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Tension on an Edge in a Graph

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ABSTRACT: In this paper we introduce the concept of tension on an edge in a graph. The tension on an edge in a graph is the number of geodesics passing through it. We investigate some results and characterizations involving tensions on edges in graphs.

Key Words: Graph, path, geodesic, tension on an edge, tension regular graph.

Contents

1. Introduction

Let $G = (V, E)$ be a graph (finite, undirected and simple). The distance between two vertices u and v in G, denoted by $d(u, v)$ is the number of edges in a shortest path (also called a graph geodesic) connecting them. We say that a graph geodesic P is passing through an edge e in G if e is an edge in P. The friendship graph F_n on $2n+1$ vertices is obtained by joining n copies of the complete graph K_2 at a shared universal vertex v.

Harary et al. [5] have defined and studied geodetic set and geodetic number of a graph in 1993. Further study on geodetic number is carried by Chartrand et al. [3] in 2002. Atici [1] has defined and studied the edge geodetic set and edge geodetic number of a graph in 2003. His works are inspired by the papers [5,3]. Motivated by the works of above mentioned authors, we define the concept of tension on an edge in a graph as the number of geodesics passing through it and obtain some related results. For standard terminology and notion in graph theory, we follow the text-book of Harary [4].

In Section 2, we give some examples and obtain some basic results about tension on edges. In Section 3, we define and study the total tension of a graph. In Section 4, we investigate the behavior of tension on edges in subgraphs. In Section 5, we obtain some results related to tension in trees. In Section 6, we define tension regular graphs and obtain some characterization results.

2. Tension on an edge

Definition 2.1. *Let G be a graph and e be an edge in G. The tension on e, denoted by $\tau_G(e)$ or simply $\tau(e)$, is defined as the number of geodesics in G passing through e.*

Example 2.2. *Consider the graph G given in Figure 1. The tensions on the edges of G are as follows: $\tau(v_1v_2) = \tau(v_2v_3) = \tau(v_2v_7) = \tau(v_2v_8) = 7$, $\tau(v_2v_4) = \tau(v_2v_6) = 6$, $\tau(v_2v_5) = 5$, $\tau(v_4v_5) = \tau(v_5v_6) = 2$.*

Proposition 2.3. *i) In the complete bipartite graph $K_{r,s}$, for any edge e, we have $\tau_{K_{r,s}}(e) = r+s-1$.*

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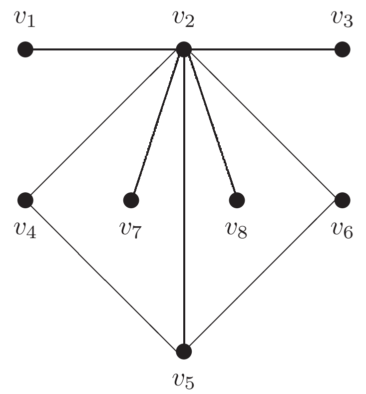

Figure 1: A graph G

ii) Let e be any edge in the cycle graph C_n on n vertices. Then

$$\tau_{C_n}(e) = \begin{cases} \frac{(n-1)(n+1)}{8}, & \text{if } n \text{ is odd} \\ \frac{n(n+2)}{8}, & \text{if } n \text{ is even.} \end{cases}$$

iii) Let e be any edge in a tree T and let C_1 and C_2 be the components of $T - e$. Then $\tau_T(e) = |V(C_1)||V(C_2)|$.

iv) Let W_n denote the wheel graph on $n \geq 5$ vertices. If e is a peripheral edge in W_n, then $\tau_{W_n}(e) = 3$ and if e is a radial edge in W_n, then $\tau_{W_n}(e) = n - 3$.

v) Let F_n denote the friendship graph on $2n+1$ vertices. If e is a peripheral edge in F_n, then $\tau_{F_n}(e) = 1$ and if e is a radial edge in F_n, then $\tau_{F_n}(e) = 2n + 1$.

Proof. Results follow by direct computations. □

Definition 2.4. *For a vertex v in a graph G, by the closed neighborhood $N[v]$ of v in G, we mean the set of all vertices in G which are adjacent to v including v.*

Proposition 2.5. *Let $e = ab$ be an edge in a graph G. Then $\tau(e) = 1$ if and only if $N[a] = N[b]$.*

Proof. Let $\tau(e) = 1$. Suppose $N[a] \neq N[b]$ then there exist $v \in V(G)$ such that $v \sim a$ or $v \sim b$ but $v \not\sim \{a, b\}$. Say $v \sim a$ and $v \not\sim b$. Then vab is a $v - b$ geodesic through e and thus $\tau(e) \geq 2$, a contradiction.

