



## Some Type of Separation Axioms in Bornological Topological Space

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**ABSTRACT:** In this paper, we introduce the concept of bornological topological continuous in bornological topological space and study some its properties. Also we define bornological topological open function, bornological topological closed function and bornological topological homeomorphism function and investigate some new properties of them. Finally some separation axioms have been studied in bornological topological space like  $B - \tau_0$ ,  $B - \tau_1$ ,  $B - \tau_2$ ,  $B - \tau_3$ ,  $B - \tau_4$  and the relationships among them.

**Key Words:** Bornological space, Bornological topological continuous, Bornological topological open, Bornological topological closed, Bornological topological homeomorphism.

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### 1. Introduction

Modern analysis is developed by the setting of functional analysis is done on various topological structures, all these spaces are special cases of uniform spaces, one of these spaces by use bornological structure to define a topological spaces. In 1977, Hogue-Nlend[7] introduced the concepts of bornology on a set. Noiri.[8] in 1984 found  $\alpha$ -continuous functions. In 1999[7] Balasubramanian, used fuzzy set to determined fuzzy  $\beta$ -open sets and fuzzy  $\beta$ -separation axioms. In 1995, Dierolf and Domanski [6] studied various bornological properties and Bornological Space Of Null Sequences. Barreira and Almeida[5] in 2002 introduced Hausdorff Dimension in Convex Bornological Space. Since that time, several methods for constructing new bornologies like forming products, subspace and quotient bornologies like that which were presented in 2007[2] by Al-Deen and Al-Shaibani, The space of entire functions over the complex field  $\mathbb{C}$  was introduced by Patwardhan who defined a metric on this space by introducing a real-valued map on. In 2018 Al-Basri [8] found the relationship between the sequentially bornological continuous map in bornological vector spaces (bvs) and sequentially bornological compact spaces have been investigated and studied as well as between them and bornological complete space in 2018 [8]. In this paper we study bornological topology spaces, introduce  $B$ -open set and  $B$ -closed set and some concepts have been defined depending on bornivorous set. The bornivorous set is the subset  $N$  of a bornological vector space  $E$  if it absorbs every bounded subset of  $E$ . So that many researchers investigated new properties like,  $B$ -base,  $B$ -sub base,  $B$ -closures set,  $B$ -interior set,  $B$ -subspace. this paper contains 6 sections, introduction of this paper in section 1, section 2 contains some basic concepts of bornological space, section 3 we introduce the concept of bornological topology continuous (written  $B$ -topology continuous), bornological topology space and bornological topology open (written  $B$ -topology open), bornological topology closed (written  $B$ -topology closed). section 3 we define bornological topology homeomorphism (written  $B$ -topology homeomorphism), some new properties of, bornological topology open, bornological topology closed and

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bornological topology homeomorphism have been investigated in bornological topology space. section 5 some separation axioms have been studied in bornological topological space like  $B - \tau_0$ ,  $B - \tau_1$ ,  $B - \tau_2$ ,  $B - \tau_3$ ,  $B - \tau_4$  and the relationships among them we prove some bornological topological separation axioms, section 6 is conclusion of this paper

## 2. Preliminaries

**Definition 2.1** (7). Let  $X$  be a non-empty set. Family  $B$  of subsets of  $X$  is called bornology on  $X$  if satisfy the following axioms:

- (1)  $X = \cup_{B \in B} B$ ; that is  $B$  is a covering of  $X$ .
- (2) If  $N \subset M$  and  $M \in B$ , then  $N \in B$ ; that is  $B$  is hereditary under inclusion.
- (3)  $M$  is stable under finite union. A pair  $(X, B)$  consisting of a set  $X$  and  $B$  bornology on  $X$  is called a Bornological space, and an elements in  $M$  is called bounded subsets of  $X$ .

**Definition 2.2** (7). Let  $(X, B)$  be a bornological space, a subset  $N$  of  $M$  is called bornological open (for brief  $B$ -open) set, if the set  $N - \{a\}$  is bornivorous for every  $a \in N$ , .The complement of bornological open set is called bornological closed (for brief  $B$  -closed) set.

**Proposition 2.3.** [3] Let  $(X, B)$  be a bornological space. Then

- (1) A finite intersection of  $B$ - open sets is  $B$  -open set.
- (2) Union of  $B$ -open sets is  $B$  -open set.

**Remark 2.4** (7). As a consequence of the (proposition 1.3 ), family  $\tau$  of all bornological open ( $B$  - open) sets on  $X$  is topological space,  $(B, \tau)$  is called bornological topological space.

