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The Existence of Renormalized Solution for Quasilinear Parabolic Problem with Variable Exponents and Measure Data

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ABSTRACT: In this paper, the study of the existence of a renormalized solution for quasilinear parabolic problem with variable exponents and measure data. The model is:

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p(x)-2}\,\nabla u) + \lambda\,|u|^{p(x)-2}\,u = \mu & \text{in} \quad Q = \Omega\times]0,T[, \\ u = 0 & \text{on} \quad \Sigma = \partial\Omega\times]0,T[, \\ u(.,0) = u_0(.) & \text{in} \quad \Omega, \end{cases}$$

where $\lambda > 0$ and T is any positive constant, $u_0 \in L^1$ and for any measure with bounded total variation over Q that do not charge the sets of zero p(.)-capacity.

Key Words: Quasilinear parabolic equations, variable exponent, renormalized solutions, measures.

Contents

1. Introduction

Variable-exponent Lebesgue and Sobolev spaces are the natural extensions of the classical constant exponent Lp-spaces. This kind of theory finds many applications, for example in nonlinear elastic mechanics, electrorheological fluids dynamics, and image restoration etc. We refer the readers to [10].
In the classical case ($p(.) = 2$ or $p(.) = p$ (a constant)), we recall that the notion of renormalized solutions was introduced by Di Perna and Lions [11] in their study of the Boltzmann equation. It has been studied by many authors under various conditions on the data the existence and uniqueness of the renormalized solution for parabolic equations with measure data in the classical Sobolev spaces (see [5], [7], [17] and [20]). In Sobolev space with variable exponents, the authors in [9] have proved the existence of entropy and renormalized solutions for strongly nonlinear elliptic equations in the framework of Sobolev spaces with variable exponents and in 2014, Chao zhang [22] provides the existence and uniqueness of entrpy solution for $p(x)$-Laplace equations with a Radon measure which is absolutely continuous with respect to the relative $p(x)$-capacity. The corresponding parabolic case equations in [12] have proved the existence of renormalized solutions for a class of nonlinear parabolic systems with variable exponents and, for the corresponding parabolic equations with L^1 data, the authors in [8] have proved the existence and uniqueness of renormalized solution to nonlinear parabolic equations with variable exponents and L^1 data. Chao Zhang and Shulin Zhou in [23] proved the existence and uniqueness results renormalized solutions and entropy solutions for nonlinear parabolic equations with variable exponents and L^1 data.
The purpose of this article is to study the existence of renormalized solutions u to the quasilinear parabolic problem involving the $p(x)$-Laplacian type operator

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$$\begin{cases} u_t - \mathrm{div}\mathcal{A}(x,t,\nabla u) + \mathcal{B}(u) = \mu & \text{in} \quad Q = \Omega \times]0,T[, \\ u = 0 & \text{on} \quad \Sigma = \partial\Omega \times]0,T[, \\ u(.,0) = u_0(.) & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where Ω be a bounded-connected domain of $\mathbb{R}^N$, $(N \geq 2)$ with lipshitz boundary $\partial\Omega$ and $Q = \Omega \times]0,T[$ for any fixed $T > 0$. Let $p : \overline{\Omega} \longrightarrow [1,+\infty)$ be a continuous real-valued function, let $p^- = \min_{x\in\overline{\Omega}} p(x)$ and $p^+ = \max_{x\in\overline{\Omega}} p(x)$ with $1 < p^- \leq p^+ < N$. The operator $-\mathrm{div}\mathcal{A}(x,t,\nabla u) = -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is a Leary lions operator (see assumption (4.1)-(4.3)), and $\mathcal{B} : \mathbb{R} \to \mathbb{R}$ with $\mathcal{B}(u) = \lambda |u|^{p(x)-2} u$ is as continuous increasing function for $\lambda > 0$ and $\mathcal{B}(0) = 0$.
The existence of renormalized solution for quasilinear parabolic problem with variable exponents and measure data of (1.1), prove in clasical Sobolev space [20] in case where $u = b(u)$, $b(u_0) \in L^1(\Omega)$ with $b : \mathbb{R} \to \mathbb{R}$ is a increasing C^1-function and $b(0) = 0$, for every μ is diffuse measure and also the purpose of this paper is to extend the results in [1] to the case of parabolic equations.
In the following section 2 is to recall some basic notations and properties of variable exponent Lebesgue-Sobolev space. Section 3 is to introduce some basic knowledge on $p(.)$-parabolic capacity and properties of measures. Section 4, is devoted to set the main assumption, to the definition of renormalized solutions of (1.1). Section 5 is prove that the formulation of renormalized solution does not depend on the decomposition of μ. Finally, to prove the main result of this paper (Theorem (5.1)), on the existence of a renormalized solution.

2. The Functional Spaces

We recall some defnitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(.)}(\Omega)$, $W^{1,p(.)}(\Omega)$ and $W_0^{1,p(.)}(\Omega)$, where Ω is an open set of $\mathbb{R}^N$. To refer to Fan and Zhao [13] for further properties of Lebesgue-Sobolev spaces with variable exponents. Let $p : \overline{\Omega} \longrightarrow [1,+\infty)$ be a continuous real-valued function, let $p^- = \min_{x\in\overline{\Omega}} p(x)$ and $p^+ = \max_{x\in\overline{\Omega}} p(x)$ with $1 < p(.) < N$. To denote the Lebesgue space with variable exponent $L^{p(.)}(\Omega)$ as the set of all measurable function $u : \Omega \longrightarrow \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u) = \int_\Omega |u|^{p(x)}\, dx, \tag{2.1}$$

is finite. If the exponent is bounded, i.e, if $p^+ < +\infty$, then the expression

$$\|u\|_{L^{p(.)}(\Omega)} = \inf\left\{\mu > 0; \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\}, \tag{2.2}$$

defines a norm in $L^{p(.)}(\Omega)$ called the Luxemburg norm. The space $(L^{p(.)}(\Omega); \|.\|_{p(.)})$ is a separable Banach space. Moreover, if $1 < p^- \leq p^+ < +\infty$, then $L^{p(.)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p'(.)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, for $x \in \Omega$. The following inequality will be used later:

$$\begin{aligned} &\min\left\{\|u\|^{p^-}_{L^{p(.)}(\Omega)}, \|u\|^{p^+}_{L^{p(.)}(\Omega)}\right\} \\ \leq &\int_\Omega |u(x)|^{p(x)}\, dx \leq \max\left\{\|u\|^{p^-}_{L^{p(.)}(\Omega)}, \|u\|^{p^+}_{L^{p(.)}(\Omega)}\right\}. \end{aligned} \tag{2.3}$$

Finally, the Hölder type inequality

$$\left|\int_\Omega uv dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p^+}\right) \|u\|_{p(.)} \|v\|_{p'(.)}, \tag{2.4}$$

for all u$\in L^{p(.)}(\Omega)$ and v $\in L^{p'(.)}(\Omega)$. Let

$$W^{1,p(.)}(\Omega)=\left\{u\in L^{p(.)}(\Omega);|\nabla u|\in L^{p(.)}(\Omega)\right\}, \tag{2.5}$$

which is Banach space equipped with the following norm

$$\|u\|_{_{1,p(.)}}=\|u\|_{_{p(.)}}+\|\nabla u\|_{_{p(.)}}. \tag{2.6}$$

The space $(W^{1,p(.)}(\Omega);\|.\|_{1,p(.)})$ is a separable and reflexive Banach space. The manipulation of the generalized Lebesgue and Sobolev spaces is plays and important role using the modular $\rho_{p(.)}$ of the space $L^{p(.)}(\Omega)$. The results as follows:

Proposition 2.1. *If $u_n,u\in L^{p(.)}(\Omega)$ and $p+<+\infty$, the following properties hold true.*

(i) $\|u\|_{_{p(.)}}>1\Longrightarrow\|u\|^{p+}_{_{p(.)}}<\rho_{p(.)}(u)<\|u\|^{p-}_{_{p(.)}}$,
(ii) $\|u\|_{_{p(.)}}<1\Longrightarrow\|u\|^{p-}_{_{p(.)}}<\rho_{p(.)}(u)<\|u\|^{p+}_{_{p(.)}}$,
(iii) $\|u\|_{_{p(.)}}<1$ *(respectively* $=1;>1$*)*$\Longleftrightarrow\rho_{p(.)}(u)<1$ *(respectively* $=1;>1$*),*
(iv) $\|u_n\|_{_{p(.)}}\longrightarrow 0$ *(respectively* $\longrightarrow+\infty$*)*$\Longleftrightarrow\rho_{p(.)}(u_n)<1$*(respectively* $\longrightarrow+\infty$*),*
(v) $\rho_{p(.)}\left(\frac{u}{\|u\|_{_{p(.)}}}\right)=1$.
For a measurable function $u:\Omega\longrightarrow\mathbb{R}$, we introduce the following notation

$$\rho_{1,p(.)}=\int_\Omega|u|^{p(x)}\,dx+\int_\Omega|\nabla u|^{p(x)}\,dx.$$

Proposition 2.2. *If $u\in W^{1,p(.)}(\Omega)$ and $p+<+\infty$, the following properties hold true.*

(i)$\|u\|_{_{1,p(.)}}>1\Longrightarrow\|u\|^{p+}_{_{1,p(.)}}<\rho_{1,p(.)}(u)<\|u\|^{p-}_{_{1,p(.)}}$,
(ii)$\|u\|_{_{1,p(.)}}<1\Longrightarrow\|u\|^{p-}_{_{1,p(.)}}<\rho_{1,p(.)}(u)<\|u\|^{p+}_{_{1,p(.)}}$,
(iii)$\|u\|_{_{1,p(.)}}<1$ *(respectively* $=1;>1$*)*$\Longleftrightarrow\rho_{1,p(.)}(u)<1$ *(respectively* $=1;>1$*).*

Extending a variable exponent $p:\overline{\Omega}\longrightarrow[1,+\infty)$ to $\overline{Q}=\overline{\Omega}\times[0,T]$ by setting $p(x,t)=p(x)$ for all $(x,t)\in\overline{Q}$. We may also consider the generalized Lebesgue space

$$L^{p(.)}(Q)=\left\{u:Q\longrightarrow\mathbb{R}\text{ mesurable such that}\int_Q|u(x,t)|^{p(x)}\,d(x,t)<\infty\right\},$$

endowed with the norm

$$\|u\|_{L^{p(.)}(Q)}=\inf\left\{\mu>0;\int_Q\left|\frac{u(x,t)}{\mu}\right|^{p(x)}d(x,t)\leq 1\right\},$$

which share the same properties as $L^{p(.)}(\Omega)$.

3. The Inportance of Parabolic Capacity And Measures

3.1. The Parabolic Capacity

The relevant notion in the study of problems as (1.1) is the notion of parabolic $p(.)$-capacity. Let $Q=\Omega\times]0,T[$ for any fixed $T>0$. We recall that for every $p>1$ and every open subset $U\subset Q$, the $p(.)$-parabolic capacity of U is given by (see [16])

$$cap_{p(.)}(U)=\inf\left\{\|u\|_{W_{p(.)}(0,T)}:u\in W_{p(.)}(0,T),u\geq\mathcal{X}_U\text{ a.e in }Q\right\},$$

where

$$W_{p(.)}(0,T)=\left\{u\in L^{p-}(0,T;V),\nabla u\in (L^{p(.)}(Q)^{\mathbb{N}},u_t\in L^{(p-)'}(0,T;V')\right\},$$

being $V=W_0^{1,p(.)}(\Omega)\cap L^2(\Omega)$ and V' its dual space. As usual $W_{p(.)}(0,T)$ is endowed with the norm

$$\|u\|_{W_{p(.)}(0,T)}=\|u\|_{L^{p-}(0,T;V)}+\|\nabla u\|_{(L^{p(.)}(Q))^{\mathbb{N}}}+\|u_t\|_{L^{(p-)'}(0,T;V')}.$$

The $p(.)$-parabolic capacity *capp* is then extended to arbitrary Borel subsets $B\subseteq Q$ as

$$cap_{p(.)}(B)=\inf\left\{cap_{p(.)}(U): B\subseteq U \text{ and } U\subset\ Q \text{ is open }\right\}.$$

3.2. The Measures

Let $\mathcal{M}(Q)$ denotes the set of all Radon measures with bounded variation on Q. Moreover, as already mentioned, by $\mathcal{M}_0(Q)$ we will denote the set of all measures with bounded total variation over Q that do not charge the sets of zero $p(.)$-capacity, that is, if $\mu\in\mathcal{M}_0(Q)$, then $\mu(E)=0$ for every Borel set $E\subset Q$ such that $cap_{p(.)}(E)=0$.