Conversely, assume the given condition. The result follows by noting that if P is a $u - v$ path through e of length greater than 1, then since $N[a] = N[b]$, P contains a $u - v$ sub-path of length one less than that of P. □

Corollary 2.6. *Let G be a connected graph. Then $\tau(e) = 1$ for every edge e in G if and only if G is a complete graph.*

Proof. Suppose $\tau(e) = 1$, for every edge e in G. Let a and b be any two vertices in G. Then, since G is connected, by Proposition 2.5, it follows that $N[a] = N[b]$. Hence a and b are adjacent. Thus G must be a complete graph.

Conversely, suppose that G is a complete graph. Then there is no geodesic of length ≥ 2 in G and so $\tau(e) = 1$, for any edge e in G. □

Proposition 2.7. *Let $e = ab$ be an edge in a graph G. If $\tau(e) = 2$, then $|N[a]\Delta N[b]| = 1$, where $N[a]\Delta N[b]$ denotes the symmetric difference between the two sets $N[a]$ and $N[b]$.*

Proof. Suppose $|N[a]\Delta N[b]| \neq 1$.

Suppose $N[a]\Delta N[b] = \phi$. Then $N[a] = N[b]$ and hence by Proposition 2.5, it follows that $\tau(e) = 1$, a contradiction.

Suppose $N[a] - N[b] = \phi$ and $|N[b] - N[a]| \geq 2$. Then we can find two distinct vertices u and v, both different from a such that both are adjacent to b but not adjacent to a (see Figure 2). But then ab, abu, abv are 3 different geodesics passing through $e = ab$, so that $\tau(e) \geq 3$, a contradiction. Similarly, the case that $N[b] - N[a] = \phi$ and $|N[a] - N[b]| \geq 2$ gives a contradiction.

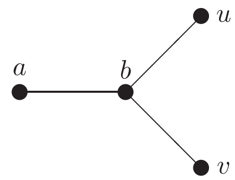

Figure 2:

Suppose $|N[a] - N[b]| \geq 1$ and $|N[b] - N[a]| \geq 1$. Then we can find a vertex u different from b and a vertex v different from a such that u is adjacent to a, but not adjacent to b and v is adjacent to b, but not adjacent to a (see Figure 3). But then ab, uab, abv are 3 different geodesics passing through $e = ab$, so that $\tau(e) \geq 3$, a contradiction.

Figure 3:

Hence $|N[a]\Delta N[b]| = 1$. ☐

Remark 2.8. *The converse of the Proposition 2.7 fails. For instance, consider the path P_n on n vertices shown in the Figure 4, with $n > 3$. Clearly $N[v_1]\Delta N[v_2] = \{v_3\}$, but $\tau(v_1v_2) = n - 1 > 2$.*

Figure 4: The path P_n on n vertices.

Definition 2.9. *Let e be an edge in a graph G and let u be any vertex. We say that u imposes a tension on e if there exists a vertex v in G and a $u - v$ geodesic in G passing through e.*

Proposition 2.10. *Let G be a connected graph. Then a vertex u imposes a tension on an edge $e = ab$ if and only if $|d(u, a) - d(u, b)| = 1$.*

Proof. Suppose u imposes a tension on e. Then there exists a vertex v and a $u - v$ geodesic, say P, in G passing through e. Then P contains a $u - a$ sub-path F_1 and a $u - b$ sub-path F_2 and clearly $|l(F_1) - l(F_2)| = 1$. (Here $l(P)$ denotes the length of P.) Now since any sub-path of a geodesic is also a geodesic, it follows that $l(F_1) = d(u, a)$ and $l(F_2) = d(u, b)$. Hence we have $|d(u, a) - d(u, b)| = 1$.