**Definition 2.5** (7). Let  $(B, \tau)$  be a bornological topological space. A collection  $U$  of subsets of  $B$  is said to be bornological base denoted by  $(B - \text{base})$  for  $B$ -topology  $\tau$  if

- (1)  $U \subset \tau$ .
- (2) For any point  $a \in X$  and each  $B$  -open set  $N$  of  $a$  there exist some  $M \in U$  such that  $a \in M \subset N$ .

**Definition 2.6** (3). Let  $(B, \tau)$  be a bornological topological space and  $N \subset E$  the intersection of all  $B$ -closed subsets of  $E$  containing  $N$  is called bornological closure set ( $B$ -closure set) denoted  $B - \overline{N}$ .

**Definition 2.7** (3). Let  $(B, \tau)$  be a bornological topological space and  $N \subset M$ . A point  $x \in X$  is said to be bornological interior point of  $N$  if there exists  $B$  -open set  $N_1$  such that  $x \in N_1 \subset N$ . The set of all bornological interior points of  $A$  is called bornological interior of  $A$  and denoted by  $B - \text{int}(A)$ .

**Definition 2.8** (3). Let  $(B, \tau)$  be a bornological topological space and  $Y \subset E$  such that  $Y$  is bornivorous set. A pair  $(Y, \tau_Y)$  is called bornological subspace of  $(B, \tau)$ .

**Definition 2.9.** Let  $(\tau_1, B)$  and  $(\tau_2, B)$  be two bornological topologies space. We say that  $\tau_1$  is  $B$ -coarser than  $\tau_2$  if  $\tau_1 \subset \tau_2$ ; that is every  $B$  - open set in  $\tau_1$  is  $B$  -open set in  $\tau_2$ .

## 3. Barnological Topological Continuous

we start this section by concept of bornological topology continuous, with some result concerning them.

**Definition 3.1.** Let  $(E, \tau_1)$  and  $(F, \tau_2)$  be  $B$ -topology spaces. A function  $f: E \rightarrow F$  is said to be bornological topology continuous ( $B\tau C$ ) at  $x_0 \in E$  iff every  $B$ -open  $M$  in  $F$  containing  $f(x_0)$  there exists  $B$  -open set  $N$  in  $E$  containing  $x_0$  such that  $f(N) \subset M$ .  $f$  is said to be  $B$  -topology continuous if and only if it is  $B$  -topology continuous for each point of  $E$ .

**Proposition 3.2.** Let  $E$  and  $F$  are  $B$  -topological spaces. A function  $f: E \rightarrow F$  is  $(B\tau C)$  if and only if the inverse image under  $f$  of every  $B$  -open set in  $F$  is  $B$  -open in  $E$ .

*Proof.* suppose that  $f$  is  $(B\tau C)$  and let  $M$  be any  $B$ -open set in  $F$  to prove  $f^{-1}(M)$  is  $B$ -open in  $E$ . If  $f^{-1}(M) = \phi$ , there is nothing to prove. So let  $f^{-1}(M) \neq \phi$  and let  $x \in f^{-1}(M)$ , then  $f(x) \in M$ . By  $B$ -topology continuous of  $f$  there exists a  $B$ -open set  $N$  in  $E$  such that  $x \in N$  and  $f(N) \subset M$  that is  $x \in N \subset f^{-1}(M)$  this shows that  $f^{-1}(M)$  is  $B$ -open in  $E$ .

Conversely, suppose  $f^{-1}(M)$  be  $B$ -open in  $E$  for every  $B$ -open set  $M$  in  $F$ . To prove  $f$  is  $(B\tau C)$  at  $x \in E$ . Let  $M$  be any  $B$ -open set in  $F$  such that  $f(x) \in M$  so that  $x \in f^{-1}(M)$ . Since  $f^{-1}(M)$  is  $B$ -open in  $E$ . If  $f^{-1}(M) = N$ , then  $N$  is a  $B$ -open set in  $E$  containing  $x$  such that  $f(N) \subset f(f^{-1}(M))$ . Hence  $f$  is  $(B\tau C)$   $\square$

**Corollary 3.3.** *Let  $E$  and  $F$  be  $B$ -topology spaces. A function  $f:E \rightarrow F$  is  $(B\tau C)$  if and only if the inverse image under  $f$  of  $B$ -closed set in  $F$  is  $B$ -closed in  $E$ .*