In [16] the authors also proved the following decomposition theorem:

Theorem 3.1. *Let μ be a bounded measure on Q. If $\mu\in\mathcal{M}_0(Q)$, then there exists $(f;F;g_1;g_2)$ such that $f\in L^1(Q)$, $F\in(L^{p'(.)}(Q))^{\mathbb{N}}$, $g_1\in L^{(p-)'}(0,T;W^{-1,p'(.)}(\Omega))$, $g_2\in L^{(p-)}(0,T;V)$ and*

$$\int_Q \varphi d\mu=\int_Q f dxdt+\int_Q F\nabla\varphi dxdt+\int_0^T\langle g_1,\varphi\rangle\, dt-\int_0^T\langle\varphi_t,g_2\rangle\, dt, \tag{3.1}$$

for any $\varphi\in C_c^\infty([0,T]\times\Omega)$. Such a triplet (f,F,g_1,g_2) will be called a decomposition of μ.

Note that the decomposition of μ is not uniquely determined.
In the proof of that result the density will be used as an argument, and so the following preliminary result can be found, for instance, in [16].

Proposition 3.2. *Let $\mu\in M_0(Q)$. Then there exists a decomposition $(f;F;div(G);g)$ of μ in the sense of Theorem (3.1) and an approximation μ^ε of μ satisfying the following conditions:*

$$\mu^\varepsilon\in C_c^\infty(Q);\qquad \|\mu^\varepsilon\|_{L^1(Q)}\le C, \tag{3.2}$$

$$\begin{aligned}\int_Q\mu^\varepsilon\varphi dxdt&=\int_Q f^\varepsilon\varphi dxdt+\int_Q F^\varepsilon\nabla\varphi dxdt+\int_0^T\langle div(G^\varepsilon),\varphi\rangle\, dt\\ &-\int_0^T\langle\varphi_t,g^\varepsilon\rangle\, dt,\quad\forall\varphi\in C_c^\infty(Q),\end{aligned} \tag{3.3}$$

and

$$\begin{array}{lll} f^\varepsilon\in C_c^\infty(Q): & f^\varepsilon\to f & L^1(Q),\\ F^\varepsilon\in (C_c^\infty(Q))^N: & F^\varepsilon\to F & \left(L^{p'(.)}(Q)\right)^N,\\ G^\varepsilon\in (C_c^\infty(Q))^N: & G^\varepsilon\to G & \left(L^{p'(.)}(Q)\right)^N,\\ g^\varepsilon\in C_c^\infty(Q): & g^\varepsilon\to g & L^{(p-)}(]0,T[;V),\end{array}$$

as $\varepsilon\to 0$.

Here are some notations that be used throughout the paper. For any nonnegative real number k denoted by $T_k(r) = \min(k; \max(r; -k))$ the truncation function at level k. By using $\langle ., . \rangle$ mean the duality between suitable spaces in which functions are involved. In particular to consider both the duality between $W_0^{1,p(.)}(\Omega)$ and $W^{-1,p'(.)}(\Omega)$ and the duality between $W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ and $W^{-1,p'(.)}(\Omega) + L^1(\Omega)$. Note that the formulation of a renormalized solution does not depend on the decomposition of μ. The proof of this fact relies on the following result.

Lemma 3.3. *Let $\mu \in \mathcal{M}_0(Q)$, and let $(f; F; g_1; g_2)$ and $(\widetilde{f}; \widetilde{F}; \widetilde{g_1}; \widetilde{g_2})$ to be two different decompositions of μ according to Theorem (3.1). Then we have $(g_2 - \widetilde{g_2})_t = \widetilde{f} - f + \widetilde{F} - F + \widetilde{g_1} - g_1$ in distributional sense, $g_2 - \widetilde{g_2} \in C(]0, T[; L^1(\Omega))$ and $(g_2 - \widetilde{g_2})(0) = 0$.*

Proof. See [16], Lemma 4.6. □

4. The Main Assumptions And The Definition of Renormalized Solution

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$, $T > 0$ is given and we set $Q = \Omega \times]0, T[$, and $\mathcal{A} : Q \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function, such that for all $\xi, \eta \in \mathbb{R}^N, \xi \neq \eta$

$$\mathcal{A}(x, t, \xi).\xi \geqslant \alpha |\xi|^{p(x)}, \tag{4.1}$$

$$|\mathcal{A}(x, t, \xi)| \leqslant \beta \left[b(x, t) + |\xi|^{p(x)-1} \right], \tag{4.2}$$

$$(\mathcal{A}(x, t, \xi) - \mathcal{A}(x, t, \eta)).(\xi - \eta) > 0, \tag{4.3}$$

$$\mu \in \mathcal{M}_0(Q), \tag{4.4}$$

$$u_0 \text{ is a measurable function in } \Omega, \text{ such that } u_0 \in L^1(\Omega). \tag{4.5}$$

Where $1 < p^- \leq p^+ < +\infty$, α, β are positives constants and b is a nonnegative function in $L^{p'(.)}(Q)$. And $\mathcal{B} : \mathbb{R} \to \mathbb{R}$ is a continuous increasing function with $\mathcal{B}(0) = 0$.
The definition of a renormalized solution for Problem (1.1) can be stated as follows.

Definition 4.1. *Let $\mu \in M_0(Q)$ and $2 - \dfrac{1}{N+1} < p^- \leq p^+ < N$, let $u_0 \in L^1(\Omega)$, $(f; F; div(G); g)$ a decomposition of μ. A measurable function u defined on Q is a renormalized solution of problem (1.1) if :*

$$T_k(u - g) \in L^{p^-}(]0, T[; W_0^{1,p(.)}(\Omega)) \text{ for any } k > 0\ ,\ \mathcal{B}(u) \in L^1(Q), \tag{4.6}$$

$$\text{and } v{=}u - g \in L^\infty\left(]0, T[; L^1(\Omega)\right) \cap L^{q^-}(]0, T[; W_0^{1,q(.)}(\Omega)), \tag{4.7}$$

for all continuous functions $q(x)$ on $\overline{\Omega}$ satisfying $q(x) \in \left[1, p(x) - \frac{N}{N+1}\right)$ for all $x \in \overline{\Omega}$,

$$\lim_{n\to\infty} \int_{\{n \leq |u-g| \leq n+1\}} |\nabla u|^{p(x)} dxdt = 0, \tag{4.8}$$

and if, for every function $S \in W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has compact support on $\mathbb{R}$, so

$$\begin{aligned} &(S(v))_t - div(\mathcal{A}(x, t, \nabla u)S'(v)) + S''(v)\mathcal{A}(x, t, \nabla u)\nabla v \\ &+\mathcal{B}(u)S'(v) = fS'(v) + FS'(v) + GS''(v)\nabla v \\ &-div(GS'(v)) \text{ in } \mathcal{D}'(Q), \end{aligned} \tag{4.9}$$

$$S(v)(t = 0) = S(u_0) \text{ in } \Omega. \tag{4.10}$$

The following are explained as shown below on definition (4.1).

Remark 4.2. *Note that, all terms in (4.9) are well defined. Indeed, let $k > 0$ such that $supp(S') \subset [K, K]$, let $S(u-g) = S(T_k(u-g)) \in L^{p^-}(]0,T[; W_0^{1;p(.)}(\Omega))$ and $\frac{\partial S(u-g)}{\partial t} \in \mathcal{D}'(Q)$. The term $S'(u-g)\mathcal{A}(x,t,\nabla u)$ identifes with $S'(T_k(v)+g)\mathcal{A}(x,t,\nabla(T_k(v)+g))$ a.e. in Q, where $v = u-g$ and $u = T_k(v)+g$ in $\{|u-g| \leq k\}$, assumptions (4.2) imply that*

$$\begin{aligned} & |S'(T_k(u-g))\mathcal{A}(x,t,\nabla u)| \\ & \leq \beta \left\|S'\right\|_{L^\infty(\mathbb{R})} \left[b(x,t) + |\nabla(T_k(v)+g)|^{p(x)-1}\right] \ a.e \ in \ Q. \end{aligned} \tag{4.11}$$

Using (4.2) and (4.6), it follows that $S'(u-g)\mathcal{A}(x,t,\nabla u) \in (L^{p'(.)}(Q))^N$. The term $S''(u-g)\mathcal{A}(x,t,\nabla u)\nabla(u-g)$ identifes with $S''(u-g)\mathcal{A}(x,t,\nabla(T_k(v)+g))\nabla T_k(u-g)$ and in view of (4.2), (4.6) and (4.11), to obtain $S''(u-g)\mathcal{A}(x,t,\nabla u)\nabla(u-g) \in L^1(Q)$ and $S'(u-g)\mathcal{B}(u) \in L^1(Q)$. Finally $f\ S'(u-g)$ and $GS''(u-g)\nabla T_k(u-g)$ belongs to $L^1(Q)$ and $FS' \in (L^{p'(.)}(Q))^N$ and $GS'(u-g) \in (L^{p'(.)}(Q))^N$ in view of (4.6) and because S' is a bounded function on $\mathbb{R}$. Also $\frac{\partial S(u-g)}{\partial t} \in L^{(p^-)'}(]0,T[; W^{-1,p'(.)}(\Omega)) + L^1(Q)$ and $S(u-g) \in L^{p^-}(]0,T[; W_0^{1,p(.)}(\Omega))$, which implies that $S(u-g) \in C(]0,T[; L^1(\Omega))$.

Let it first be proven that the formulation of renormalized solution does not depend on the decomposition of μ. This fact essentially relies on Lemma (3.3).

Proposition 4.3. *Let u be a renormalized solution of (1.1). Then u satisfies (4.6)-(4.10) for every decomposition $(f; F; div(G); g)$ of μ.*

Proof. Assume that u satisfies the conditions of Definition (4.1) for $(f; F; div(G); g)$, and let $(\widetilde{f}; \widetilde{F}; div(\widetilde{G}); \widetilde{g})$ be a different decomposition of μ. Note that since, by Lemma (3.3), then $g - \widetilde{g} \in C(]0,T[; L^1(\Omega))$ let u- $\widetilde{g} \in L^\infty(]0,T[; L^1(\Omega))$, hence it is almost everywhere finite. First of all to prove that $T_k(u-\widetilde{g}) \in L^{p^-}(]0,T[; W_0^{1,p(.)}(\Omega))$ for every $k > 0$.

Let us introduce a sequence of increasing $C^\infty(\mathbb{R})$-functions S_n such that, for any $n \geq 1$

$$\left\{ \begin{array}{c} S_n(r) = r \text{ if } |\mathrm{r}| \leq n, \\ \operatorname{supp}(S_n') \subset [-(n+1),(n+1)], \\ \|S_n''\|_{L^\infty(\mathbb{R})} \leq 1. \end{array} \right. \tag{4.12}$$

To choose as test function $T_k(S_n(u-g)+g-\widetilde{g})$ in (4.9) and use Lemma (3.3), to obtain

$$A + B + D + E = F + I + H + M. \tag{4.13}$$

Where

$$\begin{aligned} A &= \int_0^T \left\langle (S_n(u-g)+g-\widetilde{g})_t\, , T_k(S_n(u-g)+g-\widetilde{g}) \right\rangle dt, \\ B &= \int_Q S_n'(u-g)\mathcal{A}(x,t,\nabla u)\nabla T_k(S_n(u-g)+g-\widetilde{g})dxdt, \\ D &= -\int_Q S_n''(u-g)\mathcal{A}(x,t,\nabla u)\nabla(u-g)T_k(S_n(u-g)+g-\widetilde{g})dxdt, \\ E &= \int_Q S_n'(u-g)\mathcal{B}(u)T_k(S_n(u-g)+g-\widetilde{g})dxdt, \end{aligned}$$

$$\begin{aligned}
F &= \int_Q \left(\left(S_n'(u-g)-1\right)f+\widetilde{f}\right)T_k(S_n(u-g)+g-\widetilde{g})dxdt,\\
I &= \int_Q \left(\left(S_n'(u-g)-1\right)F+\widetilde{F}\right)\nabla T_k(S_n(u-g)+g-\widetilde{g})dxdt,\\
H &= \int_Q \left(\left(S_n'(u-g)-1\right)G+\widetilde{G}\right)\nabla T_k(S_n(u-g)+g-\widetilde{g})dxdt,\\
M &= \int_Q S_n''(u-g)G\nabla(u-g)T_k(S_n(u-g)+g-\widetilde{g})dxdt.
\end{aligned}$$

