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Certain Geometric Properties of the Generalized Dini Function $R_\nu^{a,k}(z)$

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abstract: In the present investigation we first introduce modified Dini function $R_\nu^{a,k}(z)$ and then find sufficient conditions so that the modified Dini function $R_\nu^{a,k}(z)$ have certain geometric properties like close-to-convexity, starlikeness and strongly starlikeness in the open unit disk. Some subordination sequences are also established.

Key Words: Analytic function, univalent function, starlike function, strongly starlike function, Dini function.

Contents

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions f defined in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ denote the subclass of $\mathcal{H}$, which are normalized by the condition $f(0) = 0 = f'(0) - 1$ and have representation of the form

$$f(z) = z + \sum_{n\geq 2} a_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

A function f is said to be univalent in a domain $\mathbb{D}$ if it is one-to-one in $\mathbb{D}$. Recall that a set $E \subset \mathbb{C}$ is said to starlike with respect to a origin $0 \in E$ if and only if the line segment joining 0 to every other point $w \in E$ lies entirely in E. A function $f \in \mathcal{A}$ is called starlike, denoted by $f \in \mathcal{S}^*$ if f is univalent in $\mathbb{D}$ and $f(\mathbb{D})$ is a starlike domain with respect to the origin. The analytic characterization of the class of starlike function is given by:

$$f \in \mathcal{S}^* \Leftrightarrow \Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \ (z \in \mathbb{D}).$$

A set E is said to be convex if and only if it is starlike with respect to each of its points, that is if and only if the line segement joining any two ponts of E lies entirely in E. A function $f \in \mathcal{A}$ is called convex, denoted by $f \in \mathcal{K}$ if f is univalent in $\mathbb{D}$ and $f(\mathbb{D})$ is a convex domain. The analytic characterization of the class of convex function is given by:

$$f \in \mathcal{K} \Leftrightarrow \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \ (z \in \mathbb{D}).$$

Further, let $\widetilde{\mathcal{S}}^*(\alpha),\ 0 < \alpha \leq 1$, be the class of strongly starlike functions of order α defined by

$$\widetilde{\mathcal{S}}^*(\alpha) = \left\{f \in \mathcal{A} : \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D}\right\}. \tag{1.2}$$

Note that $\widetilde{\mathcal{S}}^*(1) \equiv \mathcal{S}^*$. Given a convex function $g \in \mathcal{K}$ with $g(z) \neq 0$, a function $f \in \mathcal{A}$, is called close-to-convex with respect to convex function g, denoted by $\mathcal{C}_g$, if

$$\Re\left(\frac{f'(z)}{g'(z)}\right) > 0 \ (z \in \mathbb{D}). \tag{1.3}$$

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Geometrically a function $f \in \mathcal{A}$ belongs to $\mathcal{C}_g$ if the complement E of the image-region $F = \{f(z) : |z| < 1\}$ is the union of rays that are disjoint (except that the origin of one ray may lie on another one of the rays). The Noshiro-Warschawski theorem implies that close-to-convex functions are univalent in $\mathbb{D}$, but not necessarily the converse. More details about these classes can be found in Duren [7].

If $f, g \in \mathcal{H}$, then the function f is said to be subordinate to g, written as $f(z) \prec g(z)$ $(z \in \mathbb{D})$, if there exists a Schwarz function $w \in \mathcal{H}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{D})$ such that $f(z) = g(w(z))$. In particular, if g is univalent in $\mathbb{D}$, then we have the following equivalence:

$$f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}).$$

The Pochhammer (or Appell) symbol, defined in terms of Euler's gamma functions is given as $(x)_n = \Gamma(x+n)/\Gamma(x) = x(x+1)...(x+n-1)$.
It is always interesting to find sufficient conditions such that certain class of analytic functions becomes close-to-convex, starlike or convex function. Recently many special functions are studied for the above mentioned geometric properties. One can see the following papers in this direction, for Hypergeometric function [10,11,12,16,17,18], Bessel functions [2,3], Wright function [15], Mittag-Leffler function [1], Dini function [4,5,6]. In the present investigation, we are interested in some geometric properties of modified Dini function. For this we first define generalized Bessel function.