*Proof.* Let  $f$  is  $B$ -topology continuous and let  $F_1$  be any  $B$ -closed in  $F$ . To prove that  $f^{-1}(F_1)$  is  $B$ -closed in  $E$ , Since  $f$  is  $(B\tau C)$  and  $(f^{-1}(F_1))^c$  is  $B$ -open in  $F$ , that is  $f^{-1}(F_1^c) = (f^{-1}(F_1))^c$  is  $B$ -open in  $E$ ,  $f^{-1}(F_1)$  is  $B$ -closed in  $E$ .  $\square$

**Proposition 3.4.** *Let  $E$  and  $F$  be  $B$ -topology spaces. Then a function  $f:E \rightarrow F$  is  $(B\tau C)$  if and only if the inverse image under  $f$  of every member of a  $B$ -subbase for  $F$  is  $B$ -open in  $E$ .*

*Proof.* Let  $f$  be  $(B\tau C)$  and  $M_*$  a  $B$ -subbase for  $F$ . Since each member of  $M_*$  is  $B$ -open in  $F$ ,  $f^{-1}(D)$  is  $B$ -open in  $E$  for every  $D \in M_*$ , and let  $D_i$  be any  $B$ -open set in  $F$ . Let  $M$  be the family of all finite intersections of members  $M_*$  so that  $M$  is a  $B$ -base for  $F$ . If  $M_1 \in M$ , then there exists  $D_1, D_2, \dots, D_n$  ( $n$  finite) in  $M_*$  such that  $M_1 = D_1 \cap D_2 \cap \dots \cap D_n$ . Then  $f^{-1}(M_1) = f^{-1}(D_1) \cap f^{-1}(D_2) \cap \dots \cap f^{-1}(D_n)$  by given each  $f^{-1}(D_i)$ ,  $i = 1, \dots, n$  is  $B$ -open in  $E$ . There for  $f^{-1}(M_1)$  is also  $B$  open in  $E$  is a  $B$ -open for  $E$ ,  $D_1 = \cup M_1: M_1 \in M_* \subset M$ . Then  $f^{-1}(D_1) = f^{-1}\{(\cup M_1): M_1 \in \cup M_* \subset M\} = \cup\{f^{-1}(M_1): M_1 \in M\}$  which is  $B$ -open in  $E$  since each  $f^{-1}(M_1)$  is  $B$ -open in  $E$  (as shown above). Thus  $f^{-1}(D_1)$  is  $B$ -open in  $E$  for every  $B$ -open set  $D_1$  in  $F$  and there for  $f$  is  $B$ -topology continuous.

Conversely, let  $f^{-1}(F_1)$  be  $B$ -closed in  $E$  for every  $B$ -closed set  $F_1$  in  $F$ . We want to prove that  $f$  is a  $B$ -topology continuous function, let  $M$  be any  $B$ -open set in  $F$ . Then  $M^c$  is  $B$ -closed in  $F$  and  $f^{-1}(M^c) = (f^{-1}(M))^c$  is  $B$ -closed in  $E$ , that is  $f^{-1}(M)$  is  $B$ -open in  $E$ . Hence  $f$  is  $(B\tau C)$ .  $\square$

**Proposition 3.5.** *Let  $E$  and  $F$  are  $B$ -topological spaces. Then a function  $f:E \rightarrow F$  is  $(B\tau C)$  if and only if the inverse image of every member of a  $B$ -base for  $F$  is  $B$ -open in  $E$ .*

*Proof.* Let  $f$  be  $(B\tau C)$  and  $M_1$  any member of  $B$ -base  $M$  for  $F$ , since  $M_1$  is  $B$ -open in  $F$ ,  $f^{-1}(M_1)$  is  $B$ -open in  $E$ .

Conversely, let  $f^{-1}(M_1)$  be  $B$ -open in  $E$  by for every  $M_1 \in B$ , and let  $M_2$  be any  $B$ -open set in  $F$ . Then  $M_2$  is a union of members of  $M$  that is  $M_2 = \cup(M: M \in U \subset B)$  therefore  $f^{-1}(M_2) = f^{-1}(\cup M: M \in U)$  which is  $B$ -open in  $E$  since each  $f^{-1}(M_1)$  is  $B$ -open in  $E$  by given. He is  $(B\tau C)$ .  $\square$

**Proposition 3.6.** *A function  $f$  from a  $B$ -topological space  $E$  into  $B$ -topological space  $F$  is  $(B\tau C)$  if and only if  $f(B - \overline{N}) \subset (B - \overline{f(N)})$  for every  $N \subset E$ .*

*Proof.* Let  $f$  be  $(B\tau C)$ , since  $B - \overline{f(N)}$  is  $B$ -closed in  $F$ ,  $f^{-1}(B - \overline{f(N)})$  is  $B$ -closed in  $E$  and  $B - \overline{(f^{-1}(f(N)))} = f^{-1}(B - \overline{f(N)})$  .....(1)

since  $f(N) \subset (B - \overline{f(N)})$ ,  $N \subset f^{-1}(f(N) \subset \overline{f^{-1}(B - \overline{f(N)})})$ . Then  $(B - \overline{N}) \subset \overline{(f^{-1}(B - \overline{f(N))))} = B - \overline{f^{-1}(f(N))}$  by (1), that is  $f(B - \overline{N}) \subset (B - \overline{f(N)})$ .

