &= \int_0^s ((\phi_1' - \phi_2\kappa_g - \phi_4\kappa_n)^2 + (\phi_2' + \phi_1\kappa_g - \tilde{\kappa}_g\phi_3 - \phi_4\tau_g)^2 \\ &\quad + (\phi_3' + \phi_2\tilde{\kappa}_g - \phi_4\tilde{\tau}_g)^2 + (\phi_4' + \phi_1\kappa_n + \phi_2\tau_g + \phi_3\tilde{\tau}_g)^2)^{\frac{1}{2}} ds.\end{aligned}$$

If we assume that pseudo angle of each of Frenet vectors given above are fixed, then we have following relations, respectively:

$$\begin{aligned}\phi_1' - \phi_2\kappa_g - \phi_4\kappa_n &= 0, \\ \phi_2' + \phi_1\kappa_g - \tilde{\kappa}_g\phi_3 - \phi_4\tau_g &= 0, \\ \phi_3' + \phi_2\tilde{\kappa}_g - \phi_4\tilde{\tau}_g &= 0, \\ \phi_4' + \phi_1\kappa_n + \phi_2\tau_g + \phi_3\tilde{\tau}_g &= 0.\end{aligned}$$

♠ **Case 2:** *Energy of ED-frame field of second kind with Sasaki metric are given by*

$$\begin{aligned}\varepsilon nergy(\mathbf{T}) &= \frac{1}{2}\int_\beta (1 + \kappa_n^2) ds, \\ \varepsilon nergy(\mathbf{E}) &= \frac{1}{2}\int_\beta (1 + \tilde{\kappa}_g^2 + \tau_g^2) ds, \\ \varepsilon nergy(\mathbf{D}) &= \frac{1}{2}\int_\beta (1 + \tilde{\kappa}_g^2) ds, \\ \varepsilon nergy(\mathbf{N}) &= \frac{1}{2}\int_\beta (1 + \kappa_n^2 + \tau_g^2) ds.\end{aligned}$$

Putting

$$\mathbf{X} = \chi_1\mathbf{T} + \chi_2\mathbf{E} + \chi_3\mathbf{D} + \chi_4\mathbf{N}.$$

By using above equation, generalization energy of any field $\mathbf{X}$ is given by

$$\begin{aligned}\varepsilon nergy(\mathbf{X}) &= \frac{1}{2}\int_\beta (1 + (\chi_1' - \kappa_n\chi_4)^2 + (\chi_2' - \tilde{\kappa}_g\chi_3 - \tau_g\chi_4)^2 \\ &\quad + (\chi_3' + \chi_2\tilde{\kappa}_g)^2 + (\chi_4' + \chi_1\kappa_n + \tau_g\chi_2)^2) ds.\end{aligned}$$

Also, we have

$$\nabla_\mathbf{T}\mathbf{X} = (\chi_1' - \kappa_n\chi_4)\mathbf{T} + (\chi_2' - \tilde{\kappa}_g\chi_3 - \tau_g\chi_4)\mathbf{E} + (\chi_3' + \chi_2\tilde{\kappa}_g)\mathbf{D} + (\chi_4' + \chi_1\kappa_n + \tau_g\chi_2)\mathbf{N},$$

and

$$\begin{aligned}\nabla_\mathbf{T}^2\mathbf{X} &= ((\chi_1' - \kappa_n\chi_4)' - \kappa_n(\chi_4' + \chi_1\kappa_n + \tau_g\chi_2))\mathbf{T} \\ &\quad + ((\chi_2' - \tilde{\kappa}_g\chi_3 - \tau_g\chi_4)' - \tilde{\kappa}_g(\chi_3' + \chi_2\tilde{\kappa}_g) - \tau_g(\chi_4' + \chi_1\kappa_n + \tau_g\chi_2))\mathbf{E} \\ &\quad + ((\chi_3' + \chi_2\tilde{\kappa}_g)' + (\chi_2' - \tilde{\kappa}_g\chi_3 - \tau_g\chi_4)\tilde{\kappa}_g)\mathbf{D} \\ &\quad + ((\chi_4' + \chi_1\kappa_n + \tau_g\chi_2)' + (\chi_1' - \kappa_n\chi_4)\kappa_n + \tau_g(\chi_2' - \tilde{\kappa}_g\chi_3 - \tau_g\chi_4))\mathbf{N}.\end{aligned}$$

