



A Spatiotemporal SIR Epidemic Model Two-dimensional with Problem of Optimal Control

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ABSTRACT: In the context of a more realistic model, in this work, we are interested in studying a spatiotemporal two-dimensional SIR epidemic model, in the form of a system of partial differential equations (PDE). A distribution of a vaccine in the form of a control variable is considered to force immunity. The purpose is to characterize a control that minimizes the number of susceptible, infected individuals and the costs associated with vaccination over a finite space and time domain. In addition, the existence of the solution of the state system and the optimal control is proved. The characterization of the control is given in terms of state function and adjoint function. The numerical resolution of the state system shows the effectiveness of our control strategy.

Key Words: Spatiotemporal model, Distributed optimal control, Numerical method.

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1. Introduction

Mathematical modeling in the field of epidemiology has become an important tool, since it gives an approximate idea of the causes, dynamics and spread of epidemic. In addition, it can provide useful control measures to make decisions about effective control strategies [1]. SIR is among the elementary models in the mathematical modeling of diseases, it consists of divided the population into different class, depending on the stage of infection. The susceptible class (S) includes individuals who may contract the disease but are not yet infectious. The infectious class (I) includes those who have the disease and can transmit it. The recovered compartment (R) includes persons who have recovered from the disease with permanent immunity. In the literature, there are a great deal of mathematical studies of diseases that give an interesting insight into the use of mathematical models in epidemiology. For example, Baily et al. [2], Anderson et al. [3], Hethcote [4], Brauer and Castillo-Chavez [5], Keeling and Rohani [6], Huppert and Katriel [7] and [8,9,10,11,12,13,25,26]. In this contribution, we consider an epidemic SIR model, spatiotemporal in two dimensions in the work of Lotfi. et al [14] and Hattaf. et al [15], in this system we introduce a vaccine in the form of a control variable, in order to minimise susceptible, infected individuals

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and the costs associated with vaccination. The existence of the state system solution and the optimal control is proved, and the characterization of the optimal control in terms of state function and adjoint is given. For the validation of our strategy, we present the numerical results obtained.

2. The Basic Mathematical Model

2.1. The model without controls

In this paper, we consider the following SIR epidemic model:

$$\begin{cases} \frac{\partial S}{\partial t} = d_S \Delta S + \Lambda - f(S, I, R) SI - \mu S \\ \frac{\partial I}{\partial t} = d_I \Delta I + f(S, I, R) SI - (\mu + d + r)I \\ \frac{\partial R}{\partial t} = d_R \Delta R - \mu R + rI \end{cases} \quad (t, x) \in Q = [0, T] \times \Omega \quad (2.1)$$

With $f(S, I, R) = \frac{\beta}{1 + \alpha_1 S + \alpha_2 I + \alpha_3 SI}$ is the incidence rate, such as $\alpha_1, \alpha_2, \alpha_3 \geq 0$ are constants. Λ is the recruitment rate of the population, μ is the natural death rate of the population, d is the death rate due to disease, r is the recovery rate of the infective individuals, β is the infection coefficient. The positive constants $d_S, d_I,$ and d_R denote the corresponding diffusion rate for susceptible, infectious, and recovered individuals. We denote by Ω a fixed and bounded domain in IR^2 with smooth boundary $\partial\Omega$ and η is the outward unit normal vector on the boundary. The initial conditions and no-flux boundary conditions are given by

$$\frac{\partial S}{\partial \eta} = \frac{\partial I}{\partial \eta} = \frac{\partial R}{\partial \eta} = 0, \quad (t, x) \in \Sigma = [0, T] \times \partial\Omega \quad (2.2)$$

$$S(0, x) = S_0 \geq 0, \quad I(0, x) = I_0, \quad \text{and} \quad R(0, x) = R_0 \quad (2.3)$$

In this step, the numerical results obtained by using the finite difference method of the system (1) without control are given. We have adopted two situations: In the first, the disease starts from the middle (1) and in the second, the disease starts at the corner (2). Figures 1, 2, and 3 present numerical results for susceptible, infected, and recovered individuals. Results show that in both situations, susceptible individuals become infected after an incubation period, and after a period of time, the disease spreads throughout the population. In order to fight against the spread of the disease we adopted a strategy based on the introduction of a vaccine in the form of a control variable.

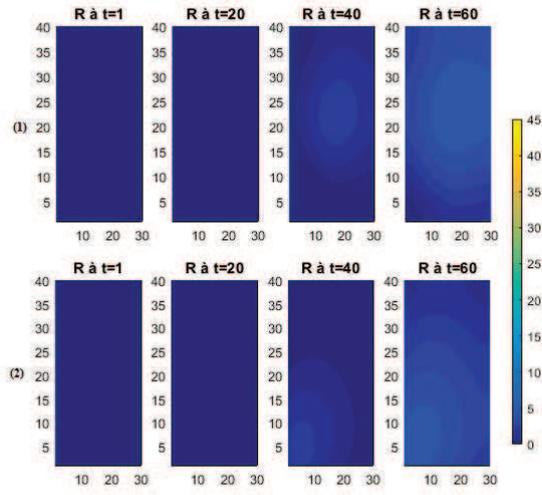


Figure 1: Susceptible behavior within Ω without control

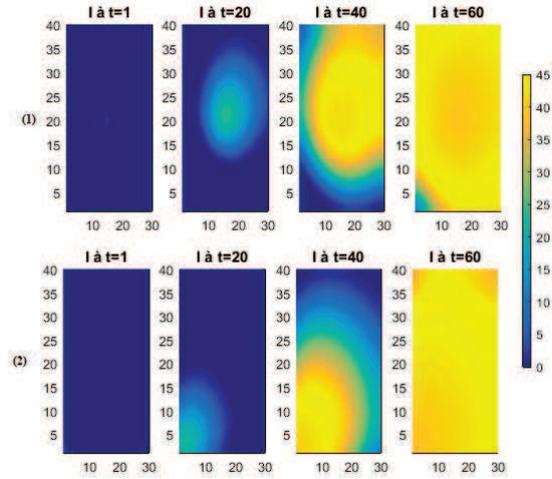
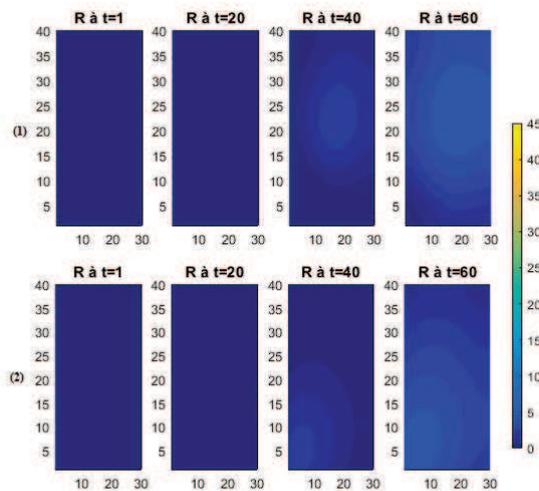