Bessel functions of the first kind play an important role in various branches of applied mathematics and engineering sciences. Their properties have been investigated by many scientists and there is a very extensive literature dealing with Bessel functions.

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+\nu}}{n!\Gamma(\nu+n+1)}. \tag{1.4}$$

The generalized Bessel function of first kind of order ν is defined by

$$J_\nu^c(z) = \sum_{n=0}^{\infty} \frac{(-c)^n (z/2)^{2n+\nu}}{n!\Gamma(\nu+n+1)} \; (c \in \mathbb{C},\ \nu > -1). \tag{1.5}$$

The generalized Dini function is the combination of the generalized Bessel function of first kind, defined by

$$d_\nu^{a,c}(z) = (a-\nu) J_\nu^c(z) + z (J_\nu^c)'(z).$$

For more details on the Dini functions see [4,5,6]. In the present paper we use the following normalized form of generalized Dini function:

$$\begin{aligned} r_\nu^{a,c}(z) &= \frac{2^\nu}{a}\Gamma(\nu+1) z^{1-\frac{\nu}{2}} \left[(a-\nu)J_\nu^c(\sqrt{z}) + \sqrt{z}(J_\nu^c)'(\sqrt{z})\right] \\ &= z + \sum_{n=1}^{\infty} \frac{(-c)^n (2n+a)\Gamma(\nu+1)}{a.\ 4^n n!\Gamma(\nu+n+1)} z^{n+1}, \end{aligned} \tag{1.6}$$

$$(c \in \mathbb{C},\ \nu > -1,\ a > 0 \text{ and } z \in \mathbb{D}\ )$$

If we take $c = -k$ where $k > 0$, we get modified Dini function, let us represent this series by:

$$R_\nu^{a,k}(z) = z + \sum_{n=1}^{\infty} \frac{k^n (2n+a)\Gamma(\nu+1)}{a.\ 4^n n!\Gamma(\nu+n+1)} z^{n+1}. \tag{1.7}$$

To prove our main results, we shall need following Lemmas and Definition:

Lemma 1.1. (Ozaki [14]). *Let* $f(z) = z + \sum_{n=2}^{\infty} A_n z^n$. *Suppose*

$$1 \geq 2A_2 \geq \cdots \geq nA_n \geq (n+1)A_{n+1} \geq \cdots \geq 0 \tag{1.8}$$

or

$$1 \leq 2A_2 \leq \cdots \leq nA_n \leq (n+1)A_{n+1} \leq \cdots \leq 2. \tag{1.9}$$

then f is close-to-convex with respect to convex function $-log(1-z)$ in $\mathbb{D}$.

Lemma 1.2. (Fejer [8]). *Let $\{a_n\}_{n\geq 1}$ be a sequence of non negative real numbers such that $a_1 = 1$. If the quantities*

$$\underline{\Delta} a_n = na_n - (n+1)a_{n+1} \quad \text{and} \quad \underline{\Delta} a_n^2 = na_n - 2(n+1)a_{n+1} + (n+2)a_{n+2}$$

are non negative, then the function $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is starlike in $\mathbb{D}$.

Lemma 1.3. (Fejer [8]). *Let $\{a_n\}_{n\geq 1}$ be a sequence of non negative real numbers such that $a_1 = 1$. If $\{a_n\}_{n\geq 2}$ is convex decreasing, i.e. $0 \geq a_{n+2} - a_{n+1} \geq a_{n+1} - a_n$, then*

$$\Re\left\{\sum_{n=1}^{\infty} a_n z^{n-1}\right\} > \frac{1}{2}, \quad (z \in \mathbb{D}).$$

Definition 1.4. *An infinite sequence $\{b_n\}_1^{\infty}$ of complex numbers will be called a subordinating factor sequence if whenever*