Conversely, let  $f(B - \overline{N}) \subset B - \overline{f(N)}$  for every  $N \subset E$ . Let  $F_1$  be any  $B$ -closed set in  $F$  so that  $B - \overline{F_1} = F_1$ .  $f^{-1}(F_1)$  is a subset of  $E$ ,  $f(B - \overline{f^{-1}(F_1)}) \subset B - \overline{F_1} = F_1$ , so that  $B - \overline{f^{-1}(F_1)} \subset f^{-1}(F_1)$ . But  $f^{-1}(F_1) \subset (B - \overline{f^{-1}(F_1)})$  always. Hence  $B - \overline{f^{-1}(F_1)} = f^{-1}(F_1)$  and so  $f^{-1}(F_1)$  is  $B$ -closed in  $E$ . Hence  $f$  is  $(B\tau C)$ .  $\square$

**Proposition 3.7.** *A function  $f$  of a  $B$ -topological space  $E$  into  $B$ -topology space  $F$  is  $(B\tau C)$  if and only if  $B - \overline{f^{-1}(N)} \subset f^{-1}(B - N)$  for ever  $N \subset F$ .*

*Proof.* Let  $f$  be  $B$ -topological continuous. Since  $B - \overline{N}$  is  $B$ -closed in  $F$ ,  $f^{-1}(B - \overline{N})$  is  $B$ -closed in  $E$  and therefore  $B - \overline{f^{-1}(B - \overline{N})} = f^{-1}(B - \overline{N})$ .....(1)

Now  $N \subset B - \overline{N}$ ,  $f^{-1}(N) \subset (B - \overline{f^{-1}(B - \overline{N})}) = f^{-1}(B - \overline{N})$ .

Conversely, let the condition hold and let  $F_1$  be any  $B$ -closed set in  $F$  so that  $\overline{(F_1)} = F_1$ . By given  $B - \overline{f^{-1}(F_1)} \subset f^{-1}(F_1)$  but  $f^{-1}(F_1) \subset B - \overline{(f^{-1}(F_1))}$  always. Hence  $B - \overline{(f^{-1}(F_1))} = f^{-1}(F_1)$  and so  $f^{-1}(F_1)$  is  $B$ -closed in  $E$ , that  $f$  is  $(B\tau C)$ .  $\square$

**Proposition 3.8.** *A function  $f$  of a  $B$ -topology space  $E$  into another  $B$ -topology space  $F$  is  $(B\tau C)$  if and only if  $B - \text{int}(f^{-1}) \supset f^{-1}(B - \text{int}(M))$  for every  $M \subset F$ .*

*Proof.* Let  $f$  be  $(B\tau C)$ ,  $B - \text{int}(M)$  is  $B$ -open in  $F$ ,  $f^{-1}(B - \text{int}(M))$  is  $B$ -open in  $E$   $B - \text{int}(f^{-1}(B - \text{int}(M))) = f^{-1}(B - \text{int}(M))$ .....(1)

Now  $M \supset B - \text{int}(M)$ , then  $f^{-1}(M) \supset f^{-1}(B - \text{int}(M))$ . Thus  $B - \text{int}(f^{-1}(M)) \supset B - \text{int}(f^{-1}(B - \text{int}(M))) = f^{-1}(B - \text{int}(M))$  by (1).