Conversely, suppose $|d(u, a) - d(u, b)| = 1$, say $d(u, a) = k$ and $d(u, b) = k + 1$. Let F be a $u - a$ geodesic. Then $l(F) = k$. Note that F cannot pass through b, since $d(u, b) = k + 1 > k$. Hence the walk P formed by F followed by the edge ab must be a $u - b$ path, where $l(P) = l(F) + 1 = k + 1 = d(u, b)$. Hence it follows that P must be a $u - b$ geodesic passing through e. Hence u imposes a tension on e. ☐

3. Total Tension of a Graph

Definition 3.1. *Let $G = (V, E)$ be a graph. The total tension of G, denoted by $N_\tau(G)$, is defined as,*

$$N_\tau(G) = \sum_{e \in E} \tau(e).$$

The following is a corollary of Proposition 2.3.

Corollary 3.2. *i) For the complete graph K_n on n vertices, $N_\tau(K_n) = \binom{n}{2}$.*

ii) For the complete bipartite graph $K_{r,s}$, $N_\tau(K_{r,s}) = rs(r + s - 1)$.

iii) For the cycle C_n on n vertices, $N_\tau(C_n) = \frac{n(n-1)(n+1)}{8}$, if n is odd and $N_\tau(C_n) = \frac{n^2(n+2)}{8}$, if n is even.

iv) For the path P_n on n vertices (shown in Figure 4), $N_\tau(P_n) = \binom{n+1}{3}$.

v) For the wheel graph W_n on n vertices, $N_\tau(W_n) = n(n - 1)$.

vi) For the friendship graph F_n on $2n + 1$ vertices, $N_\tau(F_n) = n(4n + 3)$.

Proof. Proofs follow by Definition 3.1, Corollary 2.6 and Proposition 2.3. □

Proposition 3.3. *For any graph G with n vertices and diameter d, we have*

$$N_\tau(G) = \sum_{i=1}^{d} i f_i,$$

where f_i is the number of geodesics of length i in G.

Proof. We know that, in a graph

i) every geodesic of length k has k edges;

ii) every geodesic contributes one count to the tension of each of its edges.

Since G has diameter d, the result follows from (i) and (ii) above. □

4. Tensions on edges in subgraphs

Proposition 4.1. *Let H be a subgraph of a graph G and let e be an edge in H. If all geodesics in H passing through e are also geodesics in G, then $\tau_H(e) \leq \tau_G(e)$.*

Proof. Follows from Definition 2.1. □

Corollary 4.2. *Let H be a subgraph of a tree T. Then $\tau_H(e) \leq \tau_T(e)$ for any edge e in H.*

Proof. Since, in a tree, every pair of vertices are joined by a unique path, the conditions of Proposition 4.1 are satisfied in T. Hence the result follows. □

Remark 4.3. *1. In general if e is an edge in a subgraph H of a graph G, then $\tau_H(e) \leq \tau_G(e)$ may not hold even if H is a maximal subgraph. For example, consider the cycle graph $G = C_7$ given by the Figure 5 and let H be the maximal subgraph containing the vertices v_i for $i = 1, 2, 3, 4, 5, 6$. Note that H is the path on 6 vertices. Consider the edge $e = v_3v_4$ in H. We have $\tau_H(e) = 9$ and $\tau_G(e) = 6$.*

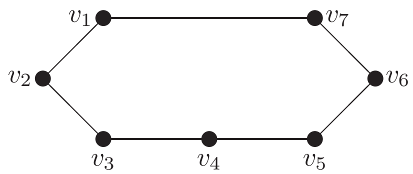

Figure 5: The cycle graph C_7

2. *The converse of the Proposition 4.1 may fail even if H is a maximal subgraph. For instance, consider the cycle graph $G = C_7$ given by the Figure 5 and let H be the maximal subgraph containing the vertices v_i for $i = 2, 3, 4, 5, 6$. Note that H is the path on 5 vertices. Consider the edge $e = v_2v_3$ in H. We have $\tau_H(e) = 4$ and $\tau_G(e) = 6$. Note that $F : v_2v_3v_4v_5v_6$ is a geodesic in H passing through e, but F is not a geodesic in G.*

Corollary 4.4. *Let G be any graph and suppose P is a geodesic in G. Then for any edge e in P, we have $\tau_P(e) \leq \tau_G(e)$.*

Proof. Follows from Proposition 4.1, by noting that any sub-path of a geodesic in G is also a geodesic in G. □

5. Some characterizations in trees

Proposition 5.1. *Let T be a tree with n vertices and let e be an edge in T. Then e is a pendant edge in T if and only if $\tau(e) = n - 1$.*

Proof. Suppose e is a pendant edge in T. Then $\tau(e) = n - 1$, using Proposition 2.3(iii).