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \tag{1.10}$$

is analytic, univalent and convex in $\mathbb{U}$, then

$$\sum_{n=1}^{\infty} a_n b_n z^n \subseteq f(z) \; (z \in \mathbb{D}, a_1 = 1). \tag{1.11}$$

Lemma 1.5. (Wilf [19]). *The sequence $\{b_n\}_1^{\infty}$ is a subordinating factor sequence if and only if*

$$\Re\left\{1 + 2\sum_{k=1}^{\infty} b_k z^k\right\} > 0 \; (z \in \mathbb{D}). \tag{1.12}$$

Lemma 1.6. (P.T. Mocanu [13]). *If $f \in \mathcal{A}$ satisfy $|f'(z) - 1| < 1$ for each $z \in \mathbb{D}$, then f is convex in $\mathbb{D}_{1/2} = \{z : \; |z| < \frac{1}{2}\}$.*

Lemma 1.7. (Halenbeck and Ruscheweyh [9]). *Let $G(z)$ be convex and univalent in $\mathbb{D}$ with $G(0) = 1$. Let $F(z)$ be analytic in $\mathbb{D}$, $F(0) = 1$ and $F(z) \prec G(z)$ in $\mathbb{D}$. Then for all $n \in \mathbb{N} \cup \{0\}$, we have*

$$(n+1)z^{-n-1}\int_0^z t^n\, F(t)\, dt \;\prec\; (n+1)z^{-n-1}\int_0^z t^n\, G(t)\, dt.$$

2. Close-to-convexity, Starlikeness and Strongly starlikeness

Theorem 2.1. *If $\nu \geq (\frac{1}{a} + \frac{1}{2})k - 1$, $a > 0$ and $k > 0$, then $R_\nu^{a,k}(z)$ is close-to-convex with respect to convex function $-log(1-z)$ in $\mathbb{D}$.*

Proof. From (1.7), we have

$$\underline{\Delta} a_n = na_n - (n+1)a_{n+1} = \frac{k^{n-1}\Gamma(\nu+1)}{4^{n-1}(n-1)!\Gamma(\nu+n)}\left[n(2n+a-2) - \frac{k(n+1)(2n+a)}{4n(\nu+n)}\right] \tag{2.1}$$

$$= \frac{k^{n-1}\Gamma(\nu+1)}{a.\; 4^n n!\Gamma(\nu+n+1)}\phi(n),$$

where

$$\phi(n) = 4n^2(2n+a-2)(\nu+n) - k(2n+a)(n+1).$$

In view of Lemma 1.1, it is sufficient to show that under the stated conditions, $\phi(n) \geq 0$ for all $n \geq 1$. For this we use the inequality

$$4n^2(2n+a-2) \geq \left(\frac{2a}{a+2}\right)(2n+a)(n+1)\ (n \in \mathbb{N}, a>0). \tag{2.2}$$

Now

$$\begin{aligned}\phi(n) &\geq 4n^2(2n+a-2)(\nu+1) - k(2n+a)(n+1)\\ &\geq \left(\frac{2a}{a+2}\right)(2n+a)(n+1)(\nu+1) - k(2n+a)(n+1)\\ &\geq 0 \quad as\ (\nu+1) \geq \left(\frac{1}{a}+\frac{1}{2}\right)k.\end{aligned}$$

□

Theorem 2.2. *If $\nu \geq \left(\frac{2}{a}+1\right)k-1$, $a>0$ and $k>0$, then $R_\nu^{a,k}(z)$ is starlike in $\mathbb{D}$.*

Proof. From (1.7), we have

$$\underline{\Delta} a_n^2 = na_n - 2(n+1)a_{n+1} + (n+2)a_{n+2}$$

$$= \frac{n(2n+a-2)k^{n-1}\Gamma(\nu+1)}{a.\ 4^{n-1}(n-1)!\Gamma(\nu+n)} - 2\frac{(n+1)(2n+a)k^n\Gamma(\nu+1)}{a.\ 4^n n!\Gamma(\nu+n+1)} + (n+2)\frac{(2n+a+2)k^{n+1}\Gamma(\nu+1)}{a.\ 4^{n+1}(n+1)!\Gamma(\nu+n+2)}.$$