By initial condition (4.10) and Lemma (3.3), to obtain

$$A=\int_\Omega \Theta_k(S_n(u-g))(T)dx-\int_\Omega \Theta_k(S_n(u_0))dx, \tag{4.14}$$

where $\Theta_k(r)=\int_0^r T_k(s)ds$ is a positive Lipschitz continuous function. Using (4.14) and the definition (4.12) of S_n, leads to

$$A\geq -k\int_\Omega |u_0|\,dx,\quad \forall n\geq 1. \tag{4.15}$$

Let $E_k=\{(x,t)\in Q:|S_n(u-g)+g-\widetilde{g}|\leq k\}$, we have

$$\begin{aligned}
B &= \int_{E_k} |S_n'(u-g)|^2\,\mathcal{A}(x,t,\nabla u)\nabla u dxdt &(4.16)\\
&- \int_{E_k} |S_n'(u-g)|^2\,\mathcal{A}(x,t,\nabla u)\nabla g dxdt\\
&+ \int_{E_k} S_n'(u-g)\mathcal{A}(x,t,\nabla u)\nabla(g-\widetilde{g})dxdt\\
&&(4.17)\\
&= B_1+B_2+B_3,
\end{aligned}$$

the properties of S_n, and because $0<S_n'<1$ let $(S_n'(s))^{p^-}\leq S_n'(s)$, $(S_n'(s))^{(p^-)'}\leq S_n'(s)$, $S_n'(s)\leq S_n'(s)^2+\chi_{\{n\leq|s|\leq n+1\}}$, to obtain

$$\begin{aligned}
B_1 &\geq \alpha\int_{E_k} |S_n'(u-g)|^{p-}\,|\nabla u|^{p(x)}\,dxdt &(4.18)\\
&- \alpha\int_{\{n\leq|v|\leq n+1\}} |\nabla u|^{p(x)}dxdt.
\end{aligned}$$

Using (4.1), (4.2), and Young's inequality, to deduce that

$$\begin{aligned}
|B_2|+|B_3| &\leq \left[C\left(\frac{1}{\varepsilon}\right)\|c\|_{L^{p'(.)}}^{(p^-)'}+C\left(\frac{1}{\varepsilon}\right)\|\nabla g\|_{L^{p(.)}}^{p^-}\right. &(4.19)\\
&\left.+C\,\|\nabla(g-\widetilde{g})\|_{L^{p(.)}}^{p^-}\right]+\frac{\alpha}{2}\int_{E_k}|S_n'(u-g)|^{p-}\,|\nabla u|^{p(x)}\\
&+\int_{\{n\leq|u-g|\leq n+1\}}|\nabla u|^{p(x)}dxdt,
\end{aligned}$$

and,

$$|D|+|M| \leq C\left[\|b\|_{L^{p'(.)}}^{(p^-)'}+\|\nabla g\|_{L^{p(.)}}^{p^-}+\|G\|_{L^{p'(.)}}^{(p^-)'}\right] + C\int_{\{n\leq|u-g|\leq n+1\}}|\nabla u|^{p(x)}dxdt. \tag{4.20}$$

Using (4.1) and Young's inequality, to obtain

$$\begin{aligned}
&|E|+|F|+|H|+|I| \leq C\Big[\|\mathcal{B}(u)\|_{L^1}+\|f\|_{L^1}+\left\|\widetilde{f}\right\|_{L^1} \\
&+\int_{E_k}\frac{1}{p'(x)(\frac{\alpha}{2}p(x))^{\frac{p(x)}{p'(x)}}}|G|^{p'(x)}dxdt+\int_{E_k}\frac{1}{p'(x)(\frac{\alpha}{2}p(x))^{\frac{p(x)}{p'(x)}}}\left|\widetilde{G}\right|^{p'(x)}dxdt \\
&+\int_{E_k}\frac{1}{p(x)(\frac{\alpha}{2}p'^{\frac{p'(x)}{p(x)}}}|\nabla g|^{p(x)}dxdt+\int_{E_k}\frac{1}{p(x)(\frac{\alpha}{2}p'^{\frac{p'(x)}{p(x)}}}|\nabla \widetilde{g}|^{p(x)}dxdt \\
&+\int_{E_k}\frac{1}{p'(x)(\frac{\alpha}{2}p(x))^{\frac{p(x)}{p'(x)}}}|F|^{p'(x)}dxdt+\int_{E_k}\frac{1}{p'(x)(\frac{\alpha}{2}p(x))^{\frac{p(x)}{p'(x)}}}\left|\widetilde{F}\right|^{p'(x)}dxdt \\
&+\|u_0\|_{L^1}\Big]+\alpha\int_{E_k}|S_n'(u-g)|^{p^-}|\nabla u|^{p(x)}dxdt \\
&+C\int_{\{n\leq|u-g|\leq n+1\}}|\nabla u|^{p(x)}dxdt.
\end{aligned} \tag{4.21}$$

Using (4.13) to (4.21), we deduce that

$$\begin{aligned}
&\alpha\int_{E_k}|S_n'(u-g)|^{p^-}|\nabla u|^{p(x)} \leq \\
&C\Big[\|\mathcal{B}(u)\|_{L^1}+\|f\|_{L^1}+\left\|\widetilde{f}\right\|_{L^1}+\|G\|_{L^{p'(.)}}^{(p^-)'}+\left\|\widetilde{G}\right\|_{L^{p'(.)}}^{(p^-)'} \\
&+\|\nabla g\|_{L^{p(.)}}^{p^-}+\|\nabla\widetilde{g}\|_{L^{p(.)}}^{p^-}+\|F\|_{L^{p'(.)}}^{(p^-)'}+\left\|\widetilde{F}\right\|_{L^{p'(.)}}^{(p^-)'}+\|u_0\|_{L^1}\Big] \\
&+C\int_{\{n\leq|u-g|\leq n+1\}}|\nabla u|^{p(x)}dxdt.
\end{aligned} \tag{4.22}$$

Using the properties of S_n and the fact that $\widetilde{g}$ belongs to $L^{p^-}(]0,T[;W_0^{1,p(.)}(\Omega))$, the result that will be deduced from preceding inequality that, for all $n\geq 1$,

$$\int_Q \chi_{E_k}|\nabla(S_n(u-g))|^{p(x)}dxdt \leq C,$$

inequality (2.3) implies that

$$\begin{aligned}
&\int_0^T \chi_{E_k}\min\left\{\|\nabla(S_n(u-g))\|_{L^{p(x)}(\Omega)}^{p-},\|\nabla(S_n(u-g))\|_{L^{p(x)}(\Omega)}^{p+}\right\} \\
&\leq \int_Q \chi_{E_k}|\nabla(S_n(u-g))|^{p(x)}dxdt \leq C,
\end{aligned}$$

then is $S_n(u-g)$ is bounded in $L^{p-}(]0,T[;W_0^{1,p(.)}(\Omega))$. Since $\nabla(T_k(S_n(u-g)+g-\widetilde{g}))=\chi_{E_k}\,\nabla(S_n(u-g)+g-\widetilde{g})$ and since $g,\widetilde{g}\in L^{p^-}(]0,T[;W_0^{1,p(.)}(\Omega))$, this implies that $v_n=T_k(S_n(u-g)+g-\widetilde{g})$ is bounded in $L^{p^-}(]0,T[;W_0^{1,p(.)}(\Omega))$ and converges, up to a subsequence converge to v weakly in $L^{p-}(]0,T[;W_0^{1,p(.)}(\Omega))$, thus also in $\mathcal{D}'(Q)$; but $v_n\to T_k(u-\widetilde{g})$ a.e. in Q and is bounded by k, so that $v_n\to T_k(u-\widetilde{g})$ in $\mathcal{D}'(Q)$. Then $T_k(u-\widetilde{g})=v\in L^{p^-}(]0,T[;W_0^{1,p(.)}(\Omega))$, for all $k>0$. To prove that (4.8) holds true for $\widetilde{g}$. Using the admissible test function $\theta_h(S_n(u-g)+g-\widetilde{g})$ in (4.9) with $S=S_n$; $\theta_h(s)=T_{h+1}(s)-T_h(s)$, the coercive character (4.1) and the use of Young's inequality it possible to obtain that

$$\begin{aligned}
&\alpha\int_{F_n}\left|S_n'\left(u-g\right)\right|^2\left|\nabla u\right|^{p(x)}dxdt \qquad (4.23)\\
\leq\;& Ck\int_Q\left(\left|\mathcal{B}(u)\right|+|f|+\left|\widetilde{f}\right|\right)\theta_h\left(S_n\left(u-g\right)+g-\widetilde{g}\right)dxdt\\
&+\int_\Omega\overline{\theta_h}\left(S_n\left(u_0\right)\right)dx+\int_{F_n}\Bigg(\frac{p^+-1}{p^+}\left|F\right|^{p'(x)}+\frac{p^+-1}{p^+}\left|\widetilde{F}\right|^{p'(x)}\\
&+\frac{p^+-1}{p^+}\left|G\right|^{p'(x)}+\frac{p^+-1}{p^+}\left|\widetilde{G}\right|^{p'(x)}+\frac{p^+-1}{p^+}\left|b\right|^{p'(x)}\\
&+\frac{C_1}{p(x)(\frac{\alpha}{2}p(x))^{\frac{p'(x)}{p(x)}}}\left|\nabla g\right|^{p(x)}+\frac{C_1}{p(x)(\frac{\alpha}{2}p(x))^{\frac{p'(x)}{p(x)}}}\left|\nabla\widetilde{g}\right|^{p(x)}\Bigg)\\
&+C\int_{\{n\leq|u-g|\leq n+1\}}\left|\nabla u\right|^{p(x)}dxdt+\omega(n).
\end{aligned}$$

Where $F_n=\{h\leq|S_n(u-g)+g-\widetilde{g}|\leq h+1\}$. Taking the limit as n tends to $+\infty$ in (4.23), using (4.8) and the convergence of χ_{F_n} to $\chi_{\{h\leq|u-g|\leq h+1\}}$ shows that for any $h>0$.

$$\begin{aligned}
&\alpha\int_{\{h\leq|u-\widetilde{g}|\leq h+1\}}\left|\nabla u\right|^{p(x)}dxdt\leq\int_{\{|u_0|>h\}}\left|u_0\right|dx+\int_{\{|u-\widetilde{g}|\geq h\}}C\left(\left|\mathcal{B}(u)\right|+|f|+\left|\widetilde{f}\right|\right)dxdt+\\
&\int_{\{h\leq|u-\widetilde{g}|\leq h+1\}}C_2\left(\left|G\right|^{p'(x)}+\left|\widetilde{G}\right|^{p'(x)}+\left|F\right|^{p'(x)}+\left|\widetilde{F}\right|^{p'(x)}+\left|b\right|^{p'(x)}+\left|\nabla g\right|^{p(x)}+\left|\nabla\widetilde{g}\right|^{p(x)}\right)dxdt, \qquad (4.24)
\end{aligned}$$

which yields, as h tends to infinity (recall that $u-\widetilde{g}$ is almost everywhere finite),

$$\lim_{h\to\infty}\int_{\{h\leq|u-\widetilde{g}|\leq h+1\}}\left|\nabla u\right|^{p(x)}dxdt=0. \qquad (4.25)$$

In the following to prove that the renormalized equation (4.9) and the initial condition (4.10) hold with $\widetilde{g}$ as well. For every function S in $W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support and let $\varphi\in C_c^\infty(Q)$, we chose $S'(S_n(u-g)+g-\widetilde{g})$ as test function in (4.9) (with $S=S_n$ in

(4.9) and to Lemma (3.3), the result is:

$$\begin{aligned}
&\int_0^T \langle (S_n(u-g)+g-\widetilde{g})_t, S'(S_n(u-g)+g-\widetilde{g})\varphi\rangle\, dt \qquad (4.26)\\
&+\int_Q S_n'(u-g)\mathcal{A}(x,t,\nabla u)\nabla\varphi S'(S_n(u-g)+g-\widetilde{g})dxdt\\
&+\int_Q S_n'(u-g)\mathcal{A}(x,t,\nabla u)\nabla(S'(S_n(u-g)+g-\widetilde{g}))\varphi dxdt\\
&+\int_Q S_n''(u-g)\mathcal{A}(x,t,\nabla u)\nabla(u-g)S'(S_n(u-g)+g-\widetilde{g})\varphi dxdt\\
&+\int_Q S_n'(u-g)\mathcal{B}(u)S'(S_n(u-g)+g-\widetilde{g})\varphi dxdt\\
=\;&\int_Q \left((S_n'(u-g)-1)\,f+\widetilde{f}\right) S'(S_n(u-g)+g-\widetilde{g})\varphi dxdt\\
&+\int_Q \left((S_n'(u-g)-1)\,F+\widetilde{F}\right)\nabla\varphi S'(S_n(u-g)+g-\widetilde{g})dxdt\\
&+\int_Q \left((S_n'(u-g)-1)\,F+\widetilde{F}\right)\nabla(S'(S_n(u-g)+g-\widetilde{g}))\varphi dxdt\\
&+\int_Q \left((S_n'(u-g)-1)\,G+\widetilde{G}\right) S'(S_n(u-g)+g-\widetilde{g})\nabla\varphi dxdt\\
&+\int_Q \left((S_n'(u-g)-1)\,G+\widetilde{G}\right)\nabla\left(S'(S_n(u-g)+g-\widetilde{g})\right)\varphi dxdt\\
&+\int_Q S_n''(u-g)G\nabla(u-g)S'(S_n(u-g)+g-\widetilde{g})\varphi dxdt.
\end{aligned}$$