Figure 2: Infected behavior within Ω without control

Figure 3: Recovered behavior within Ω without control

2.2. The model with controls

The controlled model is the following:

$$\begin{cases} \frac{\partial S}{\partial t} = d_S \Delta S + \Lambda - f(S, I, R) SI - (\mu + v(t, x)) S \\ \frac{\partial I}{\partial t} = d_I \Delta I + f(S, I, R) SI - (\mu + d + r) I \\ \frac{\partial R}{\partial t} = d_R \Delta R - \mu R + v(t, x) S + r I \end{cases} \quad (t, x) \in Q = [0, T] \times \Omega \quad (2.4)$$

With $f(S, I, R) = \frac{\beta}{1 + \alpha_1 S + \alpha_2 I + \alpha_3 SI}$ and initial conditions and no-flux boundary conditions are given by

$$\frac{\partial S}{\partial \eta} = \frac{\partial I}{\partial \eta} = \frac{\partial R}{\partial \eta} = 0, \quad (t, x) \in \Sigma = [0, T] \times \partial\Omega \quad (2.5)$$

$$S(0, x) = S_0 \geq 0, \quad I(0, x) = I_0, \quad \text{and} \quad R(0, x) = R_0 \quad (2.6)$$

$v(x, t)$ represents the vaccination rate at time and position x . We seek to minimize the functional objective

$$J(v) = \int_0^T \int_{\Omega} (\rho_1 S(t, x) + \rho_2 I(t, x)) dx dt + \frac{\alpha}{2} \|v\|_{L^2(Q)}^2 \quad (2.7)$$

Eligible controls are contained in the ensemble

$$U_{ad} = \{v \in L^\infty(Q) / 0 \leq v \leq v^{max} \leq 1\} \quad (2.8)$$

for some positive constant v^{max} .

Where ρ_1, ρ_2 are constant weights. The cost of vaccination is a nonlinear function of v , we choose a quadratic function indicating the additional costs associated with high vaccination rates.

The parameter $\frac{\alpha}{2}$, with the units $\frac{\text{Population}/\text{km}^2}{\text{vaccin}^2}$, balances the cost squared of the vaccine with the cost associated with the infected population. Our objective is to find control functions such that

$$J(v^*) = \min \{J(v), v \in U_{ad}\}$$

- We put $H(\Omega) = (L^2(\Omega))^3$, we denote by $W^{1,2}([0, T], H(\Omega))$ the space of all absolutely continuous functions $y : [0, T] \rightarrow H(\Omega)$ having the property that $\frac{\partial y}{\partial t} \in L^2([0, T], H(\Omega))$.
- $\mathcal{L}(T, \Omega) = L^2([0, T], H^2(\Omega)) \cap L^\infty([0, T], H^1(\Omega))$

3. Existence of solution

We study in this section the existence of a global strong solution, positivity, and boundedness of solutions of problem for (2.4)-(2.6). Let $y = (y_1, y_2, y_3) = (S, I, R)$ the solution of the system (2.4)-(2.6) with $y^0 = (y_1^0, y_2^0, y_3^0) = (S^0, I^0, R^0)$. A denotes the linear operator defined as following

$$\begin{aligned} A : D(A) \subset H(\Omega) &\longrightarrow H(\Omega) \\ Ay = (d_S \Delta y_1, d_I \Delta y_2, d_R \Delta y_3) &\in D(A), \forall y = (y_1, y_2, y_3) \in D(A) \end{aligned} \quad (3.1)$$

with the domain of A defined by

$$D(A) = \left\{ y \in (H^2(\Omega))^3, \frac{\partial y_1}{\partial \eta} = \frac{\partial y_2}{\partial \eta} = \frac{\partial y_3}{\partial \eta} = 0, a.e x \in \partial\Omega \right\} \quad (3.2)$$

Theorem 3.1. *Let Ω be a bounded domain from \mathbb{R}^2 , with the boundary smooth enough, $y_i^0 \geq 0$ on Ω (for $i = 1, 2, 3$), the problem (2.4-2.6) has a unique (global) strong solution $y \in W^{1,2}([0, T] : H(\Omega))$ such that $y_i \in \mathcal{L}(T, \Omega) \cap L^\infty(Q)$ for $i = 1, 2, 3$. In addition y_1, y_2 , and y_3 are nonnegative. Furthermore there exists $C > 0$ (independent of (v)) for all $t \in [0, T]$*

$$\left\| \frac{\partial y_i}{\partial t} \right\|_{L^2(Q)} + \|y_i\|_{L^2(0, T; H^2(\Omega))} + \|y_i\|_{H^1(\Omega)} + \|y_i\|_{L^\infty(Q)} \leq C, \text{ for } i = 1, 2, 3 \quad (3.3)$$

Proof. To prove the existence of a (global) strong solution for system(2.4)-(2.6), now we write system (2.4)-(2.6) as shown in ((4.1) see Appendix). Let

$$\begin{cases} g_1(y(t)) = \Lambda - f(y) y_1 y_2 - (\mu + v(t, x)) y_1 \\ g_2(y(t)) = f(y) y_1 y_2 - (\mu + d + r) y_2, & t \in [0, T] \\ g_3(y(t)) = -\mu y_3 + v(t, x) y_1 + r y_2 \end{cases} \quad (3.4)$$

The system (3.4) represent the nonlinear term of (2.4) and we consider the function $g(y(t)) = (g_1(y(t)), g_2(y(t)), g_3(y(t)))$, then we can be rewrite the system (2.4-2.6) in the space $H(\Omega)$ as follows

$$\begin{cases} \frac{\partial y}{\partial t} = Ay + g(y(t)), & t \in [0, T] \\ y(0) = y^0 \end{cases} \quad (3.5)$$

It is clear that function g is not Lipschitz continuous in $y = (y_1, y_2, y_3)$ uniformly with respect to $t \in [0, T]$. Therefore, we cannot apply Theorem (8.1) (see appendix) for our problem directly.

Step 1: This step studies the local existence of positive solutions to system (2.1)-(2.6) in view of Theorem (8.1) (see appendix). We use a truncation procedure for g . For a fixed positive integer $k > 0$, let us define the function sets $D_1 = \{z | z > k\}$, $D_2 = \{z | |z| < k\}$, $D_3 = \{z | z < -k\}$ and consider the following auxiliary problem:

$$\begin{cases} \frac{\partial y^k}{\partial t} = Ay + g^k(t, y^k(x, t)), & \text{in } Q, \\ y^k(x, 0) = y^0, & \text{in } \Omega, \end{cases}$$

where $g^k(t, y^k) = (g_1^k(t, y^k), g_2^k(t, y^k), g_3^k(t, y^k))$. Here, for each index i , $g_i^k(t, y^k)$ are defined as follows:

$$g_i^k(t, y^k) = g_i(t, [y_1]_{D_{s_1}}, [y_2]_{D_{s_2}}, [y_3]_{D_{s_3}})$$

where $[y_i]_{Ds_i}$ means that $y_i \in Ds_i$, and

$$[y_i]_{Ds_i} = \begin{cases} k & \text{if } s_i = 1, \\ y_i & \text{if } s_i = 2, \\ -k & \text{if } s_i = 3. \end{cases}$$