To show $\underline{\Delta} a_n^2$ is positive, we show that, the difference of first two term, which is equal to

$$\frac{k^{n-1}\Gamma(\nu+1)}{a.\ 4^{n-1}(n-1)!\Gamma(\nu+n)}\left[n(2n+a-2) - \frac{k(n+1)(2n+a)}{2n(\nu+n)}\right] = \frac{k^{n-1}\Gamma(\nu+1)}{2a.\ \ 4^{n-1}\ n!\Gamma(\nu+n+1)}\psi(n),$$

where

$$\psi(n) = 2n^2(2n+a-2)(\nu+n) - (n+1)(2n+a)k,$$

is positive under the stated condition. In view of Lemma 1.2, it is sufficient to show that under the stated conditions, $\psi(n) \geq 0$ for all $n \geq 1$. For this we use the inequality

$$2n^2(2n+a-2) \geq \left(\frac{a}{2+a}\right)(n+1)(2n+a) \quad (n \in \mathbb{N}, a>0).$$

Now

$$\begin{aligned}\psi(n) &= 2n^2(2n+a-2)(\nu+n) - (n+1)(2n+a)k\\ &\geq 2n^2(2n+a-2)(\nu+1) - (n+1)(2n+a)k\\ &\geq (n+1)(2n+a)\left[\left(\frac{a}{2+a}\right)(\nu+1) - k\right]\\ &\geq 0 \quad as\ (\nu+1) \geq \left(\frac{a+2}{a}\right)k.\end{aligned}$$

□

Theorem 2.3. *If $\nu \geq \frac{(4+a)k}{4(2+a)} - 2$, $a>0$ and $k>0$, then*

$$\Re\left\{\frac{R_\nu^{a,k}(z)}{z}\right\} > \frac{1}{2}\ (z \in \mathbb{D}).$$

Proof. We first prove that

$$\{a_n\}_{n=1}^{\infty} = \left\{ \frac{k^{n-1}(2n+a-2)\Gamma(\nu+1)}{a.\ 4^{n-1}(n-1)!\Gamma(\nu+n)} \right\}_{n\geq 2}$$

is a decreasing sequence. For this we calculate

$$a_n - a_{n+1} = \frac{k^{n-1}\Gamma(\nu+1)}{a.\ 4^n n!\Gamma(\nu+n+1)} X(n),$$

where

$$X(n) = 4n(2n+a-2)(\nu+n) - (2n+a)k.$$

To show that $X(n) \geq 0$ for all $n \geq 2$, we use the inequality

$$4n(2n+a-2) \geq \frac{8(2+a)}{4+a}(2n+a) \ \ (n \geq 2, a > 0). \tag{2.3}$$

Now,

$$\begin{aligned} X(n) &= 4n(2n+a-2)(\nu+n) - k(2n+a) \\ &\geq 4n(2n+a-2)(\nu+2) - k(2n+a) \\ &\geq (2n+a)\left[\frac{8(2+a)}{4+a}(\nu+2) - k\right] \\ &\geq 0 \quad as \ (\nu+2) \geq \frac{(4+a)k}{4(2+a)}. \end{aligned}$$