⊛ $\mathbf{X}$ have fixed energy with ED-frame field of second kind iff

$$\begin{aligned}&(\chi_1' - \kappa_n\chi_4)((\chi_1' - \kappa_n\chi_4)' - \kappa_n(\chi_4' + \chi_1\kappa_n + \tau_g\chi_2)) + ((\chi_2' - \tilde{\kappa}_g\chi_3 - \tau_g\chi_4)' \\ &- \tilde{\kappa}_g(\chi_3' + \chi_2\tilde{\kappa}_g) - \tau_g(\chi_4' + \chi_1\kappa_n + \tau_g\chi_2))(\chi_2' - \tilde{\kappa}_g\chi_3 - \tau_g\chi_4) \\ &+ ((\chi_3' + \chi_2\tilde{\kappa}_g)' + (\chi_2' - \tilde{\kappa}_g\chi_3 - \tau_g\chi_4)\tilde{\kappa}_g)(\chi_3' + \chi_2\tilde{\kappa}_g) + ((\chi_4' + \chi_1\kappa_n + \tau_g\chi_2)' \\ &+ (\chi_1' - \kappa_n\chi_4)\kappa_n + \tau_g(\chi_2' - \tilde{\kappa}_g\chi_3 - \tau_g\chi_4))(\chi_4' + \chi_1\kappa_n + \tau_g\chi_2) = 0.\end{aligned}$$

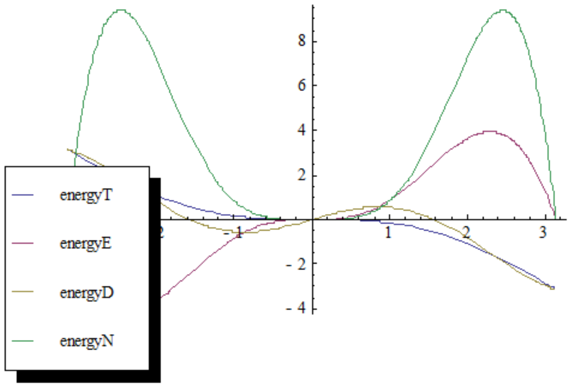

Figure 1: Plots of energy with ED-frame field of first kind

From ED-frame field of second kind, angle of Frenet vectors can be given by

$$\begin{aligned}\mathcal{A}(\mathbf{T}) &= \int_0^s (\kappa_n) ds,\\ \mathcal{A}(\mathbf{E}) &= \int_0^s (\tilde{\kappa}_g^2 + \tau_g^2)^{\frac{1}{2}} ds,\\ \mathcal{A}(\mathbf{D}) &= \int_0^s (\tilde{\kappa}_g) ds,\\ \mathcal{A}(\mathbf{N}) &= \int_0^s (\kappa_n^2 + \tau_g^2)^{\frac{1}{2}} ds.\end{aligned}$$

Generalization angle of any field $\mathbf{X}$ is given by

$$\mathcal{A}(\mathbf{X}) = \int_0^s ((\chi_1' - \kappa_n\chi_4)^2 + (\chi_2' - \tilde{\kappa}_g\chi_3 - \tau_g\chi_4)^2 + (\chi_3' + \chi_2\tilde{\kappa}_g)^2 + (\chi_4' + \chi_1\kappa_n + \tau_g\chi_2)^2)^{\frac{1}{2}} ds.$$

If we assume that angle of each of Frenet vectors given above are fixed, then we have following relations, respectively:

$$\begin{aligned}\chi_1' - \kappa_n\chi_4 &= 0,\\ \chi_2' - \tilde{\kappa}_g\chi_3 - \tau_g\chi_4 &= 0,\\ \chi_3' + \chi_2\tilde{\kappa}_g &= 0,\\ \chi_4' + \chi_1\kappa_n + \tau_g\chi_2 &= 0.\end{aligned}$$

5. Application

Recently, there has been considerable theory in energy and its comparatively geometry and its importance to a large diversity of optical applications [15]- [29]. Let M be an orientable hypersurface oriented by the unit normal vector field $\mathbf{N}$ in $\mathbb{E}^4$. If β is time helix in $\mathbb{E}^4$, then we have following applications with angle and energy.