As the operator A defined in (3.1)-(3.2) is dissipating, self-adjoint and generates a C_0 -semi-group of contractions on $H(\Omega)$ [23], it is clear that function $g^k(t, y^k)$ becomes Lipschitz continuous in y^k uniformly with respect to $t \in [0, T]$. Therefore, theorem (8.1) (see appendix) assures problem (2.1-2.6) admits a unique strong solution $y^k \in W^{1,2}([0, T], H(\Omega))$ with

$$y_1^k, y_2^k, y_3^k \in L^2([0, T], H^2(\Omega)) \quad (3.6)$$

In order to show that $y_i^k \in L^\infty(Q)$ for $i = 1, 2, 3$, we denote $M = \max\{\|g_1^k\|_{L^\infty(Q)}, \|y_1^0\|_{L^\infty(\Omega)}\}$ and $\{S(t), t \geq 0\}$ is the C_0 -semi-group generated by the operator $B : D(B) \subset L^2(\Omega) \rightarrow L^2(\Omega)$, where $By_1^k = d_1 \Delta y_1^k$ and $D(B) = \left\{ y_1^k \in H^2(\Omega), \frac{\partial y_1^k}{\partial \eta} = 0, a.e \partial\Omega \right\}$. It is clear that the function $U_1^k(t, x) = y_1^k - M_k t - \|y_1^0\|_{L^\infty(\Omega)}$ satisfies the system

$$\begin{cases} \frac{\partial U_1^k}{\partial t}(t, x) = d_S \Delta U_1^k + g_1^k(t, y(t)) - M_k, & t \in [0, T] \\ U_1^k(0, x) = y_1^0 - \|y_1^0\|_{L^\infty(\Omega)} \end{cases} \quad (3.7)$$

Note that this system has a solution given by

$$U_1^k(t) = S(t)(y_1^0 - \|y_1^0\|_{L^\infty(\Omega)}) + \int_0^t S(t-s)(g_1^k(s, y(s)) - M_k) ds,$$

As $y_1^0 - \|y_1^0\|_{L^\infty(\Omega)} \leq 0$ and $g_1^k(s, y(s)) - M_k \leq 0$, we have $U_1^k(t, x) \leq 0, \forall (t, x) \in Q$. Similarly the function $U_2^k(t, x) = y_1^k + M_k t + \|y_1^0\|_{L^\infty(\Omega)}$ satisfies $U_2^k(t, x) \geq 0, \forall (t, x) \in Q$. Then

$$|y_1^k(t, x)| \leq M_k t + \|y_1^0\|_{L^\infty(\Omega)}, \forall (t, x) \in Q$$

and analogously, we have

$$|y_i^k(t, x)| \leq M_k t + \|y_i^0\|_{L^\infty(\Omega)} \forall (t, x) \in Q \text{ for } i = 2, 3 \quad (3.8)$$

Thus we have proved that

$$y_i^k \in L^\infty(Q) \forall (t, x) \in Q \text{ for } i = 1, 2, 3. \quad (3.9)$$

By the first equation of (2.1), we obtain

$$\begin{aligned} & \int_0^t \int_\Omega \left| \frac{\partial y_1^k}{\partial s} \right|^2 ds dx + d_S^2 \int_0^t \int_\Omega |\Delta y_1^k|^2 ds dx - 2d_S \int_0^t \int_\Omega \frac{\partial y_1^k}{\partial s} \Delta y_1^k ds dx \\ & = \int_0^t \int_\Omega (\Lambda - f(y^k)) y_1^k - (\mu + v(t, x)) (y_1^k)^2 ds dx \end{aligned}$$

Using the regularity of y_1^k and the Green's formula, we can write

$$2 \int_0^t \int_\Omega \frac{\partial y_1^k}{\partial s} \Delta y_1^k dx = - \int_0^t \frac{\partial}{\partial s} \left(\int_\Omega |\nabla y_1^k|^2 dx \right) ds = - \int_\Omega |\nabla y_1^k|^2 dx + \int_\Omega |\nabla y_1^0|^2 dx$$

Then

$$\begin{aligned} & \int_0^t \int_\Omega \left| \frac{\partial y_1^k}{\partial s} \right|^2 ds dx + d_S^2 \int_0^t \int_\Omega |\Delta y_1^k|^2 ds dx + d_S \int_\Omega |\nabla y_1^k|^2 dx - d_S \int_\Omega |\nabla y_1^0|^2 dx \\ & = \int_0^t \int_\Omega (\Lambda - f(y^k)) y_1^k - (\mu + v(t, x)) (y_1^k)^2 ds dx \end{aligned}$$

Since $\|y_i^k\|_{L^\infty(Q)}$ for $i = 1, 2, 3$ are bounded independently of v and $y_1^0 \in H^2(\Omega)$, we deduce that

$$y_1^k \in L^\infty([0, T], H^1(\Omega)) \quad (3.10)$$

We make use of (3.6), (3.7), and (3.10), in order to get

$$y_1^k \in \mathcal{L}(T, \Omega) \cap L^\infty(Q)$$

and conclude that the inequality in (3.3) holds for $i = 1$, similarly for y_2^k and y_3^k .

In order to show the positiveness of y_i^k for $i = 1, 2, 3$, we start by demonstrating that y_2^k is positive to be, we set $y_2^k = y_2^{k+} - y_2^{k-}$ with $y_2^{k+} = \sup\{y_2^k(t, x), 0\}$ and $y_2^{k-} = \sup\{-y_2^k(t, x), 0\}$. we multiply the second equation of (2.4) and we integrate on Ω we obtain:

$$-\frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} (y_2^{k-})^2(t, x) dx \right) = \int_{\Omega} |d_2 \nabla y_2^{k-}(t, x)|^2 dx - \int_{\Omega} f(y^k) y_1^k (y_2^{k-}(t, x))^2 dx + \int_{\Omega} (\mu + d + r) (y_2^{k-}(t, x))^2 dx$$

wich implies

$$-\frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} (y_2^{k-})^2(t, x) dx \right) \geq - \int_{\Omega} f(y^k) y_1^k (y_2^{k-}(t, x))^2 y_1^k (y_2^{k-}(t, x))^2 dx$$

Then,

$$\frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} (y_2^{k-})^2(t, x) dx \right) \leq \int_{\Omega} f(y^k) y_1^k (y_2^{k-}(t, x))^2 y_1^k (y_2^{k-}(t, x))^2 dx$$

We put $b = f(y^k) y_1^k$ and Gronwall's inequality leads to

$$\int_{\Omega} (y_2^{k-})^2(t, x) dx \leq e^{t \|b\|_{L^\infty(\Omega)}} \int_{\Omega} (y_2^{k-})^2(0, x) dx$$

then $y_2^{k-} = 0$, on deduces that $y_2^k(t, x) \geq 0$,

To demonstrate the positivity of y_1^k and y_3^k , we write the 1st and 3rd equations of (2.1) in the form

$$\begin{cases} \frac{\partial y_1^k}{\partial t} = d_1 \Delta y_1^k + F_1^k(y_1^k, y_2^k, y_3^k), & (t, x) \in Q, \\ \frac{\partial y_3^k}{\partial t} = d_3 \Delta y_3^k + F_3^k(y_1^k, y_2^k, y_3^k). \end{cases} \quad (3.11)$$

It is obvious to see that the functions $F_1^k(y_1^k, y_2^k, y_3^k)$ and $F_3^k(y_1^k, y_2^k, y_3^k)$, are continuously differentiable satisfying $F_1(0, y_2^k, y_3^k) = \Lambda \geq 0$, and $F_3(y_1^k, y_2^k, 0) = v(t, x) y_1^k + r y_2^k \geq 0$ for all $y_1^k, y_3^k \geq 0$.