In what follows to pass to the limit as n tends to ∞ in each term of (4.26). In what follows $\omega(n)$ stands for any quantity that vanishes as n diverges. For the parabolic contribution in (4.26)

$$\begin{aligned}
&\int_0^T \langle (S_n(u-g)+g-\widetilde{g})_t\,, S'\left(S_n(u-g)+g-\widetilde{g}\right)\varphi\rangle\, dt \qquad (4.27)\\
&=\int_0^T \langle \left(S\left(S_n(u-g)+g-\widetilde{g}\right)\right)_t\,,\varphi\rangle\, dt\\
=\;&-\int_Q S\left(S_n(u-g)+g-\widetilde{g}\right)\varphi_t dxdt=\int_0^T \langle (S(u-\widetilde{g}))_t\,,\varphi\rangle\, dt+\omega(n).
\end{aligned}$$

Recall that, since $\operatorname{supp}(S')\subset[-k,k]$ and, $\operatorname{supp}\left(S_n'(u-g)S'S_n(u-g)+g-\widetilde{g}\right)\subset$ $\{|u-g|\le n+1, |u-\widetilde{g}|\le k\}$; then ∇u may be replaced by $\omega=\nabla\left(T_{k+1}\left(u-\widetilde{g}\right)+\widetilde{g}\right)\in\left(L^{p(\cdot)}\left(Q\right)\right)^N$ in all the terms of (4.26). Using the definition of S_n, ($S'_n \to 1$ and is bounded by1), we obtain

$$\begin{aligned}
&\int_Q S'_n(u-g)\mathcal{A}(x,t,\nabla u)\nabla\varphi S'(S_n(u-g)+g-\widetilde{g})dxdt\\
=&\int_Q S'_n(u-g)\mathcal{A}(x,t,\omega)\nabla\varphi S'(S_n(u-g)+g-\widetilde{g})dxdt\\
&\underset{n\to+\infty}{\longrightarrow}\int_Q \mathcal{A}(x,t,\omega)\nabla\varphi S'(u-\widetilde{g})dxdt \qquad (4.28)\\
=&\int_Q \mathcal{A}(x,t,\nabla u)\nabla\varphi S'(u-\widetilde{g})dxdt.
\end{aligned}$$

And,

$$\begin{aligned}
&\int_Q S'_n(u-g)\mathcal{A}(t,x,\nabla u)\nabla(S'(S_n(u-g)+g-\widetilde{g}))\varphi dxdt\\
=&\int_Q S'_n(u-g)\mathcal{A}(x,t,\omega)\nabla(S'(S_n(u-g)+g-\widetilde{g}))\varphi dxdt\\
&\underset{n\to+\infty}{\longrightarrow}\int_Q \mathcal{A}(x,t,\omega)\nabla(S'(u-\widetilde{g}))\varphi dxdt \qquad (4.29)\\
=&\int_Q \mathcal{A}(x,t,\nabla u)\nabla(S'(u-\widetilde{g}))\varphi dxdt,
\end{aligned}$$

and,

$$\begin{aligned}
&\int_Q S'_n(u-g)\mathcal{B}(u)S'(S_n(u-g)+g-\widetilde{g})\varphi dxdt\\
&\underset{n\to+\infty}{\longrightarrow}\int_Q \mathcal{B}(u)S'(u-\widetilde{g})\varphi dxdt, \qquad (4.30)
\end{aligned}$$

the definition of S'_n, ($S''_n \to 0$) allows to obtain that

$$\begin{aligned}
&\int_Q S''_n(u-g)\mathcal{A}(x,t,\nabla u)\nabla(u-g)S'(S_n(u-g)+g-\widetilde{g})\varphi dxdt\\
=&\int_Q S''_n(u-g)\mathcal{A}(x,t,\omega)\nabla\left(T_{k+1}\left(u-\widetilde{g}\right)+\widetilde{g}-g\right) \qquad (4.31)\\
&S'(S_n(u-g)+g-\widetilde{g})\varphi dxdt \underset{n\to+\infty}{\longrightarrow} 0,
\end{aligned}$$

repeating the arguments that lead to (4.28), (4.29), (4.30) and (4.31), we obtain

$$\begin{aligned}
&\int_Q \left(\left(S'_n(u-g)-1\right)f+\widetilde{f}\right)S'(S_n(u-g)+g-\widetilde{g})\varphi dxdt\\
&\underset{n\to+\infty}{\longrightarrow}\int_Q \widetilde{f}S'(u-\widetilde{g})\varphi dxdt, \qquad (4.32)
\end{aligned}$$

$$\int_Q \left(\left(S_n'(u-g)-1\right)F+\widetilde{F}\right)S'(S_n(u-g)+g-\widetilde{g})\nabla\varphi dxdt$$
$$\underset{n\to+\infty}{\longrightarrow}\int_Q \widetilde{F}S'(u-\widetilde{g})\nabla\varphi dxdt, \tag{4.33}$$

$$\int_Q \left(\left(S_n'(u-g)-1\right)F+\widetilde{F}\right)\nabla\left(S'(S_n(u-g)+g-\widetilde{g})\right)\varphi dxdt$$
$$\underset{n\to+\infty}{\longrightarrow}\int_Q \widetilde{F}\nabla\left(S'(u-\widetilde{g})\right)\varphi dxdt, \tag{4.34}$$

$$\int_Q \left(\left(S_n'(u-g)-1\right)G+\widetilde{G}\right)S'(S_n(u-g)+g-\widetilde{g})\nabla\varphi dxdt$$
$$\underset{n\to+\infty}{\longrightarrow}\int_Q \widetilde{G}S'(u-\widetilde{g})\nabla\varphi dxdt, \tag{4.35}$$

$$\int_Q \left(\left(S_n'(u-g)-1\right)G+\widetilde{G}\right)\nabla\left(S'(S_n(u-g)+g-\widetilde{g})\right)\varphi dxdt$$
$$\underset{n\to+\infty}{\longrightarrow}\int_Q \widetilde{G}\nabla\left(S'(u-\widetilde{g})\right)\varphi dxdt, \tag{4.36}$$

$$\int_Q S_n''(u-g)G\nabla(u-g)S'(S_n(u-g)+g-\widetilde{g})\varphi dxdt \underset{n\to+\infty}{\longrightarrow} 0. \tag{4.37}$$

As a consequence of the above convergence results in a position to pass to the limit as n tends to $+\infty$ in (4.26) and to conclude that u satisfies (4.9) (with g instead of $\widetilde{g}$). It remains to show that $S(u-\widetilde{g})$ satisfies the initial condition (4.10). To this end, for $\psi\in C_0^\infty(\Omega)$ we take $\varphi=(T-t)\psi$ in (4.26), it possible to obtain

$$\lim_{n\to+\infty}\int_0^T \left\langle (S_n(u-g)+g-\widetilde{g})_t, S'(S_n(u-g)+g-\widetilde{g})\varphi\right\rangle dt \tag{4.38}$$

$$\begin{aligned}
&+\int_Q \mathcal{A}(t,x,\nabla u)\nabla\varphi S'(u-\widetilde{g})dxdt+\int_Q \mathcal{A}(x,t,\nabla u)\nabla(S'(u-\widetilde{g}))\varphi dxdt\\
&+\int_Q \mathcal{B}(u)S'(u-\widetilde{g})\varphi dxdt=\int_Q fS'(u-\widetilde{g})\varphi dxdt+\int_Q G\nabla\varphi S'(u-\widetilde{g})dxdt\\
&+\int_Q G\varphi\nabla\left(S'(u-\widetilde{g})\right)dxdt+\int_Q F\nabla\varphi S'(u-\widetilde{g})dxdt\\
&+\int_Q F\varphi\nabla\left(S'(u-\widetilde{g})\right)dxdt.
\end{aligned}$$

As far as the parabolic contribution in (4.38) is concerned, since $\varphi(0) \neq 0$, $S_n(u-g) = S_n(u_0)$ and $(g-\widetilde{g})(0) = 0$ and the integration by parts

$$\begin{aligned} &\int_0^T \left\langle (S_n(u-g)+g-\widetilde{g})_t \,, S'\left(S_n(u-g)+g-\widetilde{g}\right)\varphi\right\rangle dt \qquad (4.39)\\ =\; &\int_0^T \left\langle (S' S_n(u-g)+g-\widetilde{g})_t \,, \varphi\right\rangle dt = -\int_\Omega S\left(S_n\left(u_0\right)\right)\varphi\left(0\right)dx\\ -\; &\int_Q S\left(S_n(u-g)+g-\widetilde{g}\right)\varphi_t dxdt = -\int_\Omega S\left(u_0\right)\varphi\left(0\right)dx\\ -\; &\int_Q S\left(u-\widetilde{g}\right)\varphi_t dxdt + \omega\left(n\right). \end{aligned}$$

Secondly, we use φ as test function in (4.9) (with $\widetilde{g}$), then leads to

$$\begin{aligned} &-\int_\Omega S\left(u-\widetilde{g}\right)(0)\,dx\\ &-\int_Q S\left(u-\widetilde{g}\right)\varphi_t dxdt + \int_Q \mathcal{A}(x,t,\nabla u)\nabla\varphi S'(u-\widetilde{g})dxdt \qquad (4.40)\\ &+\int_Q \mathcal{A}(x,t,\nabla u)\nabla(S'(u-\widetilde{g}))\varphi dxdt + \int_Q \mathcal{B}(u)S'(u-\widetilde{g})\varphi dxdt\\ =\; &\int_Q fS'(u-\widetilde{g})\varphi dxdt + \int_Q G\nabla\varphi S'(u-\widetilde{g})dxdt\\ &+\int_Q G\varphi\nabla\left(S'(u-\widetilde{g})\right)dxdt\\ &+\int_Q F\nabla\varphi S'(u-\widetilde{g})dxdt + \int_Q F\varphi\nabla\left(S'(u-\widetilde{g})\right)dxdt. \end{aligned}$$

From (4.38), (4.39) and (4.40) the conclusion is that $\int_\Omega S\left(u-\widetilde{g}\right)(0)\,\psi dx = \int_\Omega S\left(u_0\right)\psi dx$; for all $\psi \in C_0^\infty\left(\Omega\right)$, then $S\left(u-\widetilde{g}\right)(0) = S\left(u_0\right)$ in Ω.

□

5. The Existence of Result

This section is devoted to establish the existence of a renormalized solution.

Theorem 5.1. *Under assumptions (4.1)-(4.5) there exists at least a renormalized solution u of Problem (1.1).*

Proof. (of Theorem (5.1)) The proof is divided into 6 steps. In Step 1, we introduce an approximate problem. Step 2 is devoted to establish a few a priori estimates. In Step 3, the limit u of the approximate solutions u^ε is introduced and $u-g$ is shown to belongs to $L^\infty(]0,T[;L^1(\Omega))$ and to satisfy (4.6)-(4.7). In Step 4, the definition of a time regularization of the field $T_k(u)$ and to establish Lemma (5.2), which allows to control the parabolic contribution that arises in the monotonicity method when passing to the limit. Step 5 is devoted to prove an energy estimate (Lemma (5.3)). At last, Step 6 is devoted to prove that u satisfies (4.8)-(4.10) of Definition (4.1).

- **Step 1.** Let us introduce the following regularization of the data: for $\varepsilon > 0$ fixed

$$u_0^\varepsilon \text{ are a sequence of } C_c^\infty(\Omega) \text{ functions such that} \tag{5.1}$$
$$u_0^\varepsilon \to u_0 \text{ in } L^1(\Omega) \text{ as } \varepsilon \text{ tends to } 0.$$

In view of Proposition (3.2), we can find

$$\begin{aligned} &\mu^\varepsilon \in C_c^\infty(\Omega) : \|\mu^\varepsilon\|_{L^1(Q)} \leq C \text{ and} \\ &\mu^\varepsilon = f^\varepsilon + F^\varepsilon - div(G^\varepsilon) + (g^\varepsilon)_t \,, \end{aligned} \tag{5.2}$$

and such that

$$f^\varepsilon \in C_c^\infty(\Omega) : f^\varepsilon \to f \text{ in } L^1(Q) \text{ as } \varepsilon \text{ tends to } 0, \tag{5.3}$$
$$F^\varepsilon \in C_c^\infty(\Omega) : F^\varepsilon \to F \text{ in } \left(L^{p'(.)}(Q) \right)^N \text{ as } \varepsilon \text{ tends to } 0, \tag{5.4}$$
$$G^\varepsilon \in C_c^\infty(\Omega) : G^\varepsilon \to G \text{ in } \left(L^{p'(.)}(Q) \right)^N \text{ as } \varepsilon \text{ tends to } 0, \tag{5.5}$$
$$g^\varepsilon \in C_c^\infty(\Omega) : g^\varepsilon \to g \text{ in } L^{p-}\left( ]0,T[; V \right) \text{ as } \varepsilon \text{ tends to } 0\ . \tag{5.6}$$

Let us now consider the following regularized problem

$$\begin{aligned} &(u^\varepsilon)_t - div\mathcal{A}(x,t,\nabla u^\varepsilon) + \mathcal{B}(u^\varepsilon) \\ &= \mu^\varepsilon = f^\varepsilon + F^\varepsilon - div(G^\varepsilon) + (g^\varepsilon)_t \text{ in } Q, \end{aligned} \tag{5.7}$$
$$u^\varepsilon = 0 \text{ on }]0,T[\times \partial\Omega, \tag{5.8}$$
$$u^\varepsilon(t=0) = u_0^\varepsilon \text{ in } \Omega. \tag{5.9}$$

As a consequence, proving existence of a weak solution $u^\varepsilon \in L^{p^-}(0,T; W_0^{1,p(.)}(\Omega))$ of (5.7)-(5.9) is an easy task (see [15]).