Next, we prove that $\{a_n\}_{n\geq 2}$ is a convex decreasing sequence. For this, we show

$$a_{n+2} - a_{n+1} \geq a_{n+1} - a_n \ (\forall\ n \geq 2).$$

To show $a_n - 2a_{n+1} + a_{n+2}$ is positive, we show that, the difference of first two term, which is equal to

$$a_n - 2a_{n+1} = \frac{k^{n-1}\Gamma(\nu+1)}{2a.\ 4^{n-1}n!\Gamma(\nu+n+1)} Y(n), \tag{2.4}$$

where

$$Y(n) = 2n(2n+a-2)(\nu+n) - k(2n+a).$$

To show $Y(n) \geq 0$ for all $n \geq 2$, we use the inequality

$$2n(2n+a-2) \geq \frac{4(2+a)}{4+a}(2n+a) \ (n \geq 2, a > 0). \tag{2.5}$$

Now,

$$\begin{aligned} Y(n) &= 2n(2n+a-2)(\nu+n) - k(2n+a) \\ &\geq \frac{4(2+a)}{4+a}(2n+a)(\nu+2) - k(2n+a) \\ &= (2n+a)\left[\frac{4(2+a)}{4+a}(\nu+2) - k\right] \geq 0 \quad as \ (\nu+2) \geq \frac{4+a}{4(2+a)}. \end{aligned}$$

In view of Lemma 1.3, we have

$$\Re\left\{\sum_{n=1}^{\infty} a_n z^{n-1}\right\} > \frac{1}{2} \ \ (z \in \mathbb{D}),$$

which is equivalent to

$$\Re\left\{\frac{R_\nu^{a,k}(z)}{z}\right\} > \frac{1}{2}, \quad (z \in \mathbb{D}).$$

This proves the Theorem 2.3. □

Corollary 2.4. *If $\nu \geq \frac{4+a}{4(2+a)} - 2$, $a > 0$ and $k > 0$, then the sequence*

$$\left\{ \frac{k^n (2n+a)\Gamma(\nu+1)}{a.\ 4^n\ n!\ \Gamma(\nu+n+1)} \right\}_{n=1}^{\infty} \tag{2.6}$$

is a subordinating factor sequence for the class $\mathcal{K}$.

Proof. Result directly follows in view of Theorem 2.3 and Lemma 1.5. □

Theorem 2.5. *If $\nu \geq \frac{3k}{8}\left(\frac{4+a}{2+a}\right) - 2$, $a > 0$ and $k > 0$ then*

$$\Re\left\{(R_\nu^{a,k}(z))'\right\} > \frac{1}{2}\ (z \in \mathbb{D}).$$

Proof. From (1.7),

$$(R_\nu^{a,k}(z))' = 1 + \sum_{n=2}^{\infty} \frac{k^{n-1} n(2n+a-2)\Gamma(\nu+1)}{a.\ 4^{n-1}\ (n-1)!\Gamma(\nu+n)} z^{n-1}. \tag{2.7}$$

So taking

$$a_n = \frac{k^{n-1} n(2n+a-2)\Gamma(\nu+1)}{a.\ 4^{n-1}\ (n-1)!\Gamma(\nu+n)}$$

and proceeding similarly as in Theorem 2.3, we get the proof. □

Corollary 2.6. *If $\nu \geq \frac{3}{8}\left(\frac{4+a}{2+a}\right)k - 2$ and $k, a > 0$, then*

$$\left\{ \frac{k^n (n+1)(2n+a)\Gamma(\nu+1)}{a.\ 4^n\ n!\Gamma(\nu+n+1)} \right\}_{n=1}^{\infty} \tag{2.8}$$

a subordinating factor sequence for the class $\mathcal{K}$.

Proof. Result directly follows in view of Theorem 2.5 and Lemma 1.5. □

Theorem 2.7. *If $\nu \geq 2|c| - 1$ and $a \geq 1$, then $r_\nu^{a,c}(z)$ is convex in $\mathbb{D}_{1/2}$.*

Proof. To prove this theorem, we use the well-known triangle inequality and the inequalities

$$a.\ 4^m \geq \frac{2}{3}\left[2m^2 + m(2+a) + a\right],\quad (\nu+1)_m \geq (\nu+1)^m,\quad m! \geq 2^{m-1}\quad (m \in \mathbb{N}),$$

we have,

$$\begin{aligned} |(r_\nu^{a,c})'(z) - 1| &= \sum_{m=1}^{\infty} \frac{\left[2m^2 + m(2+a) + a\right]|c|^m}{a.\ 4^m m!(v+1)_m}|z|^m \\ &\leq \frac{3}{2}\sum_{m=1}^{\infty} \frac{|c|^m}{2^{m-1}(\nu+1)^m} \\ &= \frac{3|c|}{2\nu+2-|c|} = \beta. \end{aligned} \tag{2.9}$$