Since initial data of system (3.11) are nonnegative, we deduce the positivity of y_1^k, y_2^k and y_3^k (see [24]). Now we particularize $k > 0$ large enough such that

$$M_k \theta + \|y_i^0\|_{L^\infty(\Omega)} \leq k, \quad i = 1, 2, 3, \quad \text{for some } \theta \in [0, T] \quad (3.12)$$

For example, we can take $k > 2 \max\{\|y\|_{L^\infty(\Omega)}, i = 1, 2, 3\}$. Let $\theta \in (0, T)$ be maximal with property (3.12). By (3.8)-(3.12), it is clear that $|y_i^k(t, x)| < k$, for $(t, x) \in [0, \theta] \times \Omega$ and $i = 1, 2, 3$. So, $g^k(t, y_1, y_2, y_3)$ coincides with $g(t, y_1, y_2, y_3)$ for $(t, x) \in [0, \theta] \times \Omega$, and consequently $y^k = (y_1^k, y_2^k, y_3^k)$ is a local solution for (2.4)-(2.6) defined on $[0, \theta] \times \Omega$.

Step 2. It remains to show that the above local positive solution of problem (2.4)-(2.6) is in fact a global one in $[0, \theta] \times \Omega$. Indeed, it is sufficient to show the uniformly boundedness of $y_i, i = 1, 2, 3$, in $[0, \theta] \times \Omega$. To this end, we first introduce

$N = y_1 + y_2 + y_3$ then $\frac{\partial N}{\partial t} = d_1 \Delta y_1 + d_2 \Delta y_2 + d_3 \Delta y_3 - \Lambda - d y_2 - \mu N$, there exists m such that: $\frac{\partial N}{\partial t} \leq d_m \Delta N + mN$ with $d_m = \max\{d_1, d_2, d_3\}$. This leads to the estimate $0 < N(t, x) \leq e^m S(t) N_0(x)$,

$(t, x) \in [0, \theta] \times \Omega$, where $S(t)$, $t \geq 0$ is the C_0 -semi group of contractions on $L^2(\Omega)$ generated by the operator $BN = d\Delta N$, with the domain $D(B) = \left\{ N \in H^2(\Omega), \frac{\partial N}{\partial \eta} = 0, \text{ a.e in } \partial\Omega \right\}$. Therefore, $\|N\|_{L^\infty([0, \theta] \times \Omega)} \leq m_1$ for some $m_1 > 0$ independent of k and of v . Next, we can deduce the boundedness of y_1, y_2 and y_3 on $[0, \theta] \times \Omega$. Consequently, y_i are defined on the whole set Q (and also positive and bounded). Thus (y_1, y_2, y_3) is a global positive strong solution of system (2.4)-(2.6) and it satisfies (3.3). This completes the proof. \square

4. The existence of the optimal solution

In this section, we will prove the existence of an optimal control for the problem (2.7) subject to reaction diffusion system (2.4)-(2.6) and $(v) \in U_{ad}$. The main result of this section is the following theorem.

Theorem 4.1. *Under the hypotheses of theorem (3.1), the optimal control problem (2.4-2.7) admits an optimal solution $(y^*, (v^*))$.*

Proof. From Theorem 3.1, we know that, for every $v \in U_{ad}$, there exists a unique solution y to system (2.4-2.6). Assume that

$$\inf_{v \in U_{ad}} J((v)) > -\infty$$

Let $\{(v^n)\} \subset U_{ad}$ be a minimizing sequence such that

$$\lim_{n \rightarrow \infty} J(v^n) = \inf_{v \in U_{ad}} J(v)$$

where (y_1^n, y_2^n, y_3^n) is the solution of system (2.4-2.6) corresponding to the control (v^n) for $n = 1, 2, \dots$. That is

$$\begin{cases} \frac{\partial y_1^n}{\partial t} = d_1 \Delta y_1^n + \Lambda - f(y^n) y_1^n - (\mu + v^n(t, x)) y_1^n \\ \frac{\partial y_2^n}{\partial t} = d_2 \Delta y_2^n + f(y^n) y_1^n - (\mu + d + r) y_2^n, (t, x) \in Q \\ \frac{\partial y_3^n}{\partial t} = d_3 \Delta y_3^n - \mu y_3^n + v^n(t, x) y_1^n + r y_2^n \end{cases} \quad (4.1)$$

$$\frac{\partial y_1^n}{\partial \eta} = \frac{\partial y_2^n}{\partial \eta} = \frac{\partial y_3^n}{\partial \eta} = 0 \quad (t, x) \in \Sigma, (t, x) \in \Sigma \quad (4.2)$$

$$y_i^n(0, x) = y_i^0 \text{ for } i = 1, 2, 3 \quad x \in \Omega \quad (4.3)$$

and By theorem (3.1) using the estimate (3.3) of the solution y_i^n , there exists a constant $C > 0$ such that for all $n \geq 1, t \in [0, T]$

$$\left\| \frac{\partial y_i^n}{\partial t} \right\|_{L^2(Q)} \leq C, \|y_i^n\|_{L^2(0, T; H^2(\Omega))} \leq C, \|y_i^n\|_{H^1(\Omega)} \leq C, i = 1, 2, 3 \quad (4.4)$$

$H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, so we deduce that $y_1^n(t)$ is compact in $L^2(\Omega)$.

Let's Show that $\{y_1^n(t), n \geq 1\}$ is equicontinuous in $C([0, T] : L^2(\Omega))$. As $\frac{\partial y_1^n}{\partial t}$ is bounded in $L^2(Q)$, this implies that for all $s, t \in [0, T]$

$$\left| \int_{\Omega} (y_1^n)^2(t, x) dx - \int_{\Omega} (y_1^n)^2(s, x) dx \right| \leq K |t - s| \quad (4.5)$$

The Ascoli-Arzelà Theorem(See [22]) implies that y_1^n is compact in $C([0, T] : L^2(\Omega))$. Hence, selecting further sequences, if necessary, we have

$y_1^n \rightarrow y_1^*$ in $L^2(\Omega)$, uniformly with respect to t and analogously, we have for $y_i^n \rightarrow y_i^*$ in $L^2(\Omega)$ for $i = 2, 3$, uniformly with respect to t .