Conversely, suppose e is not a pendant edge in T. Then $\tau(e) = n_1n_2$ for some $n_1, n_2 \geq 2$, by Proposition 2.3(iii). But then $1/n_1, 1/n_2 \leq 1/2$, so that $1/n_1 + 1/n_2 \leq 1$ which implies that $n = n_1 + n_2 \leq n_1n_2 = \tau(e)$. Hence $\tau(e) > n - 1$ if e is not a pendant edge. Thus, if $\tau(e) = n - 1$, then e must be a pendant edge in T. □

Corollary 5.2. *A tree T, with n vertices, is a star graph if and only if $\tau(e) = n - 1$ for every edge e in T.*

Proof. Follows from Proposition 5.1. □

Remark 5.3. 1. *If e is a pendant edge in a connected graph G with n vertices, then $\tau(e) \geq n - 1$. Equality may not hold. For example, consider the graph G with $n = 7$ vertices given in the Figure 6. Here e is a pendant edge, but $\tau(e) = 9 > 6 = n - 1$.*

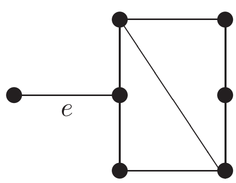

Figure 6: A Graph G with 7 vertices

2. *If e is an edge in a connected graph G with n vertices, having $\tau(e) = n - 1$, then e need not be a pendant edge in G. For example, consider the graph G with $n = 7$ vertices given in the Figure 6. Here $\tau(e') = 6 = n - 1$, but e' is not a pendant edge.*

6. Tension regular graphs

Definition 6.1. *A graph G is said to be k-tension regular if all its edges are of tension k.*

Observation: For every natural number k, there is a k-tension regular graph, namely, the star graph on n vertices with $n = k+1$.

Proposition 6.2. *A connected graph G is 1-tension regular if and only if it is a complete graph.*

Proof. Follows from Corollary 2.6. □

Remark 6.3. *1. A tension regular graph may not be a regular graph. For instance, the star graph on $n \geq 3$ vertices is $(n-1)$-tension regular graph, but it is not a regular graph.*

2. A regular graph may not be tension regular. For instance, the cubic Sylvester graph given in the Figure 7 is regular but it is not tension regular. The numbers near the edges indicate the tension on the corresponding edge.

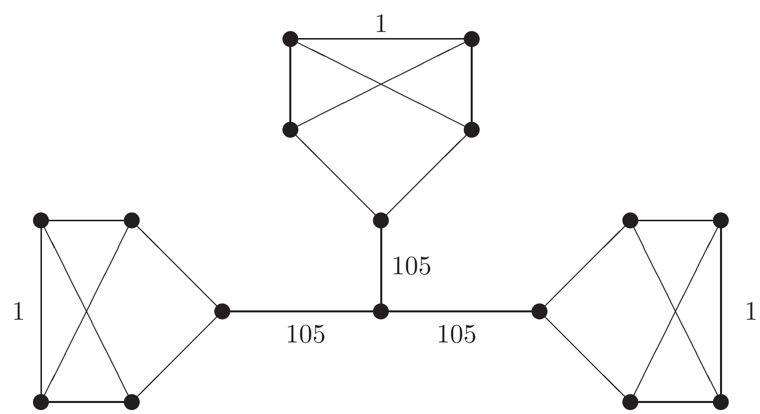

Figure 7: Cubic Sylvester Graph

Proposition 6.4. *A tree T is tension regular if and if T is a star graph.*

Proof. Suppose a tree T on n vertices is k-tension regular. Since T is a tree, it must have a pendant edge e. Then by Proposition 5.1, it follows that $\tau(e) = n-1$. Hence, we must have $k = n-1$ and thus every edge in T has $n-1$ tension. Now again by the Proposition 5.1, it follows that every edge in T must be a pendant edge in T and hence T is a star graph.