- **Step 2** Using $T_k(u^\varepsilon - g^\varepsilon)$ as a test function in (5.7) leads to

$$\begin{aligned} &\int_\Omega \overline{T}_k(u^\varepsilon - g^\varepsilon)(t)dx + \int_0^t \int_\Omega \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon - g^\varepsilon)dxds \\ &+ \int_0^t \int_\Omega \mathcal{B}(u^\varepsilon)\, T_k(u^\varepsilon - g^\varepsilon)dxds = \int_0^t \int_\Omega f^\varepsilon T_k(u^\varepsilon - g^\varepsilon)dxds \\ &+ \int_0^t \int_\Omega F^\varepsilon \nabla T_k(u^\varepsilon - g^\varepsilon)dxds \int_0^t \int_\Omega G^\varepsilon \nabla T_k(u^\varepsilon - g^\varepsilon)dxds \\ &+ \int_\Omega \overline{T}_k(u_0^\varepsilon)dx, \end{aligned} \tag{5.10}$$

for almost every t in $(0,T)$, and where $\overline{T}_k(r) = \int_0^r T_k(s)ds$. Using assumptions (4.1)-(4.2) and the definition of $\overline{T}_k(r)$ in (5.10), to obtain

$$\begin{aligned} &\int_\Omega \overline{T}_k(u^\varepsilon - g^\varepsilon)(t)dx + \alpha \int_{E_k} |\nabla u^\varepsilon|^{p(x)} dxds \\ &\leq k\,\|\mu^\varepsilon\|_{L^1(Q)} + k\,\|\mathcal{B}(u^\varepsilon)\|_{L^1(Q)} + \beta \int_{E_k} b(t,x)\,|\nabla g^\varepsilon|\,dxdt \\ &+ \beta \int_{E_k} |\nabla u^\varepsilon|\,|\nabla g^\varepsilon|\,dxds + k\,\|u_0^\varepsilon\|_{L^1(Q)}\,, \end{aligned} \tag{5.11}$$

where $E_k = \{(x,t) \in Q : |u^\varepsilon - g^\varepsilon| \le k\}$, using young's inequality, we get

$$\begin{aligned}
&\int_\Omega \overline{T}_k(u^\varepsilon - g^\varepsilon)(t)dx + \left(\alpha - \beta \frac{p^+ - 1}{p^+}\right) \int_{E_k} |\nabla u^\varepsilon|^{p(x)} \, dxds \\
\le\ & k \|\mu^\varepsilon\|_{L^1(Q)} + k \|\mathcal{B}(u^\varepsilon)\|_{L^1(Q)} + \beta \|b\|_{L^{p'(.)}(Q)} \|\nabla g^\varepsilon\|_{L^{p(.)}(Q)} \\
& + \frac{\beta}{p^+} \int_{E_k} |\nabla g^\varepsilon|^{p(x)} \, dxdt + k \|u_0^\varepsilon\|_{L^1(Q)} . \qquad (5.12)
\end{aligned}$$

Also, to obtain

$$\begin{aligned}
& k \int_{\{(x,t)\in Q : |u^\varepsilon - g^\varepsilon| > k\}} |\mathcal{B}(u^\varepsilon)| \, dxdt \le k \|\mu^\varepsilon\|_{L^1(Q)} \\
& + \beta \|b\|_{L^{p'(.)}(Q)} \|\nabla g^\varepsilon\|_{L^{p(.)}(Q)} \qquad (5.13) \\
& + \frac{\beta}{p^+} \int_{E_k} |\nabla g^\varepsilon|^{p(x)} \, dxdt + k \|u_0^\varepsilon\|_{L^1(Q)} .
\end{aligned}$$

Now, let $T_1(s - T_k(s)) = T_{k,1}(s)$ and take $T_{k,1}(u^\varepsilon - g^\varepsilon)$ as test function in (5.7). Reasoning as above, using that $\nabla T_{k,1}(s) = \nabla s \chi_{\{k \le |s| \le k+1\}}$ and appling young's inequality, we obtain

$$\begin{aligned}
& \int_{\{k \le |u^\varepsilon - g^\varepsilon| \le k+1\}} |\nabla (u^\varepsilon - g^\varepsilon)|^{p(x)} \, dxdt \\
& \le Ck \int_{|u_0^\varepsilon| > k} |u_0^\varepsilon| \, dx + Ck \int_{|u^\varepsilon - g^\varepsilon| > k} |\mathcal{B}(u^\varepsilon)| \, dxdt \\
& + Ck \int_{|u^\varepsilon - g^\varepsilon| > k} |f^\varepsilon| \, dxdt + C(\int_{|u^\varepsilon - g^\varepsilon| > k} |F^\varepsilon|^{p'(x)} \, dxdt \\
& + \int_{|u^\varepsilon - g^\varepsilon| > k} |G^\varepsilon|^{p'(x)} \, dxdt) \le C,
\end{aligned}$$

inequality (2.3) implies that

$$\begin{aligned}
& \int_0^T \chi_{\{k \le |u^\varepsilon - g^\varepsilon| \le k+1\}} \min \left\{ \|\nabla (u^\varepsilon - g^\varepsilon)\|_{L^{p(x)}(\Omega)}^{p-}, \|\nabla (u^\varepsilon - g^\varepsilon)\|_{L^{p(x)}(\Omega)}^{p+} \right\} \\
\le & \int_{\{k \le |u^\varepsilon - g^\varepsilon| \le k+1\}} |\nabla (u^\varepsilon - g^\varepsilon)|^{p(x)} \, dxdt \le C. \qquad (5.14)
\end{aligned}$$

From the estimation (5.12), (5.14) and the properites of $\overline{T}_k$ and u_0^ε, we have

$$u^\varepsilon - g^\varepsilon \text{ is bounded in } L^\infty \left(]0,T[; L^1(\Omega)\right), \qquad (5.15)$$

and

$$u^\varepsilon - g^\varepsilon \text{ is bounded in } L^{p-}(]0,T[; W_0^{1,p(x)}(\Omega)), \qquad (5.16)$$

by Lemma 2.1 in [8] and by (5.12), (5.14) and si $2 - \dfrac{1}{N+1} < p(.) < N$, we obtain

$$v^\varepsilon = u^\varepsilon - g^\varepsilon \text{ is bounded in } L^{q-}(]0,T[; W_0^{1,q(x)}(\Omega)), \qquad (5.17)$$

for all continuous variable exponents $q \in C(\overline{\Omega})$ satisfying

$$1 \leq q(x) < \frac{N(p(x)-1)+p(x)}{N+1},$$

for all $x \in \Omega$. And

$$T_k\left(u^{\varepsilon}-g^{\varepsilon}\right) \text{ is bounded in } L^{p^-}\left(]0,T[;W_0^{1,p(.)}(\Omega)\right), \tag{5.18}$$

and by (5.13), to obtain

$$\mathcal{B}(u^{\varepsilon}) \text{ is bounded in } L^1\left(]0,T[;L^1(\Omega)\right), \tag{5.19}$$

independently of ε.
Proceeding as in [2], [3] that for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' is compact (supp $S' \subset [-k,k]$)

$$S\left(u^{\varepsilon}-g^{\varepsilon}\right) \text{ is bounded in } L^{p-}\left(]0,T[;W_0^{1,p(.)}(\Omega)\right), \tag{5.20}$$

and

$$\left(S\left(u^{\varepsilon}-g^{\varepsilon}\right)\right)_t \text{ is bounded in } L^1(Q)+L^{(p-)'}\left(]0,T[;W^{-1,p'(.)}(\Omega)\right). \tag{5.21}$$

In fact, as a consequence of (5.18), by Stampacchia's Theorem, we obtain (5.20). To show that (5.21) holds true, we multiply the equation (5.7) by $S'(u^{\varepsilon}-g^{\varepsilon})$ to obtain

$$\begin{aligned}
&\left(S\left(u^{\varepsilon}-g^{\varepsilon}\right)\right)_t = div(S'\left(u^{\varepsilon}-g^{\varepsilon}\right)\mathcal{A}(x,t,\nabla u^{\varepsilon}))\\
&-\mathcal{A}(x,t,\nabla u^{\varepsilon})\nabla\left(S'\left(u^{\varepsilon}-g^{\varepsilon}\right)\right)\\
&-\mathcal{B}\left(u^{\varepsilon}\right)S'\left(u^{\varepsilon}-g^{\varepsilon}\right)+f^{\varepsilon}S'\left(u^{\varepsilon}-g^{\varepsilon}\right)\\
&+F^{\varepsilon}S'\left(u^{\varepsilon}-g^{\varepsilon}\right)-div\left(G^{\varepsilon}S'\left(u^{\varepsilon}-g^{\varepsilon}\right)\right)\\
&+G^{\varepsilon}\nabla\left(S\left(u^{\varepsilon}-g^{\varepsilon}\right)\right) \text{ in } \mathcal{D}'(Q).
\end{aligned} \tag{5.22}$$

Since supp(S') and supp(S'') are both included in $[-k;k]$; u^{ε} may be replaced by $(T_k(v^{\varepsilon})+g^{\varepsilon})$ in $\{|u^{\varepsilon}-g^{\varepsilon}| \leq k\}$, where $v^{\varepsilon}=u^{\varepsilon}-g^{\varepsilon}$. To have

$$\begin{aligned}
&\left|S'\left(u^{\varepsilon}-g^{\varepsilon}\right)\mathcal{A}(x,t,\nabla u^{\varepsilon})\right|\\
&\leq \beta\left\|S'\right\|_{L^{\infty}}\left[b(x,t)+|T_k(v^{\varepsilon})+g^{\varepsilon}|^{p(x)-1}\right].
\end{aligned} \tag{5.23}$$

As a consequence, each term in the right hand side of (5.22) is bounded either in $L^{(p-)'}\left(]0,T[;W^{-1,p'(.)}(\Omega)\right)$ or in $L^1(Q)$, and obtain (5.21).