From the boundedness of Δy_i^n in $L^2(Q)$, which implies it is weakly convergent in $L^2(Q)$ on a subsequence denoted again Δy_i^n then for all distribution φ

$$\int_Q \varphi \Delta y_i^n = \int_Q y_i^n \Delta \varphi \rightarrow \int_Q y_i^* \Delta \varphi = \int_Q \varphi \Delta y_i^*$$

Which implies that $\Delta y_i^n \rightarrow \Delta y_i^*$ weakly in $L^2(Q)$, $i = 1, 2, 3$, In addition, the estimates (4.4) leads to

$$\frac{\partial y_i^n}{\partial t} \rightarrow \frac{\partial y_i^*}{\partial t} \text{ weakly in } L^2(Q), i = 1, 2, 3$$

$$y_i^n \rightarrow y_i^* \text{ weakly in } L^2(0, T; H^2(\Omega)), i = 1, 2, 3$$

$$y_i^n \rightarrow y_i^* \text{ weakly star in } L^\infty(0, T; H^1(\Omega)), i = 1, 2, 3$$

We now show that $y_1^n y_2^n \rightharpoonup y_1^* y_2^*$ and $f(y^n) y_1^n y_2^n \rightarrow f(y^*) y_1^* y_2^*$ strongly in $L^2(Q)$, we write

$$y_1^n y_2^n - y_1^* y_2^* = (y_1^n - y_1^*) y_2^n + (y_2^n - y_2^*) y_1^*$$

$$f(y^n) y_1^n y_2^n - f(y^*) y_1^* y_2^* = f(y^n) (y_1^n y_2^n - y_1^* y_2^*) + y_1^* y_2^* (f(y^n) - f(y^*))$$

and

$$f(y^n) - f(y^*) = \frac{\beta}{1 + \alpha_1 y_1^n + \alpha_2 y_2^n + \alpha_3 y_1^n y_2^n} - \frac{\beta}{1 + \alpha_1 y_1^* + \alpha_2 y_2^* + \alpha_3 y_1^* y_2^*}$$

and we make use of the convergences $y_i^n \rightarrow y_i^*$ strongly in $L^2(Q)$, $i = 1, 2$, and of the boundedness of y_1^n, y_2^n in $L^\infty(Q)$, then $y_1^n y_2^n \rightarrow y_1^* y_2^*$ and $f(y^n) y_1^n \rightarrow f(y^*) y_1^*$ strongly in $L^2(Q)$.

Since v^n is bounded, we can assume that $v^n \rightarrow v^*$ weakly in $L^2(Q)$ on a subsequence denoted again v^n . Since U_{ad} is a closed and convex set in $L^2(Q)$, it is weakly closed, so $v^* \in U_{ad}$

We now show that

$$v^n y_1^n \rightarrow v^* y_1^* \text{ weakly in } L^2(Q)$$

Writing

$$v^n y_1^n - v^* y_1^* = (y_1^n - y_1^*) v^n + (v^n - v^*) y_1^*$$

and making use of the convergences $y_i^n \rightarrow y_i^*$ strongly in $L^2(Q)$, and $v^n \rightarrow v^*$ weakly in $L^2(Q)$, one obtains that $v^n y_1^n \rightarrow v^* y_1^*$ weakly in $L^2(Q)$.

By taking $n \rightarrow \infty$ in (4.1-4.3), we obtain that y^* is a solution of (2.4-2.6) corresponding to $(v_1^*) \in U_{ad}$. Therefore

$$\begin{aligned} J(y^*, v^*) &= \rho_1 \int_0^T \int_\Omega y_1^*(t, x) dx dt + \rho_2 \int_0^T \int_\Omega y_2^*(t, x) dx dt + \frac{\eta}{2} \|v^*\|_{L^2(Q)}^2 \\ &\leq \lim_{n \rightarrow \infty} \inf \left(\rho_1 \int_0^T \int_\Omega y_1^n(t, x) dx dt + \rho_2 \int_0^T \int_\Omega y_2^n(t, x) dx dt + \frac{\eta}{2} \|v^n\|_{L^2(Q)}^2 \right) \\ &= \lim_{n \rightarrow \infty} \left(\rho_1 \int_0^T \int_\Omega y_1^n(t, x) dx dt + \rho_2 \int_0^T \int_\Omega y_2^n(t, x) dx dt + \frac{\eta}{2} \|v^n\|_{L^2(Q)}^2 \right) \\ &= \inf_{(y, v) \in U_{ad}} J((y, v)) \end{aligned}$$

This shows that J attains its minimum at (y^*, v^*) , we deduce that (y^*, v^*) verifies problem (2.4-2.6) and minimizes the objectif functional (2.7). The proof is complet \square

5. Necessary optimality conditions

Let $v \in U_{ad}$ and $v^\varepsilon = v^* + \varepsilon v \in U_{ad}$, in this section, we show the optimality conditions to problem (2.4-2.6), and we find the characterization of optimal control. First, we need the Gateaux differentiability of the mapping $v \rightarrow y(v)$. For this reason, denoting by $y^\varepsilon = (y_1^\varepsilon, y_2^\varepsilon, y_3^\varepsilon) = (y_1, y_2, y_3)(v^\varepsilon)$

and $y^* = (y_1^*, y_2^*, y_3^*) = (y_1, y_2, y_3)(v^*)$ the solution of (2.4-2.6) corresponding to v^ε and v^* respectively. $H = \begin{pmatrix} -\frac{\beta y_2^*(1+\alpha_2 y_2^*)}{(1+\alpha_1 y_1^* + \alpha_2 y_2^* + \alpha_3 y_1^* y_2^*)^2} - (\mu + v^*) & -\frac{\beta y_1^*(1+\alpha_1 y_1^*)}{(1+\alpha_1 y_1^* + \alpha_2 y_2^* + \alpha_3 y_1^* y_2^*)^2} & 0 \\ \frac{\beta y_2^*(1+\alpha_2 y_2^*)}{(1+\alpha_1 y_1^* + \alpha_2 y_2^* + \alpha_3 y_1^* y_2^*)^2} & \frac{\beta y_1^*(1+\alpha_1 y_1^*)}{(1+\alpha_1 y_1^* + \alpha_2 y_2^* + \alpha_3 y_1^* y_2^*)^2} - (\mu + d + r) & 0 \\ v^* & r & -\mu \end{pmatrix}$ and $G = \begin{pmatrix} -y_1^* \\ 0 \\ y_1^* \end{pmatrix}$.

Proposition 5.1. *The mapping $y : U_{ad} \rightarrow W^{1,2}([0, T]; H(\Omega))$ with $y_i \in \mathcal{L}(T, \Omega)$ for $i = 1, 2, 3$ is Gateaux differentiable with respect to v^* . For all direction $u \in U_{ad}$, $y'(v^*)v = Y$ is the unique solution in $W^{1,2}([0, T]; H(\Omega))$ with $Y_i \in \mathcal{L}(T, \Omega)$ of the following equation*

$$\begin{cases} \frac{\partial Y}{\partial t} = AY + HY + Gv, & t \in [0, T] \\ Y(0) = 0 \end{cases} \quad (5.1)$$

Proof. Put $Y_i^\varepsilon = \frac{y_i^\varepsilon - y_i^*}{\varepsilon}$ for $i = 1, 2, 3$, $Q(y_1, y_2) = \frac{\beta y_1 y_2}{1 + \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_1 y_2}$, $M_1^\varepsilon = \frac{Q(y_1^\varepsilon, y_2^\varepsilon) - Q(y_1^*, y_2^*)}{y_1^\varepsilon - y_1^*}$, and $M_2^\varepsilon = \frac{Q(y_1^*, y_2^\varepsilon) - Q(y_1^*, y_2^*)}{y_2^\varepsilon - y_2^*}$.