Converse follows from Corollary 5.2. □

Lemma 6.5. *Let G be any graph and let $k = max\{\tau(e) : e \in E(G)\}$. If n is a positive integer such that $\left\lfloor \frac{n+1}{2} \right\rfloor \left\lceil \frac{n+1}{2} \right\rceil > k$, then all geodesics in G are of length less than n.*

Proof. Suppose P is a geodesic in G with $m = l(P) \geq n$. If e is an edge incident on a centre vertex of P, then by Proposition 2.3(iii), we have

$$\tau_P(e) = \left\lfloor \frac{m+1}{2} \right\rfloor \left\lceil \frac{m+1}{2} \right\rceil \geq \left\lfloor \frac{n+1}{2} \right\rfloor \left\lceil \frac{n+1}{2} \right\rceil > k.$$

But by Corollary 4.4, we have $\tau_P(e) \leq \tau_G(e)$, and hence $\tau_G(e) > k$, which contradicts the definition of k. Hence the result follows. □

Theorem 6.6. *A connected graph G is 2-tension regular if and only if $G = P_3$, the path on 3 vertices.*

Proof. The converse holds from Proposition 2.3(iii).

Let a connected graph G be 2-tension regular. Then by Lemma 6.5, it follows that all geodesics in G are of length less than 3. Further clearly, since G is 2-tension regular, all geodesics in G cannot be of length 1. Hence G contains a geodesic of length 2, say $F = abc$ (see Figure 8). Note that now there are exactly 2 geodesics of G passing through each of the edges ab and bc.

Figure 8: A geodesic F of length 2 in G.

We prove that $G = F$. Suppose not. Then, since G is connected, we can find a vertex u different from a, b, c and which is adjacent to either a or b or c.

If u is adjacent to a, then u must be adjacent to b, for otherwise, uab will be a geodesic passing through ab and so $\tau_G(ab) \geq 3$, a contradiction. Similarly u must be adjacent c also.

Similarly, if u is adjacent to b or c, then it must be adjacent to remaining two vertices. In any case u must be adjacent all 3 vertices a, b and c. The current situation and the number of geodesics in G passing through various edges are shown in the Figure 9.

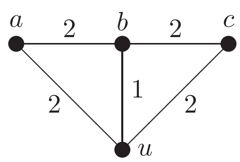

Figure 9:

Now, since $\tau_G(bu) = 2$, by Proposition 2.7, we must have $N[b]\Delta N[u] = \{v\}$ for some vertex v in G.

Suppose v is adjacent to b, but not adjacent to u. If v is not adjacent to a, then vba will be a geodesic passing through the edge ab, so that $\tau_G(ab) \geq 3$, a contradiction. Hence v must be adjacent to a (see Figure 10). But then the path vau will be a $v - u$ geodesic passing through the edge au, so that $\tau_G(au) \geq 3$, a contradiction.

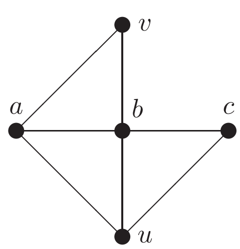

Figure 10:

Thus v must be adjacent to u, but not adjacent to b. Now if v is not adjacent to a, we will have $\tau_G(au) \geq 3$, which is not possible. Hence v must be adjacent a (see Figure 11). Now the path vab will be a $v - b$ geodesic passing through the edge ab, so that $\tau_G(ab) \geq 3$, a contradiction.
Thus, we must have $G = F$ and hence $G = P_3$, the path on 3 vertices. □

Theorem 6.7. *A connected graph G, which is not self centered, is 3-tension regular if and only if G is isomorphic to the star graph $K_{1,3}$ on 4 vertices or the wheel graph W_6 on 6 vertices.*

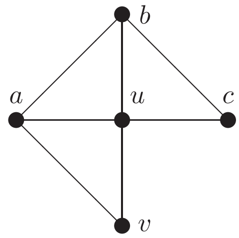

Figure 11:

Proof. The converse holds from Proposition 2.3.

Let a connected graph G be 3-tension regular, which is not self centered. Then by Lemma 6.5, it follows that all geodesics in G are of length less than 3. Consequently, $e(v) \leq 2$ for every $v \in V(G)$. (Here $e(v)$ denotes the eccentricity of a vertex v of G.) Since G is not self centered, it follows that $e(a) = 1$ for some $a \in V(G)$. Then a must be adjacent to every vertex in G. Consider an edge $e = ab$ in G. Since $\tau(e) = 3$ and since a is adjacent to every vertex in G, we can find two distinct vertices c and d in G which are both adjacent to a, bot not adjacent to b.

Suppose c and d are not adjacent. Then cad is a geodesic in G. Already cab and dab are geodesics in G (see Figure 12).