Now an estimate on a sort of energy at infinity of the approximating solutions. For any integer $n \geq 1$, consider the Lipschitz continuous function θ_n defined through $\theta_n(s)=T_{n+1}(s)-T_n(s)$. Remark that $||\theta_n||_{L^{\infty}} \leq 1$ for any $n \geq 1$ and that $\theta_n(s) \to 0$, for any s when n tends to infinity. Using the admissible test function $\theta_n(u^{\varepsilon}-g^{\varepsilon})$ in (5.7) leads to

$$\begin{aligned}
&\int_{\Omega}\widetilde{\theta_n}\left(u^{\varepsilon}-g^{\varepsilon}\right)(t)\,dx+\int_{Q}\mathcal{A}(x,t,\nabla u^{\varepsilon})\nabla\left(\theta_n(u^{\varepsilon}-g^{\varepsilon})\right)dxdt\\
&+\int_{Q}\mathcal{B}\left(u^{\varepsilon}\right)\theta_n(u^{\varepsilon}-g^{\varepsilon})dxdt\\
&+\int_{Q}F^{\varepsilon}\nabla\left(\theta_n(u^{\varepsilon}-g^{\varepsilon})\right)dxdt\\
&+\int_{Q}G^{\varepsilon}\nabla\left(\theta_n(u^{\varepsilon}-g^{\varepsilon})\right)dxdt\\
&+\int_{\Omega}\widetilde{\theta_n}\left(u_0^{\varepsilon}\right)dx,
\end{aligned} \tag{5.24}$$

for almost every t in $]0,T[$ and where $\widetilde{\theta_n}(r)=\int_0^r \theta_n(s)ds \geq 0$. Hence, dropping a nonnegative term

$$
\begin{aligned}
&\int_{\{n\leq|u^\varepsilon-g^\varepsilon|\leq n+1\}} \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon dxdt \qquad (5.25)\\
\leq &\int_{\{n\leq|u^\varepsilon-g^\varepsilon|\leq n+1\}} \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla g^\varepsilon dxdt - \int_Q \mathcal{B}\left(u^\varepsilon\right)\theta_n(u^\varepsilon-g^\varepsilon)dxdt\\
&+\int_Q f^\varepsilon\theta_n(u^\varepsilon-g^\varepsilon)dxdt+\int_Q F^\varepsilon\nabla\left(\theta_n(u^\varepsilon-g^\varepsilon)\right)dxdt\\
&+\int_Q G^\varepsilon\nabla\left(\theta_n(u^\varepsilon-g^\varepsilon)\right)dxdt+\int_\Omega \widetilde{\theta_n}\left(u_0^\varepsilon\right)dx\\
\leq &\ \beta\int_{\{n\leq|u^\varepsilon-g^\varepsilon|\leq n+1\}} \left[|b(x,t)|\,|\nabla g^\varepsilon|+|\nabla v^\varepsilon+\nabla g^\varepsilon|^{p(x)-1}|\nabla g^\varepsilon|\right]dxdt\\
&+\int_Q \left|\mathcal{B}\left(u^\varepsilon\right)\right|\theta_n(u^\varepsilon-g^\varepsilon)dxdt+\int_Q |f^\varepsilon|\,\theta_n(u^\varepsilon-g^\varepsilon)dx\\
&+\int_Q F^\varepsilon\nabla\left(\theta_n(u^\varepsilon-g^\varepsilon)\right)dxdt+\int_Q G^\varepsilon\nabla\left(\theta_n(u^\varepsilon-g^\varepsilon)\right)dxdt\\
&+\int_\Omega \widetilde{\theta_n}\left(u_0^\varepsilon\right)dx.
\end{aligned}
$$

Using assumption (5.2), (5.25) and applying Young's inequality, to obtain

$$
\begin{aligned}
&\frac{\alpha}{2}\int_{\{n\leq|u^\varepsilon-g^\varepsilon|\leq n+1\}} |\nabla u^\varepsilon|^{p(x)}dxdt \qquad (5.26)\\
&\leq C\left(\int_{\{|u^\varepsilon-g^\varepsilon|\geq n\}} \frac{p^+-1}{p^+}|b|^{p'(x)}\right.\\
&\left.+\frac{C_1}{p(x)(\frac{\alpha}{2}p'(x))^{\frac{p(x)}{p'(x)}}}|\nabla g^\varepsilon|^{p(x)}+|G^\varepsilon|^{p'(x)}+|F^\varepsilon|^{p'(x)}\right)dxdt\\
&+\int_{\{|u^\varepsilon-g^\varepsilon|\geq n\}} \left|\mathcal{B}\left(u^\varepsilon\right)\right|dxdt+\int_{\{|u^\varepsilon-g^\varepsilon|\geq n\}} |f^\varepsilon|\,dxdt\\
&+\int_{\{|u_0^\varepsilon|\geq n\}} |u_0^\varepsilon|\,dx.
\end{aligned}
$$

- **Step 3** Arguing again as in [[2],[3],[4]] estimate (5.20) and (5.21) implies that, for a subsequence still indexed by ε,

$$u^\varepsilon-g^\varepsilon \text{ converges a.e where to } u-g \text{ in } Q, \qquad (5.27)$$

using (5.7), (5.18) and (5.23), we get

$$u^\varepsilon \text{ converge almost every where to } u \text{ in } Q, \qquad (5.28)$$

$$T_k(u^\varepsilon - g^\varepsilon) \text{ converge weakly to } T_k(u-g) \\ \text{in } L^{p-}\left(]0,T[\,,W_0^{1,p(.)}(\Omega)\right), \tag{5.29}$$

$$\chi_{\{|u^\varepsilon - g^\varepsilon|\le k\}}\mathcal{A}(t,x,\nabla u^\varepsilon) \rightharpoonup \eta_k \text{ weakly in } \left(L^{p'(.)}(Q)\right)^N, \tag{5.30}$$

as ε tends to 0 for any $k>0$ and any $n\ge 1$ and where for any $k>0$, η_k belongs to $\left(L^{p'(.)}(Q)\right)^N$. Since $\mathcal{B}(u^\varepsilon)$ is a continuous incrassing function, from the monotone convergence theorem, by (5.18) and (5.28), to obtain that

$$\mathcal{B}(u^\varepsilon) \text{ converge weakly to } \mathcal{B}(u) \text{ in } L^1(Q). \tag{5.31}$$

Now establish that $u-g$ belongs to $L^\infty\left(]0,T[\,;L^1(\Omega)\right)$. Indeed using (5.12) and $\left|\overline{T}_k(s)\right| \ge |s|-1$ leads to

$$\begin{aligned}\int_\Omega |u^\varepsilon - g^\varepsilon|(t)dx \quad &\le \quad meas(\Omega) + k\left\|\mu^\varepsilon\right\|_{L^1(Q)} + k\left\|\mathcal{B}(u^\varepsilon)\right\|_{L^1(Q)} \\ &+\left(\frac{1}{p^-}+\frac{1}{p'^-}\right)\|b\|_{L^{p'(.)}(Q)}\left\|\nabla g^\varepsilon\right\|_{L^{p(.)}(Q)} \\ &+\left(\frac{1}{p^-}+\frac{1}{p'^-}\right)\left\|\nabla g^\varepsilon\right\|_{L^{p(.)}(Q)}\left\|\nabla u^\varepsilon\right\|_{L^{p(.)}(Q)} \\ &+k\left\|u_0^\varepsilon\right\|_{L^1(\Omega)}.\end{aligned}$$

Using (5.27) and (5.1)-(5.6) , to have $u-g$ belongs to $L^\infty\left(]0,T[\,;L^1(\Omega)\right)$. Now in a position to exploit (5.26). Since $u^\varepsilon - g^\varepsilon$ is bounded in $L^\infty\left(]0,T[\,;L^1(\Omega)\right)$, we get

$$\lim_{n\to+\infty}\left(\sup_\varepsilon meas\left\{|u^\varepsilon - g^\varepsilon| \ge n\right\}\right) = 0, \tag{5.32}$$

using the equi-integrability of the sequences $|\nabla g^\varepsilon|^{p(x)}$, $|G^\varepsilon|^{p'(x)}$, $|F^\varepsilon|^{p'(x)}$, $|f^\varepsilon|$, $|\mathcal{B}(u^\varepsilon)|$ and $|u_0^\varepsilon|$ in $L^1(\Omega)$, we deduce that

$$\lim_{n\to+\infty}\left(\sup_\varepsilon \int_{\{n\le|u^\varepsilon - g^\varepsilon|\le n+1\}} |\nabla u^\varepsilon|^{p(x)}dxdt\right) = 0. \tag{5.33}$$

- **Step 4** The specific time regularization of $T_k(u)$ (for fixed $k\ge 0$) is defined as follows. Let $(v_0^\mu)_\mu$ be a sequence in $L^\infty(\Omega)\cap W_0^{1,p(.)}(\Omega)$ such that $\|v_0^\mu\|_{L^\infty(\Omega)} \le k$, $\forall\mu>0$, and $v_0^\mu \to T_k(u_0)$ a.e in Ω with $\frac{1}{\mu}\|v_0^\mu\|_{L^{p(.)}(\Omega)} \to 0$ as $\mu\to+\infty$.
  For fixed $k\ge 0$ and $\mu>0$, let us consider the unique solution

$$T_k(u)_\mu \in L^\infty(\Omega)\cap L^{p-}\left(]0,T[;W_0^{1,p(.)}(\Omega)\right)$$

of the monotone problem

$$\frac{\partial T_k(u)_\mu}{\partial t} + \mu\left(T_k(u)_\mu - T_k(u)\right) = 0 \text{ in } \mathcal{D}'(Q), \tag{5.34}$$

$$T_k(u)_\mu(t=0) = v_0^\mu. \tag{5.35}$$

The behavior of $T_k(u)_\mu$ as $\mu \to +\infty$ is investigated in [14] and we just recall here that (5.34)-(5.35) imply that

$$\begin{aligned} &T_k(u)_\mu \to T_k(u) \text{ strongly in } L^{p-}\left(]0,T[;W_0^{1,p(.)}(\Omega)\right) \\ &\text{a.e in, } Q \text{ as } \mu \to +\infty, \end{aligned} \tag{5.36}$$

with $\|T_k(u)_\mu\|_{L^\infty(\Omega)} \leq k$, for any μ, and $\frac{\partial T_k(u)_\mu}{\partial t} \in L^{(p-)'}\left(]0,T[;W^{-1,p'(.)}(\Omega)\right)$.
The main estimate is the following

Lemma 5.2. *Let $v^\varepsilon = u^\varepsilon - g^\varepsilon$. Let S be an increasing $C^\infty(\mathbb{R})-$ function such that $S(r) = r$ for $r \leq k$, and suppS' is compact. Then*

$$\liminf_{\mu\to+\infty} \lim_{\varepsilon\to 0} \int_0^T \left\langle \frac{\partial v^\varepsilon}{\partial t}, S'(v^\varepsilon)\left(T_k(v^\varepsilon)_\mu - T_k(v)\right)\right\rangle dt \geq 0,$$

where here $\langle .,.\rangle$ denotes the duality pairing between $L^1(\Omega) + W^{-1,p'(.)}(\Omega)$ and $L^\infty(\Omega) \cap V(\Omega)$.

Proof. See [4], Lemma 1. □

- **Step 5** Here to prove that the weak limit η_k and to prove the weak L^1 convergence of the "truncted" energy $\mathcal{A}(x,t,\nabla T_k(v^\varepsilon))$ as ε tends to 0. In order to show this result recall the Lemma below.

Lemma 5.3. *The subsequence of u^ε defined in step 3 satisfies*

$$\limsup_{\varepsilon\to 0} \int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)dxdt \leq \int_Q \eta_k \nabla T_k(v)dxdt, \tag{5.37}$$

$$\begin{aligned} \lim_{\varepsilon\to 0}\int_Q &\left[\mathcal{A}\left(x,t,\nabla u^\varepsilon_{\chi_{\{|v^\varepsilon|\leq k\}}}\right) - \mathcal{A}\left(x,t,\nabla u_{\chi_{\{|v|\leq k\}}}\right)\right] \\ &\times\left[\nabla u^\varepsilon_{\chi_{\{|v^\varepsilon|\leq k\}}} - \nabla u_{\chi_{\{|v|\leq k\}}}\right]dxdt = 0 \end{aligned} \tag{5.38}$$

$\eta_k = \mathcal{A}\left(x,t,\nabla u_{\chi_{\{|v|\leq k\}}}\right)$ *a.e in Q, for any $k \geq 0$, as ε tends to 0.*

$$\begin{aligned} &\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon) \to \mathcal{A}(x,t,\nabla u)\nabla T_k(v) \\ &\textit{weakly in } L^1(Q). \end{aligned} \tag{5.39}$$

Proof. For $k \geq 0$, to consider the test function $S_n'(v^\varepsilon)\left(T_k(u_\varepsilon) - (T_k(u))_\mu\right)$ in (5.7), and use the definition (4.12) of S'_n and we definie $W^\varepsilon_\mu = T_k(u_\varepsilon) - (T_k(u))_\mu$, to get

$$
\begin{aligned}
& \int_0^T \left\langle (u^\varepsilon - g^\varepsilon)_t \,, S'_n(v^\varepsilon) W^\varepsilon_\mu \right\rangle dt + \int_Q S'_n(v^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla W^\varepsilon_\mu dxdt \\
& + \int_Q S''_n(v^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon W^\varepsilon_\mu dxdt \\
+ \quad & \int_Q \mathcal{B}(u^\varepsilon) S'_n(v^\varepsilon) W^\varepsilon_\mu dxdt \\
& + \int_Q F^\varepsilon S'_n(v^\varepsilon)\nabla W^\varepsilon_\mu dxdt \\
& + \int_Q F^\varepsilon S''_n(v^\varepsilon) W^\varepsilon_\mu \nabla v^\varepsilon dxdt \\
& + \int_Q G^\varepsilon S'_n(v^\varepsilon)\nabla W^\varepsilon_\mu dxdt \\
& + \int_Q G^\varepsilon S''_n(v^\varepsilon) W^\varepsilon_\mu \nabla v^\varepsilon dxdt.
\end{aligned} \tag{5.40}
$$