Conversely, let the condition hold and let  $N$  be any  $B$ -open set in  $F$  so that  $B - \text{int}(N) = N$ . By given  $B - \text{int}(A) \supset f^{-1}(B - \text{int}(M)) = f^{-1}(N)$  but  $f^{-1}(N) \supset B - \text{int}(f^{-1}(N))$  always and so  $B - \text{int}(f^{-1}(N)) = f^{-1}(N)$ . Therefore  $f^{-1}(N)$  is  $B$ -open in  $E$  and consequently  $f$  is  $(B\tau C)$   $\square$

**Proposition 3.9.** *Let  $E, F, H$  be  $B$ -topological spaces and the functions  $f: E \rightarrow F$  and  $g: F \rightarrow H$  be  $(B\tau C)$  then the composition function  $g \circ f: E \rightarrow H$  is  $(B\tau C)$ .*

*Proof.* Let  $A$  be any  $B$ -open set in  $H$ . Since  $g$  is  $B$ -topology continuous  $g^{-1}(A)$  is  $B$ -open set in  $F$ . Since  $f$  is  $(B\tau C)$ ,  $f^{-1}(g^{-1}(A))$  is  $B$ -open in  $E$ ,  $f^{-1}(g^{-1}(A)) = (f^{-1} \circ g^{-1})(A) = (g \circ f)^{-1}(A)$ . Thus the inverse image under  $g \circ f$  of every  $B$ -open set in  $H$  is  $B$ -open in  $E$  and therefore  $g \circ f$  is  $B$ -topology continuous.  $\square$

**Proposition 3.10.** *Let  $E$  and  $F$  are  $B$  topology spaces and  $N$  non-empty subset of  $E$ . If  $f: E \rightarrow F$  is  $(B\tau C)$  then the restriction  $f_N$  of  $f$  to  $N$  is  $(B\tau C)$ .*

*Proof.* Since  $f_N: N \rightarrow F$  is defined by  $f_{N(x)} = f(x)$  for every  $x \in N$ . Let  $N_1$  be any  $B$ -open subset of  $F$ . Then  $f_N$  it is evident that  $f_N^{-1}(N_1) = N \cap f^{-1}(N_1)$ . Since  $f$  is  $B$ -topology continuous  $f^{-1}(N_1)$  is  $B$ -open in  $E$ . Hence  $N \cap f^{-1}(N_1)$  is  $B$ -open in  $N$ . It follows  $f_N$  is a  $(B\tau C)$ .  $\square$

**Proposition 3.11.** *Let  $\tau_i$  be an arbitrary collection of  $B$ -topologies on a set  $E$  and  $(F, V)$  any other  $B$ -topology space. If the function  $f: E \rightarrow F$  is  $(B\tau C)$  for all  $i = 1, \dots, n$ , then  $f$  is  $B$ -topology continuous with respect to the intersection  $B$ -topologies  $\tau \cap \tau_i$ .*

*Proof.* Let  $N$  be any  $B$ -open set in  $F$ . Since  $f$  is  $(B\tau C)$ ,  $f^{-1}(N) \in \tau_i$  for all  $i = 1, \dots, n$ . This implies that  $f^{-1}(N) \in \cap \tau_i = \tau$ , the inverse image of every  $B$ -open set under  $f$  is  $B$ -open in  $E$  with respect to  $\tau$ . Hence  $f$  is  $(B\tau C)$   $\square$

#### 4. Bornological Topological Homeomorphism

We define bornological topology open function, bornological topology closed function in this section and we define bornological topology homeomorphism function and investigate some properties on them.

**Definition 4.1.** *Let  $(E, \tau_1)$  and  $(F, \tau_2)$  be  $B$ -topology spaces, and let  $f$  be a function of  $E$  into  $F$ . Then (i)  $f$  is said to be a  $B$ -topology open and denoted by  $(B\tau O)$  if  $f(N)$  is  $B$ -open in  $F$  when ever  $N$  is  $B$ -open in  $E$ .*

*(2)  $f$  is said to be  $B$ -topology closed iff and denoted by  $(B\tau D)$ ,  $f(H)$  is  $B$ -closed in  $F$  when ever  $H$  is  $B$ -closed in  $E$ .*

*(3)  $f$  is said to be  $B$ -topology homeomorphism and denoted by  $(B\tau H)$  iff:*

- (i)  $f$  is one-one and onto.
- (ii)  $f$  is  $B$  - topology continuous.
- (iii)  $f^{-1}$  is  $B$ - topology continuous.

**Example 4.2.** Let  $\tau_1$  and  $\tau_2$  be two  $B$ -topological spaces on  $E$  and  $\tau_2$  be  $B$ -finer than  $\tau_1$ . If  $\tau_1$  is a  $B - \tau_0$  then  $\tau_2$  is  $B - \tau_0$ .

**Proposition 4.3.** Let  $(E, \tau_1)$  and  $(F, \tau_2)$  be  $B$ -topology spaces.  $f$  be an one-one and onto function of  $E$  to  $F$ . Then the following statements are equivalent:

- (1)  $f$  is  $(B\tau H)$  .
- (2)  $f$  is  $(B\tau C)$  and  $(B\tau O)$  .
- (3)  $f$  is a  $B$ - topology continuous and  $B$ - topology closed.