Under the hypothesis, $0 < \beta \leq 1$, and in view of Lemma 1.6, $r_\nu^{a,c}(z)$ is convex in $\mathbb{D}_{1/2}$. □

Theorem 2.8. *If $\nu \geq 2|c| - 1$ and $a \geq 1$ then $r_\nu^{a,c}(z) \in \widetilde{\mathcal{S}}^*(\alpha)$, where*

$$\alpha = \frac{2}{\pi}\arcsin\left(\beta\sqrt{1 - \frac{\beta^2}{4}} + \frac{\beta}{2}\sqrt{1-\beta^2}\right), \tag{2.10}$$

for $\beta = \frac{3|c|}{2\nu+2-|c|}$.

Proof. In view of Theorem 2.7, we conclude that

$$(r_\nu^{a,c})'(z) \prec 1 + \beta z \quad z \in \mathbb{D}.$$

which gives

$$|\arg (r_\nu^{a,c})'(z)| < \arcsin \beta, \quad z \in \mathbb{D}. \tag{2.11}$$

Using Lemma 1.7, for $F(z) = (r_\nu^{a,c})'(z)$ and $G(z) = 1 + \beta z$ with $n = 0$, we get

$$\frac{r_\nu^{a,c}(z)}{z} \prec 1 + \frac{\beta}{2} z, \quad z \in \mathbb{D},$$

consequently

$$\left| \arg \left(\frac{r_\nu^{a,c}(z)}{z} \right) \right| < \arcsin \frac{\beta}{2}, \quad z \in \mathbb{D}. \tag{2.12}$$

Now from (2.11) and (2.12), we conclude that

$$\begin{aligned} \left| \arg \left(\frac{z (r_\nu^{a,c})'(z)}{r_\nu^{a,c}(z)} \right) \right| &= \left| \arg \left(\frac{z}{r_\nu^{a,c}(z)} \right) + \arg \left((r_\nu^{a,c})'(z) \right) \right| \\ &\le \left| \arg \left(\frac{z}{r_\nu^{a,c}(z)} \right) \right| + \left| \arg \left((r_\nu^{a,c})'(z) \right) \right| \\ &< \arcsin \frac{\beta}{2} + \arcsin \beta \\ &= \arcsin \left(\beta \sqrt{1 - \frac{\beta^2}{4}} + \frac{\beta}{2} \sqrt{1 - \beta^2} \right) \end{aligned}$$

i.e. $r_\nu^{a,c}(z) \in \widetilde{\mathcal{S}}^*(\alpha)$, for α given in (2.10). □

Corollary 2.9. *Let* $\nu \ge 2|c| - 1$, $0 < \alpha \le 1$, $a \ge 1$ *and*

$$\beta = \frac{3|c|}{2\nu + 2 - |c|} = 2l \sqrt{\frac{5 - 4\sqrt{1 - l^2}}{16 l^2 + 9}}, \tag{2.13}$$

where $l = \sin\left(\frac{\alpha\pi}{2}\right)$. *Then* $r_\nu^{a,c}(z) \in \widetilde{\mathcal{S}}^*(\alpha)$.

Proof. If we put β from (2.13) to (2.10), we obtain α. □

Putting $\alpha = 1$ in Corollary 2.9, we get

$$l = 1 \Rightarrow \beta = \frac{3|c|}{2\nu + 2 - |c|} = \frac{2}{\sqrt{5}}.$$

Corollary 2.10. *If*

$$\nu \ge \frac{|c|(3\sqrt{5} + 2) - 4}{4},$$

then $r_\nu^{a,c}(z) \in \mathcal{S}^*$.

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