We denote S^ε the system (2.4) corresponding to v^ε and S^* the system (2.4) corresponding to v^* , subtracting system S^ε from S^* , we have

$$\begin{cases} \frac{\partial Y_1^\varepsilon}{\partial t} = d_1 \Delta Y_1^\varepsilon - (M_1^\varepsilon + \mu + v^\varepsilon) Y_1^\varepsilon - M_2^\varepsilon Y_2^\varepsilon - v y_1^* \\ \frac{\partial Y_2^\varepsilon}{\partial t} = d_2 \Delta Y_2^\varepsilon + M_1^\varepsilon Y_1^\varepsilon + (M_2^\varepsilon - d - \mu - r) Y_2^\varepsilon, & (x, t) \in Q \\ \frac{\partial Y_3^\varepsilon}{\partial t} = d_3 \Delta Y_3^\varepsilon + v^\varepsilon Y_1^\varepsilon + r Y_2^\varepsilon - \mu Y_3^\varepsilon + v y_1^* \end{cases} \quad (5.2)$$

with the homogeneous Neumann boundary conditions

$$\frac{\partial Y_1^\varepsilon}{\partial \eta} = \frac{\partial Y_2^\varepsilon}{\partial \eta} = \frac{\partial Y_3^\varepsilon}{\partial \eta} = 0 \quad (x, t) \in \Sigma \quad (5.3)$$

$$Y_i^\varepsilon(0, x) = 0 \quad x \in \Omega, \text{ for } i = 1, 2, 3 \quad (5.4)$$

We prove that Y_i^ε are bounded in $L^2(Q)$ uniformly with respect to ε . For this end, denoting by

$$Y^\varepsilon = (Y_1^\varepsilon, Y_2^\varepsilon, Y_3^\varepsilon), H^\varepsilon = \begin{pmatrix} -M_1^\varepsilon - \mu - v^\varepsilon & -M_2^\varepsilon & 0 \\ M_1^\varepsilon & M_2^\varepsilon - (d + \mu + r) & 0 \\ v^\varepsilon & r & -\mu \end{pmatrix}, \text{ and } G = \begin{pmatrix} -y_1^* \\ 0 \\ y_1^* \end{pmatrix}. \text{ Then}$$

(5.2) given by

$$\begin{cases} \frac{\partial Y^\varepsilon}{\partial t} = AY^\varepsilon + H^\varepsilon Y^\varepsilon + Gv, & t \in [0, T] \\ Y^\varepsilon(0) = 0 \end{cases} \quad (5.5)$$

$(S(t), t \geq 0)$ be the semi-group generated by A , then the solution of (5.5) can be expressed as

$$Y^\varepsilon(t) = \int_0^t S(t-s) H^\varepsilon(s) Y^\varepsilon(s) ds + \int_0^t S(t-s) Gv(s) ds, \quad (5.6)$$

On the other hand the coefficients of the matrix H^ε are bounded uniformly with respect to ε , using Gronwall's inequality, we have

$$\|Y_i^\varepsilon\|_{L^2(Q)} \leq \Gamma \quad (5.7)$$

where $\Gamma > 0$ ($i = 1, 2, 3$). Then

$$\|y_i^\varepsilon - y_i^*\|_{L^2(Q)} = \varepsilon \|Y_i^\varepsilon\|_{L^2(Q)} \quad (4.7) \quad (5.8)$$

Hence $y_i^\varepsilon \rightarrow y_i^*$ in $L^2(Q)$, $i = 1, 2, 3$.

Denoting by $H = \begin{pmatrix} -M_1^* - \mu - v^* & -M_2^* & 0 \\ M_1^* & M_2^* - (d + \mu + r) & 0 \\ v^* & r & -\mu \end{pmatrix}$ where $M_1^* = \frac{\partial Q(y_1^*, y_2^*)}{\partial y_1}$, $M_2^* = \frac{\partial Q(y_1^*, y_2^*)}{\partial y_2}$, and $Y = (Y_1, Y_2, Y_3)$. Hence, then system (5.2-5.4) can be written in the form

$$\begin{cases} \frac{\partial Y}{\partial t} = AY + HY + Gv, & t \in [0, T] \\ Y(0) = 0 \end{cases} \quad (5.9)$$

and its solution can be expressed as

$$Y(t) = \int_0^t S(t-s) H(s) Y(s) ds + \int_0^t S(t-s) Gv(s) ds, \quad (5.10)$$

By (5.6) and (5.10) we deduce that

$$Y^\varepsilon(t) - Y(t) = \int_0^t S(t-s) H^\varepsilon(s) (Y^\varepsilon - Y) + Y(s) (H^\varepsilon(s) - H(s)) ds \quad (5.11)$$

Thus all the coefficients of the matrix H^ε tend to the corresponding coefficients of the matrix H in $L^2(Q)$. An application of Gronwall's Inequality yields to $Y_i^\varepsilon \rightarrow Y_i$ in $L^2(Q)$ as $\varepsilon \rightarrow 0$, for $i = 1, 2, 3$. \square

Let v^* be an optimal control of (2.4-2.8), $y^* = (y_1^*, y_2^*, y_3^*)$ be the optimal state, D is the matrix defined by $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\rho = (\rho_1, \rho_2, 0)$, D^* is the adjoint matrix associated to D , H^* is the adjoint matrix associated to H and $p = (p_1, p_2, p_3)$ is the adjoint variable, we can write the dual system associated to system (2.4-2.8):

$$\begin{cases} -\frac{\partial p}{\partial t} - Ap - H^*p = D^*D\rho, & t \in [0, T] \\ p(T, x) = 0 \end{cases} \quad (5.12)$$

Lemma 5.2. *Under hypotheses of theorem (3.1), if $(y^*, (v^*))$ is an optimal pair, then there exists a unique strong solution $p \in W^{1,2}([0, T]; H(\Omega))$ to the system (5.12) with $p_i \in \mathcal{L}(T, \Omega)$ for $i = 1, 2, 3$.*

Proof. Like in Theorem (3.1), by making the change of variable $s = T - t$ and the change of functions $q_i(s, x) = p_i(T - s, x) = p_i(t, x)$, $(t, x) \in Q$, $i = 1, 2, 3$. we can easily prove the existence of the solution to this lemma. \square

To obtain the necessary conditions for the optimal control problem, applying standard optimality techniques, analyzing the objective functional and utilizing relationships between the state and adjoint equations, we obtain a characterization of the control optimal.