*Proof.* (1) $\implies$ (2): assume (1), let  $g$  be the inverse function of  $f$  so that  $f^{-1} = g$  and  $g^{-1} = f$ . Since  $f$  is one-to-one and onto,  $g$  is also one-to-one and onto. Let  $N$  be a  $B$  -open set in  $E$ . Since  $g$  is  $(B\tau C)$ ,  $g^{-1}(N)$  is  $B$ -open in  $F$ . But  $g^{-1} = f$  so that  $g^{-1}(N) = f(N)$  is  $B$ -open in  $F$ . It follows that  $f$  is a  $B$  - topology open function. Also  $f$  is  $(B\tau C)$  by given. Hence (1)  $\implies$  (2).

It's clear that (2)  $\implies$  (3). Now assume (3) to prove that  $g = f^{-1}$  is  $(B\tau C)$ . Let  $N$  be any  $B$ - open set in  $E$ . then  $E - A$  is  $B$ -closed since  $f$  is a  $(B\tau D)$  function  $f(E - N) = g^{-1}(E - N) = F - g^{-1}(N)$  is  $B$  -closed in  $F$ , that is  $g^{-1}(N)$  is  $B$ -open in  $F$ . Thus inverse image under  $gf$  every  $B$ -open set in  $E$  is  $B$ -open set in  $F$ . hence  $g = f^{-1}$  is  $(B\tau C)$  and so (3)  $\implies$  (1). □

**Proposition 4.4.** Let  $(E, \tau_1)$  and  $(F, \tau_2)$  two  $B$ - topological spaces. Then a function  $f: E \rightarrow F$  is  $(B\tau O)$  if and only if  $f(B - int(N)) \subset B - int(f(N))$  for every  $N \subset E$ .

*Proof.* Let  $f$  be  $(B\tau O)$  and let  $N \subset E$ . We know that  $B - int(N)$  is  $B$ -open set in  $E$ . since  $f$  is a  $(B\tau O)$  function,  $f(B - int(N))$  is  $B$ -open in  $F$ . Since  $B - int(N) \subset N$ , we have  $f(B - int(N)) \subset f(N)$ . since  $f(B - int(N))$  is a  $B$  -open in  $F$  we have  $B - int(f(B - int(N))) = f(B - int(N))$ .....(1)

Also  $f(B - int(N)) \subset f(N) \implies B - int(f(B - int(N))) \subset B - int(f(N))$ .....(2)

From (1) and (2), we have  $f(B - int(N)) \subset B - int(f(N))$ .

Conversely, let  $f(B - int(N)) \subset B - int(f(N))$  for all  $N \subset E$  and let  $N_1$  be any  $B$ -open set in  $E$ , so that  $B - int(N_1) = N_1$ . Then  $f(N_1) = f(B - int(N_1)) \subset B - int(f(N_1))$  (by given). But  $B - int(f(N_1)) \subset f(N_1)$ . Hence  $f(N_1) = B - int(f(N_1))$ . Therefore  $f(N_1)$  is  $B$ -open set in  $F$ . □

**Proposition 4.5.** Let  $(E, \tau_1)$  and  $(F, \tau_2)$  be  $B$ - topology spaces. A function  $f: E \rightarrow F$  is  $(B\tau O)$  if and only if  $B - \overline{f(N)} \subset f(B - \overline{N})$  for every  $N \subset E$

*Proof.* Let  $f$  be  $(B\tau O)$  and let  $N \subset E$ . Since  $B - \overline{N}$  is  $B$  -closed set in  $E$  and  $f$  is a  $(B\tau D)$  function, it follows that  $f(B - \overline{N})$  is  $B$ -closed set in  $F$  and  $B - (f(B - \overline{N})) = f(B - \overline{N})$ .....(1)

$N \subset \overline{N}$ ,  $f(N) \subset f(B - \overline{N})$  Also,  $f(N) \subset f(B - \overline{N})$ ,  $(B - \overline{f(N)}) \subset B - (f(B - \overline{N})) \subset B - ((f(B - \overline{N})))$ . Hence  $f$  is a  $B$  - topology closed function. □

**Proposition 4.6.** Let  $(E, \tau_1)$  and  $(F, \tau_2)$  be two  $B$ - topology spaces. A function  $f: E \rightarrow F$  be one- one and onto. Then  $f$  is a  $B$ - topology homeomorphism if and only if  $f(B - \overline{N}) = (B - \overline{f(N)})$  for every  $N \subset E$ .