Now pass to the limit in (5.40) as $\varepsilon \to 0$, $\mu \to +\infty$, $n \to +\infty$ for k fixed real number. In order to perform this task, to prove below the following results for any $k \geq 0$

$$\liminf_{\mu\to+\infty} \lim_{\varepsilon\to 0} \int_0^T \left\langle (u^\varepsilon - g^\varepsilon)_t \,, S'_n(v^\varepsilon) W^\varepsilon_\mu \right\rangle dt \geq 0 \text{ for any } n \geq k, \tag{5.41}$$

$$\lim_{n\to+\infty} \lim_{\mu\to+\infty} \lim_{\varepsilon\to 0} \int_Q S''_n(v^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon W^\varepsilon_\mu dxdt = 0, \tag{5.42}$$

$$\lim_{\mu\to+\infty} \lim_{\varepsilon\to 0} \int_Q \mathcal{B}(u^\varepsilon) S'_n(v^\varepsilon) W^\varepsilon_\mu dxdt = 0, \text{ for any } n \geq 1, \tag{5.43}$$

$$\lim_{\mu\to+\infty} \lim_{\varepsilon\to 0} \int_Q f^\varepsilon S'_n(v^\varepsilon) W^\varepsilon_\mu dxdt = 0, \text{ for any } n \geq 1, \tag{5.44}$$

$$\lim_{\mu\to+\infty} \lim_{\varepsilon\to 0} \int_Q F^\varepsilon S'_n(v^\varepsilon)\nabla W^\varepsilon_\mu dxdt = 0, \text{ for any } n \geq 1, \tag{5.45}$$

$$\lim_{\mu\to+\infty} \lim_{\varepsilon\to 0} \int_Q F^\varepsilon S''_n(v^\varepsilon) W^\varepsilon_\mu \nabla v^\varepsilon dxdt = 0, \text{ for any } n \geq 1, \tag{5.46}$$

$$\lim_{\mu\to+\infty} \lim_{\varepsilon\to 0} \int_Q G^\varepsilon S'_n(v^\varepsilon)\nabla W^\varepsilon_\mu dxdt = 0, \text{ for any } n \geq 1, \tag{5.47}$$

$$\lim_{\mu\to+\infty} \lim_{\varepsilon\to 0} \int_Q G^\varepsilon S''_n(v^\varepsilon) W^\varepsilon_\mu \nabla v^\varepsilon dxdt = 0, \text{ for any } n \geq 1. \tag{5.48}$$

Proof of (91). In view of the definition W_μ^ε, we apply lemma (5.2) with $S = S_n$ for fixed $n \geq k$. As a consequence, (5.41) hold true. $\square$

Proof of (92). For $n \geq 1$, we have $supp(S_n'') \subset [-(n+1), -n] \cup [n, n+1]$, $\left\|W_\mu^\varepsilon\right\|_{L^\infty(Q)} \leq 2k$ and $\|S_n''\|_{L^\infty(\mathbb{R})} \leq 1$. Using assumptions (4.2) and applying Young's inequality, we get

$$\begin{aligned} &\left|\int_Q S_n''(v^\varepsilon)\mathcal{A}\left(x,t,\nabla u^\varepsilon\right)\nabla v^\varepsilon W_\mu^\varepsilon dxdt\right| \\ &\leq C_3 \int_{\{n \leq |v^\varepsilon| \leq n+1\}} \mathcal{A}\left(x,t,\nabla u^\varepsilon\right)\nabla u^\varepsilon dxdt \\ &+C_4 \int_{\{n \leq |v^\varepsilon|\}} \left(|b(x,t)|^{p'(x)} + |\nabla g^\varepsilon|^{p(x)}\right) dxdt, \end{aligned} \tag{5.49}$$

for $n \geq 1$, using assumptions (5.33)-(5.32) and the equi-integrability of the sequences $|\nabla g^\varepsilon|^{p(x)}$ in $L^1(Q)$, permits to pass to the limit as n tends to $+\infty$ in (5.49) and to etablish (5.42) $\square$

Proof of (93). For $n \geq 1$ and in view (5.31). Lebesgue's convergence theorem implies that for any $\mu > 0$ and $n \geq 1$

$$\begin{aligned} &\lim_{\varepsilon \to 0}\int_Q \mathcal{B}(u^\varepsilon)S_n'(v^\varepsilon)W_\mu^\varepsilon \; dxdt \\ &= \int_Q \mathcal{B}(u)S_n'(v)(T_k(v) - T_k\left(v\right)_\mu)dxdt. \end{aligned} \tag{5.50}$$

Appealing now to (5.36) and passing to the limit as $\mu \to +\infty$ in (5.50) allows to conclude that (5.43) holds true. $\square$

Proof of (94). By (5.3), (5.27) and Lebesgue's convergence theorem implies that for any $\mu > 0$ and $n \geq 1$, it is possible to pass to the limit for $\varepsilon \to 0$

$$\lim_{\varepsilon \to 0}\int_Q f^\varepsilon S_n'(v^\varepsilon)W_\mu^\varepsilon \; dxdt = \int_Q f S_n'(v)(T_k(v) - T_k\left(v\right)_\mu)dxdt,$$

using (5.36) permits to the limit as μ tends to $+\infty$ in the above equality to obtain (5.44). $\square$

Proof of (95). By (5.4), we have

$$\begin{aligned} &F^\varepsilon S_n'(v^\varepsilon) \to F S_n'(v) \text{ a.e. in } Q, \text{and} \\ &|F^\varepsilon S_n'(v^\varepsilon)| \leq (n+1)\left\|F^\varepsilon\right\|_{L^{p'(.)}(Q)} \text{ a.e. in } Q. \end{aligned}$$

Let us recal the main properrties of W_μ^ε. For $\mu > 0$, W_μ^ε converges to $(T_k(v) - T_k\left(v\right)_\mu)$ weakly in $L^{p^-}(]0,T[;W_0^{1,p}(\Omega))$ as $\varepsilon \to 0$. Taking into account that

$$\left\|W_\mu^\varepsilon\right\|_{L^\infty(Q)} \leq 2k \text{ for any } \varepsilon > 0,\; \mu > 0, \tag{5.51}$$

to be able to deduce that

$$W_\mu^\varepsilon \to (T_k(v) - T_k\,(v)_\mu) \qquad \text{a.e. in } Q \text{ and}$$
$$L^\infty(Q) \text{ weakly-* as } \varepsilon \to 0, \tag{5.52}$$

also to deduce that

$$\lim_{\varepsilon\to 0}\int_Q F^\varepsilon S_n'(v^\varepsilon)\nabla W_\mu^\varepsilon dxdt = \int_Q F S_n'(v)\nabla(T_k(v) - T_k\,(v)_\mu)dxdt, \tag{5.53}$$

and the strong convergence of $T_k(v)_\mu$ to $T_k(v)$ in $L^{p^-}(]0,T[;W_0^{1,p}(\Omega))$ as $\mu\to+\infty$, as consequece (5.45) holds true. □

Proof of (96). For $n\geq 1$ and from (5.4), (5.29), it follows that

$$\begin{aligned}&\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q F^\varepsilon S_n''(v^\varepsilon)\nabla v^\varepsilon W_\mu^\varepsilon dxdt\\ &= \lim_{\mu\to+\infty}\int_Q F S_n''(v)\nabla v(T_k(v) - T_k\,(v)_\mu)dxdt = 0.\end{aligned}$$

□

Proof of (97). Using (5.29) and (5.5) lead to $G^\varepsilon S_n'(v^\varepsilon)$ tends to $GS_n'(v)$ strongly in $\left(L^{p'(\cdot)}(Q)\right)^N$ as $\varepsilon\to 0$. We deduce that

$$\lim_{\varepsilon\to 0}\int_Q G^\varepsilon S_n'(v^\varepsilon)\nabla W_\mu^\varepsilon dxdt = \int_Q G S_n'(v)\nabla(T_k(v) - T_k\,(v)_\mu)dxdt,$$

for $\mu>0$, by (5.52) and the strong convergence of $T_k(v)_\mu$ to $T_k(v)$ in $L^{p^-}(]0,T[;W_0^{1,p}(\Omega))$ as $\mu\to+\infty$ allows to conclude (5.47). □

Proof of (98). From (5.5) and (5.29), it follows that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_Q G^\varepsilon\nabla S_n'(v^\varepsilon)W_\mu^\varepsilon dxdt &= \int_Q G\nabla S_n'(v)(T_k(v) - T_k\,(v)_\mu)dxdt\\ &= 0, \text{ for } n\geq 1.\end{aligned}$$

□

Now turn back to the proof of Lemma (5.3), due to (5.41)-(5.48), in a position to pass to the limit-sup when $\varepsilon\to 0$, then to the limit-sup when $\mu\to+\infty$ and to the limit as $n\to+\infty$ in (5.40). Using the definition of W_μ^ε, we deduce that for $k\geq 0$,

$$\begin{aligned}&\lim_{n\to+\infty}\limsup_{\mu\to+\infty}\limsup_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)S_n'(v^\varepsilon)\\ &\nabla\left(T_k(v^\varepsilon) - T_k(v)_\mu\right)dxdt \leq 0.\end{aligned}$$

Since $\mathcal{A}(x,t,\nabla u^\varepsilon)S_n'(v^\varepsilon)\nabla T_k(v^\varepsilon) = \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)$ for $k \leq n$, the above inequality implies that for $k \leq n$,

$$\begin{aligned} &\limsup_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)dxdt \\ \leq\ &\lim_{n\to+\infty}\limsup_{\mu\to+\infty}\limsup_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)S_n'(v^\varepsilon)\nabla T_k(v)_\mu dxdt. \end{aligned} \tag{5.54}$$

Due to (5.30), $\mathcal{A}(x,t,\nabla u^\varepsilon)S_n'(v^\varepsilon) \to \eta_{n+1}S_n'(v)$ weakly in $\left(L^{p'(\cdot)}(Q)\right)^N$ as $\varepsilon \to 0$ and the strong convergence of $T_k(v)_\mu$ to $T_k(v)$ in $L^{p^-}(]0,T[;W_0^{1,p}(\Omega))$ as $\mu \to +\infty$, we get

$$\begin{aligned} &\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)S_n'(v^\varepsilon)\nabla T_k(v)_\mu dxdt \\ =\ &\int_Q S_n'(v)\eta_{n+1}\nabla T_k(v)dxdt = \int_Q \eta_{n+1}\nabla T_k(v)dxdt, \end{aligned} \tag{5.55}$$

for $k \leq n$, since $S_n'(s) = 1$ for $|s| \leq n$. Now for $k \leq n$, we have

$$S_n'(v^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)_{\chi_{\{|v^\varepsilon|\leq k\}}} = \mathcal{A}(x,t,\nabla u^\varepsilon)_{\chi_{\{|v^\varepsilon|\leq k\}}} \text{ a.e in } Q.$$

Letting $\varepsilon \to 0$, to obtain

$$\eta_{n+1}\chi_{\{|v|\leq k\}} = \eta_k\chi_{\{|v|\leq k\}} \text{ a.e in } Q - \{|v| = k\} \text{ for } k \leq n.$$

Recalling (5.54) and (5.55) allows to conclude that (5.37) holds true. □

Proof of (88). Let $k \geq 0$ be fixed. We use the monotone character (4.3) of $\mathcal{A}(x,t,\xi)$ with respest to ξ, to obtain

$$\begin{aligned} I^\varepsilon = \int_Q &\Big(\mathcal{A}(x,t,\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\leq k\}}) - \mathcal{A}(x,t,\nabla u\chi_{\{|v|\leq k\}})\Big) \\ &\Big(\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\leq k\}} - \nabla u\chi_{\{|v|\leq k\}}\Big)\,dxdt \geq 0. \end{aligned} \tag{5.56}$$

Inequality (5.56) is split into $I^\varepsilon = I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon$ where

$$\begin{aligned} I_1^\varepsilon &= \int_Q \mathcal{A}(x,t,\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\leq k\}})\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\leq k\}}dxdt, \\ I_2^\varepsilon &= -\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\leq k\}})\nabla u\chi_{\{|v|\leq k\}}dxdt, \\ I_3^\varepsilon &= -\int_Q \mathcal{A}(x,t,\nabla u\chi_{\{|v|\leq k\}})\Big(\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\leq k\}} - \nabla u\chi_{\{|v|\leq k\}}\Big)\,dxdt. \end{aligned}$$

To pass to the limit-sup as $\varepsilon \to 0$ in I_1^ε, I_2^ε and I_3^ε. Let us remark that $v^\varepsilon = u^\varepsilon - g^\varepsilon$ and $\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\leq k\}} = \Big(\nabla T_k(v^\varepsilon) - g^\varepsilon\chi_{\{|v^\varepsilon|\leq k\}}\Big)$ a.e in Q, assume that k is such that $\chi_{\{|v^\varepsilon|\leq k\}}$ almost everywhere converges to $\chi_{\{|v|\leq k\}}$(in fact this is true for almost every k, see Lemma 3.2 in [6]).