Theorem 5.3. *Let v^* be an optimal control of (2.4)-(2.8) and let $y^* \in W^{1,2}([0, T]; H(\Omega))$ with $y_i^* \in \mathcal{L}(T, \Omega)$ for $i = 1, 2, 3$, be the optimal state, that is y^* is the solution to (2.4)-(2.8) with the control u^* . Then,*

$$v^* = \min \left(v^{max}, \max \left(0, \frac{y_1^* p_1 - y_1^* p_3}{\alpha} \right) \right) \quad (5.13)$$

Proof. We suppose v^* is an optimal control and $y^* = (y_1^*, y_2^*, y_3^*) = (y_1, y_2, y_3)(v^*)$ are the corresponding state variables. Consider $v^\varepsilon = v^* + \varepsilon h \in U_{ad}$ and corresponding state solution $y^\varepsilon = (y_1^\varepsilon, y_2^\varepsilon, y_3^\varepsilon) = (y_1, y_2, y_3)(v^\varepsilon)$, we have

$$\begin{aligned}
J'(v^*)(h) &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (J(v^\varepsilon) - J(v^*)) \\
&= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left(\int_0^T \int_\Omega \rho_1 (y_1^\varepsilon - y_1^*)(t, x) dx dt + \int_0^T \int_\Omega \rho_2 (y_2^\varepsilon - y_2^*)(t, x) dx dt \right. \\
&\quad \left. + \frac{\alpha}{2} \int_0^T \int_\Omega ((v^\varepsilon)^2 - (v^*)^2)(t, x) dx dt \right) \\
&= \lim_{\varepsilon \rightarrow 0} \left(\int_0^T \int_\Omega \rho_1 \left(\frac{y_1^\varepsilon - y_1^*}{\varepsilon} \right) (t, x) dx dt + \int_0^T \int_\Omega \rho_2 \left(\frac{y_2^\varepsilon - y_2^*}{\varepsilon} \right) (t, x) dx dt \right. \\
&\quad \left. + \frac{\alpha}{2} \int_0^T \int_\Omega ((\varepsilon h)^2 + 2h v^*) (t, x) dx dt \right) \tag{5.14} \\
&= \int_0^T \int_\Omega \rho_1 Y_1(t, x) dx dt + \int_0^T \int_\Omega \rho_2 Y_2(t, x) dx dt + \alpha \int_0^T \int_\Omega (h^* v^*)(t, x) dx dt \\
&= \int_0^T \langle D\rho, DY \rangle_{H(\Omega)} dt + \int_0^T \langle \alpha v^*, h \rangle_{L^2(\Omega)} dt
\end{aligned}$$

We use (5.1) and (5.12), we have

$$\begin{aligned}
\int_0^T \langle D\rho, DY \rangle_{H(\Omega)} dt &= \int_0^T \langle D^* D\rho, Y \rangle_{H(\Omega)} dt \\
&= \int_0^T \left\langle -\frac{\partial p}{\partial t} - Ap - H^* p, Y \right\rangle_{H(\Omega)} dt \\
&= \int_0^T \left\langle p, \frac{\partial Y}{\partial t} - AY - HY \right\rangle_{H(\Omega)} dt \\
&= \int_0^T \langle p, Gh \rangle_{H(\Omega)} dt \tag{5.15} \\
&= \int_0^T \langle G^* p, h \rangle_{L^2(\Omega)} dt
\end{aligned}$$

Since J is Gateaux differentiable at v^* and U_{ad} is convex, as the minimum of the objective functional is attained at v^* it is seen that $J'(v^*)(u - v^*) \geq 0$ for all $u \in U_{ad}$.

We take $h = u - v^*$ and we use (5.14)-(5.15) then $J'(v^*)(u - v^*) = \int_0^T \langle G^* p + \alpha u^*, (u - v^*) \rangle_{L^2(\Omega)} dt$. We conclude that $J'(v^*)(u - v^*) \geq 0$ equivalent to $\int_0^T \langle G^* p + \alpha v^*, (u - v^*) \rangle_{L^2(\Omega)} dt \geq 0$ for all $u \in U_{ad}$. By standard arguments varying u , we obtain

$$\alpha v^* = -G^* p$$

Then

$$v^* = \frac{y_1^* p_1 - y_1^* p_3}{\alpha}$$

As $v^* \in U_{ad}$, we have

$$v^* = \min \left(v^{max}, \max \left(0, \frac{y_1^* p_1 - y_1^* p_3}{\alpha} \right) \right)$$

□

Notations	Value	Description (<i>Units</i>)
$S_0(x, y)$	40 for $(x, y) \in \Omega_i \quad i = 1, 2$ 45 for $(x, y) \notin \Omega_i$	Initial susceptible population (people/km ²)
$I_0(x, y)$	5 for $(x, y) \in \Omega_i \quad i = 1, 2$ 0 for $(x, y) \notin \Omega_i$	Initial infected population (people/km ²)
$R_0(x, y)$	0 for $(x, y) \in \Omega$	Initial recovered population (people/km ²)
Λ	0.5	Recruitment rate (day ⁻¹)
μ	0.1	rate (day ⁻¹)
β	0.6	The infection coefficient
r	0.02	rate (day ⁻¹)
d	0.01	TB induced mortality rate (day ⁻¹)
d_S	0.5	diffusion rate for susceptible
d_I	0.9	diffusion rate for infected
d_R	0.9	diffusion rate for recovered
α_1	0.1	Constant
α_2	0.02	Constant
α_3	0.03	Constant
t	[1, 60]	time period (day)

Table 1: Initial conditions and parameters values

6. Numerical results

We present the results obtained, by numerical resolution using the forward-backward sweep method (FBSM) [21], of our optimality system, which is formulated by state equations with initial and boundary conditions (2.4-2.6), adjoint equations with transversality conditions(5.12), and optimal control characterization (5.13).our strategy is to apply two types of treatment respectively to susceptible and infected individuals, in order to fight the spread of the disease. We will keep the same situations described previously in section 2.1: the first, the disease starts from the middle of the domain Ω (1) and in the second, the disease begins in the lower left corner of Ω (2). In this work, we take the density of 45 in order to model a situation of high contacts. Concerning the choice of the domain Ω , we take a rectangular grid of size 30 km \times 40 km: The parameter values and the initial values are given in table 1. These values are extracted from [14].Moreover, the upper limits of the optimality condition are considered to be $v^{max} = 1$ [19] and the constant weighting values in the objective function are $\rho_1 = 1, \rho_2 = 1, \alpha = 2$, taken from [20].

6.1. Optimal control simulation

To validate our vaccination strategy, we will proceed in two different ways:

- 1- Start vaccination against the disease after 20 days.
- 2- Vaccination against the disease starts from the first day.

In the first case, when introducing vaccination after 20 days, it can be seen in Figures 4 and 5 that the number of susceptible and infected individuals decreases rapidly.on the other hand, in Figure 6, we can clearly see the increase in the number of individuals recovered.