*Proof.* Let  $f$  be  $(B\tau H)$ . Then  $f$  is one-one, onto,  $f$  is a  $(B\tau C)$  and  $f$  is a  $(B\tau D)$ . Let  $N$  be any subset of  $E$ . Then  $f(B - \overline{N}) \subset B - (f(N))$  by (Proposition 2.8),  $N \subset B - \overline{N}$ ,  $f(N) \subset (B - \overline{N})$ , then  $B - \overline{f(N)} \subset \overline{B - f(N)}$ .....(2)

Since  $f$  is  $B$ - topology closed and  $N$  is  $B$  -closed in  $E$ , therefore  $f(B - \overline{N})$  is  $B$ -closed in  $F$ . hence  $B - \overline{(f(B - \overline{N}))} = (f(B - \overline{N}))$ .....(2)

from (2) and (3) we have  $B - \overline{(f(N))} \subset f(B - \overline{N})$ ..... (4)

Hence from (1) and (4), we have  $f(B - \overline{N}) = B - \overline{f(N)}$ .

Conversely, let  $B - \overline{f(N)} = B - \overline{(f(N))}$  for all  $N \subset E$ .....(5)

Then  $f(B - \overline{N}) \subset B - \overline{(f(N))}$  so that  $f$  is  $(B\tau C)$ . Now let  $M$  be any  $B$  -closed set in  $E$ . so that

$B - \bar{M} = M$ . Therefore  $f(B - \bar{M}) = f(M)$  then from (5), we get  $f(M) = B - \overline{f(M)}$ . It follows that  $f(M)$  is  $B$ -closed set in  $F$ . Hence  $f$  is a  $(B\tau D)$  function. Thus it is shown that  $f$  is  $(B\tau C)$  as well as  $B$ -topology closed.  $f$  is one-one and onto. Hence  $f$  is  $(B\tau H)$ .  $\square$

**Proposition 4.7.** *Let  $(E, \tau_1)$  and  $(F, \tau_2)$  be two  $B$ -topological spaces. A function  $f: E \rightarrow F$  be  $(B\tau O)$  and onto. If  $M$  is a  $B$ -base for  $\tau_1$  then  $f(N): N \in M$  is a  $B$ -base for  $\tau_2$ .*

*Proof.* Let  $H$  be any  $B$ -open set in  $\tau_2$  and let  $y$  be any arbitrary point of  $H$ . since  $f$  is onto, there exists  $x \in E$  such that  $f(x) = y$ . Since  $M$  is a  $B$ -base for  $\tau_1$ , there exists a member of  $M$  containing  $x$ . Let  $N_*$  be the smallest member of  $M$  containing  $x$ . Then  $N_*$  is the smallest  $B$ -open containing  $x$ . Since  $f$  is  $B$ -open,  $f(N_*)$  is a  $B$ -open set in  $\tau_2$ . Also,  $x \in N_*$  that is  $f(x) \in f(N_*)$ . Since  $N_*$  is the smallest  $B$ -open in  $E$  containing  $x$ ,  $f(N_*)$  is the smallest  $B$ -open set in  $F$  containing  $f(x)$ . Now since  $H$  is one of the  $B$ -open set in  $F$  containing  $y = f(x)$ , we have  $y = f(x) \in f(N_*) \subset H$ . Hence the collection  $\{f(N): N \in M\}$  is a  $B$ -base for  $\tau_2$ .  $\square$

**Proposition 4.8.** *Let  $(E, \tau_1)$  and  $(F, \tau_2)$  be two  $B$ -topology spaces and let  $M$  be a  $B$ -base for  $\tau_1$ . If the function  $f: E \rightarrow F$  satisfy  $f(N)$  is  $B$ -open in  $F$  for every  $N \in M$ , then  $f$  is a  $(B\tau O)$  function.*

*Proof.* Let  $G$  be any  $B$ -open in  $E$ . Since  $M$  is a  $B$ -base for  $\tau_1$  we have  $G = \cup N_\lambda: N_\lambda \in M$ . Then  $f(G) = f(\cup N_\lambda \in M)$  By hypothesis each  $f(N_\lambda)$  is  $B$ -open in  $F$  it follows that  $f(G)$  is  $B$ -open in  $F$ . Hence  $f$  is a  $(B\tau O)$  function.  $\square$