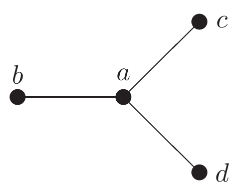

Figure 12: A subgraph H of G

Hence the number of geodesics in G passing through each of the edges ab, ac and ad is already 3. Consequently, G must be equal to the subgraph H of G, given in the Figure 12. For, otherwise, since G is connected, we can find a vertex x in G which is adjacent to any one of the vertices of H. Since already every edge of H has tension 3 in G, it follows, by Proposition 2.10, that x must be adjacent to every vertex in H. Currently there is only one geodesic in G passing through ax. Since $\tau_G(ax) = 3$, and since a is adjacent to every vertex in G, we can find a vertex y in G such that y is adjacent to a, but not adjacent to x. Now since $\tau_G(ab) = \tau_G(ac) = \tau_G(ad) = 3$, by Proposition 2.10, it follows that y must be adjacent to b, c and d. Now note that cyb, dyb, xby and yb are 4 different geodesics in G passing through the edge yb, which is a contradiction. Hence $G = H = K_{1,3}$.

Suppose c and d are adjacent. Note that currently, there are only 2 geodesics in G passing through ac and ad (see Figure 13). Now, since $\tau_G(ac) = 3$ and since a is adjacent to every vertex in G, we can find

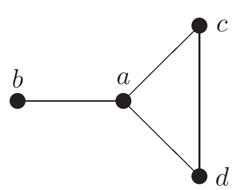

Figure 13: A subgraph H of G

a vertex u adjacent to a, but not to c. Since already, there are 3 geodesics in G are passing through ab, it follows, from Proposition 2.10, that u must be adjacent to b. Now we claim that u must be adjacent to d also. Suppose not. The current situation and the number of geodesics of G passing through various edges are as shown in the Figure 14.

Since $\tau_G(cd) = 3$, we can find a vertex x which is adjacent to either c or d, but not to both. Say, x

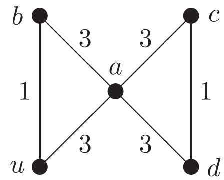

Figure 14:

is adjacent to c, but not to d. Clearly a is adjacent to x. Now since $\tau_G(ad) = 3$, by Proposition 2.10, it follows that x must be adjacent to d, a contradiction. Hence u must be adjacent to d as claimed.

The current situation and the number of geodesics in G passing through various edges are as shown in the Figure 15.

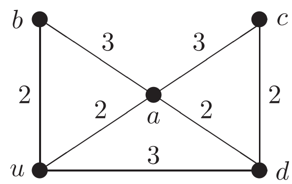

Figure 15:

Now, since $\tau_G(cd) = 3$, we can find a vertex v in G which is adjacent to c or d, but not adjacent to the other. Suppose v is adjacent to d, but not to c. Clearly a is adjacent to v. Now since $\tau_G(ac) = 3$, by Proposition 2.10, it follows that v must be adjacent to c, a contradiction. Hence v must be adjacent to c, but not to d. Clearly a is adjacent to v. Now since $\tau_G(ab) = 3$, by Proposition 2.10, it follows that v must be adjacent to b. Further v cannot be adjacent to u. For otherwise, since $\tau_G(ud) = 3$, by Proposition 2.10, it follows that v must be adjacent to d, which is not true. Hence we obtain a subgraph H of G as shown in the Figure 16.

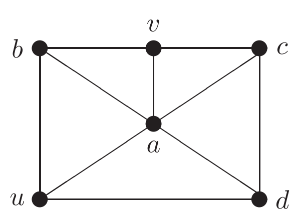

Figure 16: A subgraph H of G.

Note that every edge of H has tension 3 in G. We claim that $G = H$. For, otherwise, since G is connected, we can find a vertex x in G which is adjacent to a vertex in H. Now since every edge of H has tension 3 in G, by Proposition 2.10, it follows that x must be adjacent to every vertex in H. Currently there is only one geodesic in G passing through ax. Since $\tau_G(ax) = 3$, and since a is adjacent to every vertex in G, we can find a vertex y in G such that y is adjacent to a, but not to x. Now since every edge in H has tension 3 in G, by Proposition 2.10, it follows that y must be adjacent to b, c, d, u and v. Now note that cyb, dyb, xby and yb are 4 different geodesics in G passing through the edge yb, which is a contradiction. Hence $G = H = W_6$. □

Acknowledgments

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