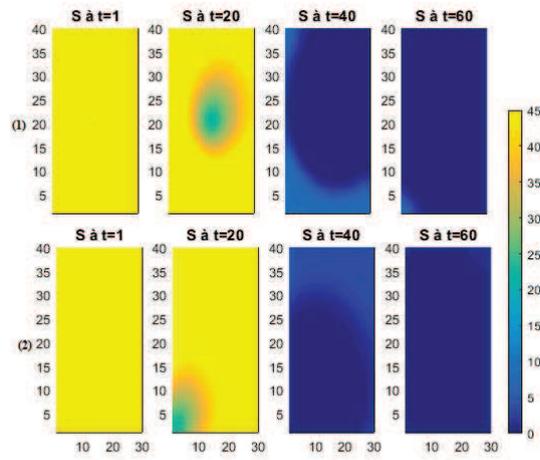


Figure 4: Susceptible behavior within Ω with control (vaccine strategy starts after the 20th day)

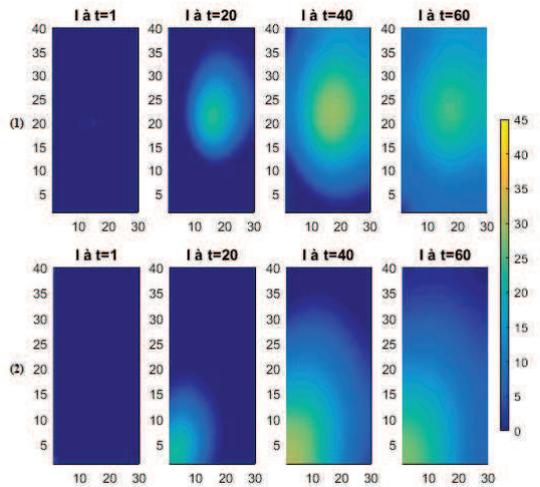


Figure 5: Infected behavior within Ω with control (vaccine strategy starts after the 20th day)

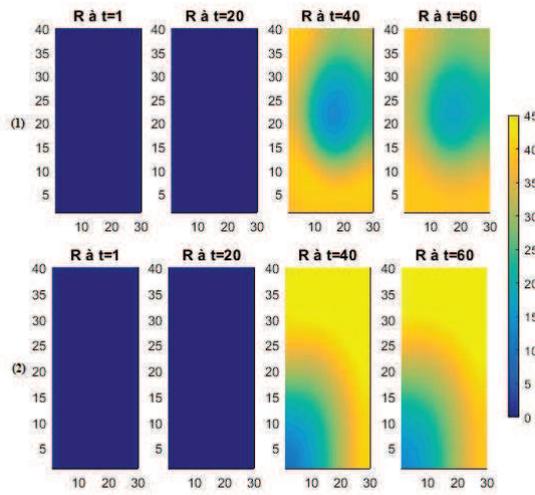


Figure 6: Recovered behavior within Ω with control (vaccine strategy starts after the 20th day)

In the second case, when the vaccination against the disease starts from the first day, the effectiveness of our vaccination strategy is clear, since the disease disappears quickly (figure 7 and 8).

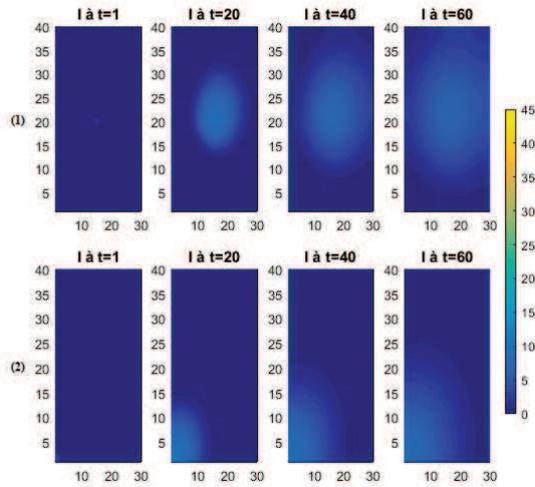


Figure 7: Infected behavior within Ω with control (vaccine strategy starts from the first day)

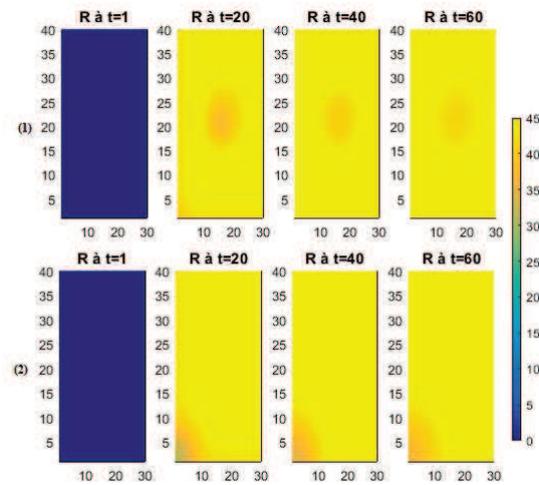


Figure 8: Recovered behavior within Ω with control (vaccine strategy starts from the first day)

7. Conclusion

In this contribution, we presented an SIR model in the form of a system of partial derivative equations with initial and boundary conditions. We have shown the existence of the solution of our state system and optimal control, so we have given a characterization of this control. Numerical simulation has proven the positive impact of our vaccination strategy. In fact Figures 1, 2 and 3 in the absence of the vaccine, have shown the spread of the disease in the entire domain, especially when the disease begins in the middle of the domain. However, when we introduced the vaccine, we observed that the number of infected individuals has decreased and the number of recovered has increased, which is very beneficial and reflects the importance of our control strategy. It should be noted that it is preferable to apply the vaccine during the first days of the onset of the disease, in order to block the spread in the population (see Figures 7 and 8).

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

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8. Appendix

First recall a general existence result which we use in the sequel (Proposition 1.2, p. 175, [16]; see also [17,18]). Consider the initial value problem

$$\begin{cases} \frac{\partial z}{\partial t} = Az(t) + g(t, z(t)), & t \in [0, T] \\ z(0) = z_0 \end{cases} \quad (8.1)$$

where A is a linear operator defined on a Banach space X , with the domain $D(A)$ and $g : [0, T] \times X \rightarrow X$ is a given function. If X is a Hilbert space endowed with the scalar product (\cdot, \cdot) , then the linear operator A is called dissipative if $(Az, z) \leq 0$, $(\forall z \in D(A))$.

Theorem 8.1. *X be a real Banach space, $A : D(A) \subseteq X \rightarrow X$ be the infinitesimal generator of a C_0 -semigroup of linear contractions $S(t)$, $t \geq 0$ on X , and $g : [0, T] \times X \rightarrow X$ be a function measurable in t and Lipschitz continuous in $x \in X$, uniformly with respect to $t \in [0, T]$.*

(i) *If $z_0 \in X$, then problem (8.1) admits a unique mild solution, i.e. a function $z \in C([0, T]; X)$ which verifies the equality $z(t) = S(t)z_0 + \int_0^t S(t-s)g(s, z(s))ds$, $(\forall t \in [0, T])$.*

(ii) *If X is a Hilbert space, A is self-adjoint and dissipative on X and $z_0 \in D(A)$, then the mild solution is in fact a strong solution and $z \in W^{1,2}([0, T]; X) \cap L^2(0, T; D(A))$*

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