Using (5.37), we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0} I_1^\varepsilon &= \lim_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)dxdt\\ &+ \lim_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\le k\}})\nabla g^\varepsilon dxdt\\ &\le \int_Q \eta_k\nabla T_k(v)dxdt + \int_Q \eta_k\nabla g\chi_{\{|v|\le k\}}dxdt. \end{aligned}\tag{5.57}$$

In view of (5.29) and (5.30),

$$\begin{aligned}\lim_{\varepsilon\to 0} I_2^\varepsilon &= -\lim_{\varepsilon\to 0}\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\le k\}})\left(\nabla T_k(v)+\nabla g\right)dxdt\\ &= -\int_Q \eta_k\left(\nabla T_k(v)+\nabla g\right)dxdt. \end{aligned}\tag{5.58}$$

As a consequence of (5.6) and (5.29), for all $k>0$

$$\begin{aligned}\lim_{\varepsilon\to 0} I_3^\varepsilon &= -\int_Q \mathcal{A}(x,t,\nabla u\chi_{\{|v|\le k\}})\\ &\qquad\Big(\nabla T_k(v^\varepsilon)+\nabla g^\varepsilon\chi_{\{|v^\varepsilon|\le k\}}-\nabla T_k(v)+\nabla g\chi_{\{|v|\le k\}}\Big)dxdt=0. \end{aligned}\tag{5.59}$$

Taking the limit-sup as $\varepsilon\to 0$ in (5.56) and using (5.57), (5.58) and (5.59) show that (5.38) holds true. □

Proof of (89). Using (5.38) and the usual Minty argument applies it follows that (5.39) holds true. □

- **Step 6:** In this step to prove that u satisfies (4.8)-(4.10). To this end, remark that $v^\varepsilon = u^\varepsilon - g^\varepsilon$ and for fixed $n\le 0$ one has

$$\begin{aligned}&\int_{\{n\le|u^\varepsilon-g^\varepsilon|\le n+1\}} \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon dxdt\\ &= \int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_{n+1}(v^\varepsilon)dxdt - \int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_n(v^\varepsilon)dxdt\\ &\quad+\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le n+1\}}\nabla g^\varepsilon dxdt\\ &\quad-\int_Q \mathcal{A}(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le n\}}\nabla g^\varepsilon dxdt. \end{aligned}$$

According to (5.30) and (5.39) one is at liberty to pass to the limit as ε tends to 0 for fixed $n\ge 1$, to obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int\limits_{\{n\le|u^\varepsilon-g^\varepsilon|\le n+1\}}\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon dxdt \qquad (5.60)\\
=\;&\int\limits_Q\mathcal{A}(x,t,\nabla u)\nabla T_{n+1}(v)dxdt-\int\limits_Q\mathcal{A}(x,t,\nabla u)\nabla T_n(v)dxdt\\
&+\int\limits_Q\mathcal{A}(x,t,\nabla u)\chi_{\{|v|\le n+1\}}\nabla g dxdt\\
&-\int\limits_Q\mathcal{A}(x,t,\nabla u)\chi_{\{|v|\le n\}}\nabla g dxdt\\
=\;&\int\limits_{\{n\le|u^\varepsilon-g^\varepsilon|\le n+1\}}\mathcal{A}(x,t,\nabla u)\nabla u dxdt.
\end{aligned}$$

Taking the limit as n tends to $+\infty$ in (5.60) and using the estimate (5.33) , that u satisfies (4.8). Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact. Let k be a positive real number such that $\text{supp}(S')\subset[-k,k]$. Pontwise multiplication of that approximate equation (5.7) by $S'(u^\varepsilon-g^\varepsilon)$ leads to

$$\begin{aligned}
&(S(u^\varepsilon-g^\varepsilon))_t-div(S'(u^\varepsilon-g^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)) \qquad (5.61)\\
&+S''(u^\varepsilon-g^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla(u^\varepsilon-g^\varepsilon)+\mathcal{B}(u^\varepsilon)S'(u^\varepsilon-g^\varepsilon)\\
&=f^\varepsilon S'(u^\varepsilon-g^\varepsilon)+F^\varepsilon S'(u^\varepsilon-g^\varepsilon)-div(S'(u^\varepsilon-g^\varepsilon)G^\varepsilon)\\
&+S''(u^\varepsilon-g^\varepsilon)G^\varepsilon\nabla(u^\varepsilon-g^\varepsilon)\text{ in }\mathcal{D}'(Q).
\end{aligned}$$

In what follows to pass to the limit as ε tends to 0 in each term of (5.61). Since S is bounded, and $S(u^\varepsilon-g^\varepsilon)$ converges to $S(u-g)$ a.e in Q and in $L^\infty(Q)*-$weak, then $(S(u^\varepsilon-g^\varepsilon))_t$ converges to $(S(u^\varepsilon-g^\varepsilon))_t$ in $\mathcal{D}'(Q)$ as ε tends to 0. Since $\text{supp}(S')\subset[-k,k]$, we have $S'(u^\varepsilon-g^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)=S'(u^\varepsilon-g^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}}$ a.e in Q. The pointwise convergence of u^ε to u as ε tends to 0, the bounded character of S and (5.39) of Lemma(5.3) imply that $S'(u^\varepsilon-g^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)$ converges to $S'(u-g)\mathcal{A}(x,t,\nabla u)$ weakly in $\left(L^{p'(.)}(Q)\right)^N$ as ε tends to 0, because $S'(u-g)=0$ for $|u-g|\ge k$ a.e in Q. The pointwise convergence of $u^\varepsilon-g^\varepsilon$ to $u-g$, the bounded character of S', S'' and (5.39) of Lemma (5.3) allow to conclude that

$$\begin{aligned}
&S''(u^\varepsilon-g^\varepsilon)\mathcal{A}(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon-g^\varepsilon)\\
&\to S''(u-g)\mathcal{A}(x,t,\nabla u)\nabla T_k(u-g)\text{ weakly in }L^1(Q)
\end{aligned}$$

as $\varepsilon\to 0$. The use of (5.31) to obtain that $\mathcal{B}(u^\varepsilon)S'(u^\varepsilon-g^\varepsilon)$ converges to $\mathcal{B}(u)S'(u-g)$ in $L^1(Q)$, and we use (5.3), (5.4), (5.5), (5.6) and (5.29) and we obtain that $f^\varepsilon S'(u^\varepsilon-g^\varepsilon)$ converges to $fS'(u-g)$ in $L^1(Q)$, the term $F^\varepsilon S'(u^\varepsilon-g^\varepsilon)$ converges to $FS'(u-g)$ weakly in $\left(L^{p'(.)}(Q)\right)^N$ and the term $G^\varepsilon S'(u^\varepsilon-g^\varepsilon)$ converges to $GS'(u-g)$ strongly in $\left(L^{p'(.)}(Q)\right)^N$ and $S''(u^\varepsilon-g^\varepsilon)G^\varepsilon\nabla(u^\varepsilon-g^\varepsilon)$ converges to $S''(u-g)G\nabla(u-g)$ weakly in $L^1(Q)$. As a consequence of the above convergence result, the position to pass to the limit as ε tends to 0 in equation (5.61) and to conclude that u satisfies (4.9). It remains to show that $S(u-g)$ satisfies the initial condition (4.10). To this end, firstly remark that, S being bounded, $S(u^\varepsilon-g^\varepsilon)$ is bounded in $L^\infty(Q)$. Secondly, (5.61) and the above considerations on the behavior of the terms of this equation show that $\frac{\partial S(u^\varepsilon-g^\varepsilon)}{\partial t}$ is bounded in $L^1(Q)+L^{(p-)'}(]0,T[;W^{-1,p'(.)}(\Omega))$. As a consequence, an Aubin's type Lemma ([21], Corollary 4) implies that $S(u^\varepsilon-g^\varepsilon)$ lies in a compact set of $C^0(]0,T[;L^1(\Omega))$. It follows that, on one hand, $S(u^\varepsilon-g^\varepsilon)(t=0)$ converges to $S(u-g)(t=0)$ strongly in $L^1(\Omega)$. Due to (5.1), to conclude that (4.10) holds true. As a conclusion of **Step 3** and **Step 6**, the proof of Theorem (5.1) is complete.

□

References

1. E. Azroula, M.B. Benboubkera, M. Rhoudaf, *On some $p(x)$-quasilinear problem with right-hand side measure*, Mathematics and Computers in Simulation, Volume 102 , 117- 130, (2014).

2. D. Blanchard, and F. Murat, *Renormalised solutions of nonlinear parabolic problems with L^1 data, Existence and uniqueness,* Proc. Roy. Soc. Edinburgh Sect., A127, 1137-1152,(1997).

3. D. Blanchard , F. Murat, and H. Redwane, *Existence et unicité de la solution reormalisée d'un probléme parabolique assez général,* C. R. Acad. Sci. Paris Sér., I329 , 575-580,(1999).

4. Blanchard, D., Murat, F., and Redwane, H., *Existence and Uniqueness of a Renormalized Solution for a Fairly General Class of Nonlinear Parabolic Problems*, J. Differential Equations, 177, 331-374 (2001).

5. D.Blanchard,F. Petitta and H. Redwane, *Renormalized solutions of nonlinear parabolic equations with diffuse measure data*, Manuscripta Math. 141, 601-635, (2013).

6. Boccardo, L., Dall'Aglio, A., Gallouët T., and Orsina, L., *Nonlinear parabolic equa- tions with measure data,* J. Funct. Anal., 147, 237-258, (1997).

7. L. Boccardo, T. Gallouët, *Nonlinear elliptic and parabolic equations involving measure data*, J. Funct. Anal. 87 , no. 1, 149-169,(1989).

8. T. M. Bendahmane, P. Wittbold, A.Zimmermann, *Renormalized solutions for a nonlinear parabolic equation with variable exponents and L^1-data* . J.Differential Equations 249, 1483-1515, (2010).

9. M. Badr Benboubker, H. Chrayteh, M. EL Moumni and H. Hjiaj, *Entropy and Renormalized Solutions for Nonlinear Elliptic Problem Involving Variable Exponent and Measure Data*, Acta Mathematica Sinica, English Series Jan., Vol. 31, No. 1, pp. 151-169, (2015).

10. Y.M. Chen, S. Levine, M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math. 66, 1383-1406, (2006).

11. Di Perna, R.-J. and P.-L. Lions, P.-L., *On the Cauchy problem for Boltzmann equations : Global existence and weak stability,* Ann. Math., 130, 321-366, (1989).

12. B. El Hamdaoui, J. Bennouna, and A. Aberqi, *Renormalized Solutions for Nonlinear Parabolic Systems in the Lebesgue Sobolev Spaces with Variable Exponents*, Journal of Mathematical Physics, Analysis, Geometry, Vol. 14, No. 1, pp. 27-53, (2018).

13. X.L. Fan and D. Zhao, *On the spaces $L^{p(x)}(U)$ and $W^{m;p(x)}(U)$* ,J. Math. Anal. Appl. 263, 424-446, (2001).

14. R. Landes, *On the existence of weak solutions for quasilinear parabolic initial- boundary problems*, Proc. Roy. Soc. Edinburgh Sect. A 89, 321-366, (1981).

15. J.-L. Lions, *Quelques méthodes de résolution des problémes aux limites non linéaires* . Dunod et Gauthier-Villars, (1969).

16. S. Ouaro and U. Traore, *$p(.)$-parabolic capacity and decomposition of measures* , Annals of the University of Craiova, Mathematics and Computer Science Series. Volume 44(1), , Pages 30-63, (2017).

17. F. Petitta, *Renormalized solutions of nonlinear parabolic equations with general measure data* , Ann. Mat. Pura Appl. Vol. 187, no. 4, 563-604, (2008).

18. F. Petitta., A.C. Ponce, , A. Porretta, *Approximation of diffuse measures for parabolic capacities*, C. R. Math. Acad. Sci. Paris 346 , no. 3-4(2008) , 161-166.

19. F. Petitta., A.C. Ponce, , A. Porretta, *Diffuse measures and nonlinear parabolic equations*, J. Evol. Equations, 11 (4) , 861-905 (2011).

20. F. Petitta and H. Redwane, *Renormalized solutions of nonlinear parabolic equations with diffuse measure data*, Manuscripta Mathematica volume 141, pages601–635,(2013).

21. J. Simon, *Compact sets in Lp(0, T;B)*, Ann. Mat. Pura Appl., 146, 65-96, (1987).

22. C. Zhang, *Entropy solutions for nonlinear elliptic equations with variable exponents* . Electronic Journal of Differential Equations, Vol. 2014, No. 92, pp. 1-14,(2014).

23. C. Zhang, S. Zhou, *Renormalized and entropy solution for nonlinear parabolic equations with variable exponents and L^1 data* , J.Differential Equations 248, 1376-1400, (2010).

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