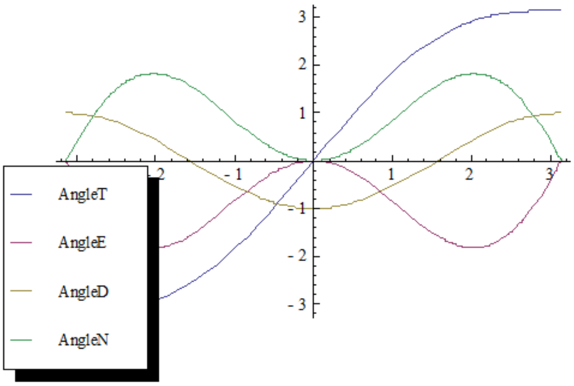

Figure 2: Plots of angle with ED-frame field of first kind

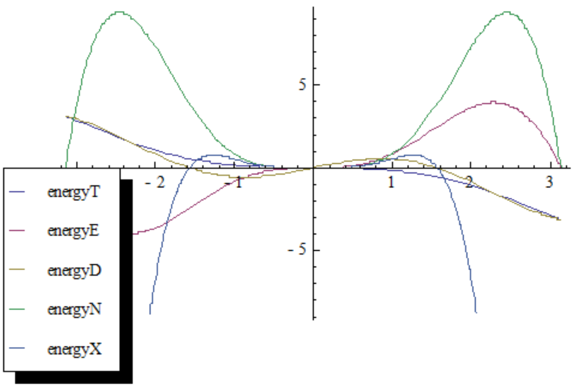

Figure 3: Plots of energy with X, ED-frame field of first kind

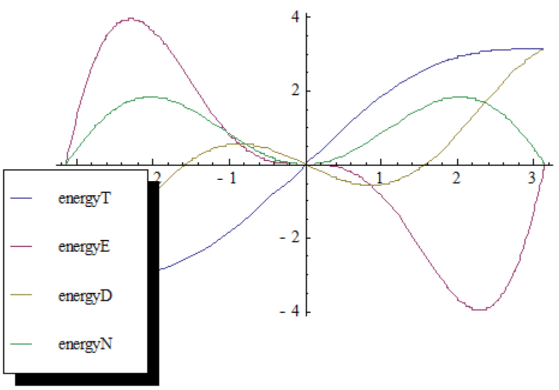

Figure 4: Plots of energy with ED-frame field of second kind

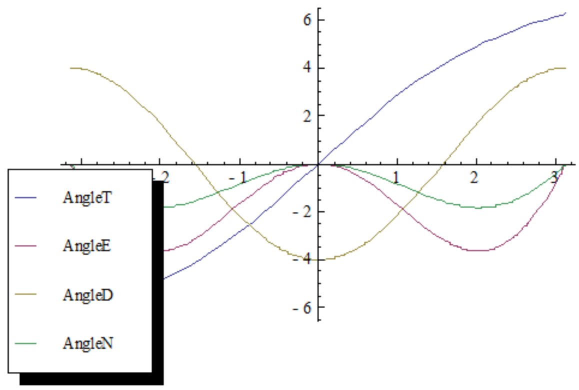

Figure 5: Plots of angle with ED-frame field of second kind

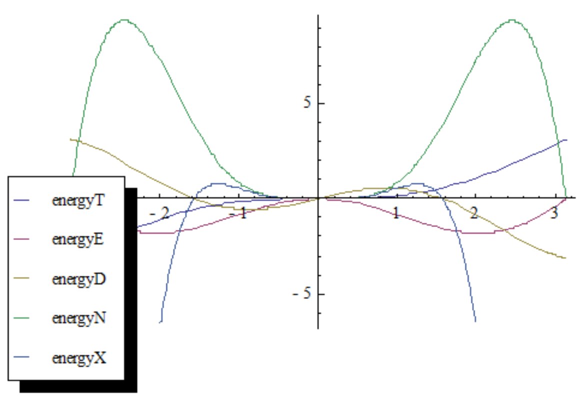

Figure 6: Plots of energy with X, ED-frame field of second kind

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