**Proposition 4.9.** *Let  $(E, \tau_1)$  and  $(F, \tau_2)$  be two  $B$ -topology spaces and let  $f: E \rightarrow F$  be a  $(B\tau H)$ . Let  $N \in E$  and let  $M \subset F$  such that  $f(N) = M$  then the function  $f_N: (N, \tau_{1N}) \rightarrow (M, \tau_{2M})$  is also a  $(B\tau H)$  where  $f_N$  denoted the restriction of  $f$  to  $N$  where  $\tau_{1N}$  and  $\tau_{2M}$  are the relative  $B$ -topologies.*

*Proof.* Since  $f$  is one-to-one and  $f_N$  is also one-to-one and  $f(N) = M$  we have  $f_N(N) = M$  so that  $f_N$  is onto also. We now show that  $f_N$  is  $(B\tau O)$  and  $(B\tau C)$  Let  $H$  be any  $B$ -open in  $M$ . Then by definition of relative  $B$ -topology, we have  $H = N \cap G$  where  $G$  is  $B$ -open in  $E$ . Since  $f$  is one-one we have  $f(N \cap G) = f(N) \cap f(G)$ . Hence  $f_N(H) = f(H) = f(N \cap G) = f(N) \cap f(G) = M \cap f(G) \dots \dots (1)$  Since  $f$  is  $(B\tau O)$  and  $G$  is  $B$ -open in  $E$ , it follows that  $f(G)$  is  $B$ -open in  $F$ . Hence from (1)  $f_N(H)$  is  $B$ -open in  $M$ . It follows that  $f_N$  is a  $(B\tau O)$  function. Also since the restriction of any  $(B\tau C)$  function  $(B\tau C)f_N$  is  $(B\tau C)$ . Hence  $f_N$  is a  $(B\tau H)$ .  $\square$

## 5. Bornological Topological Separation Axioms

**Definition 5.1.** *A  $B$ -topological space  $(E, \tau)$  is said to be a bornological topology  $(B - \tau_0)$  space if and only if given any distinct points  $x, y$  there exist a  $B$ -open set  $N$  such that  $x \in N$  and  $y \notin N$  or there exists a  $B$ -open set  $M$  such that  $y \in M$  and  $x \notin M$ . Here you can put your definition.*

**Example 5.2.** *Let  $\tau_1$  and  $\tau_2$  be two  $B$ -topology spaces on  $E$  and let  $\tau_2$  be  $B$ -finer than  $\tau_1$ . If  $\tau_1$  is a  $B - \tau_0$  space then  $\tau_2$  is  $B - \tau_0$  space.*

**Theorem 5.3.** *Every sub space of a  $B - \tau_0$  space is a  $B - \tau_0$  space.*

*Proof.* Let  $(E, \tau)$  be a  $B - \tau_0$  space and  $(Y, \tau_y)$  be a sub space of  $(E, \tau)$ . Let  $y_1, y_2$  be two distinct points of  $Y$ . Since  $Y \subset E$ ,  $y_1, y_2$  are also distinct points of  $E$ . Since  $(E, \tau)$  is a  $B - \tau_0$  space, there exists a  $B$ -open  $N$  in  $E$  of  $y_1$  and  $y_2 \notin N$ . Then  $N \cap Y$  is a  $B$ -open set in  $Y$  such that  $y_1 \in (N \cap Y)$  and  $y_2 \notin (N \cap Y)$ . Hence  $(Y, \tau_y)$  is a  $B - \tau_0$  space.  $\square$

**Theorem 5.4.** *A  $B$ -topology space  $(E, \tau)$  is a  $B - \tau_0$  space if and only if for any distinct points  $x, y$  of  $E$ , the  $B$ -closure of  $x$  and  $y$  are distinct.*

*Proof.* Let  $x \notin y \implies B - (\bar{x}) \not\subseteq B - (\bar{y})$  where  $x, y$  are points of  $E$ . Since  $B - (\bar{x}) \not\subseteq B - (\bar{y})$ , there exists at least one point  $z$  of  $E$  which belongs to one of them, say  $B - (\bar{x})$ , and does not belong to  $B - (\bar{y})$ . We claim that  $x \notin B - (\bar{y})$ . Let  $x \in B - (\bar{y})$ . Then  $B - (\bar{x}) \subseteq B - \overline{B - (\bar{y})} = B - (\bar{y})$ , and so  $z \in B - (\bar{x}) \subset B - (\bar{y})$  which is a contradiction. Accordingly  $x \notin B - (\bar{y})$  and consequently  $x \in E - (B - (\bar{y}))$ . Also since  $B - (\bar{y})$  is  $B$ -closed,  $E - (B - (\bar{y}))$  is  $B$ -open set. Hence  $E - (B - (\bar{y}))$  is a  $B$ -open set containing  $x$  and not containing  $y$ , then  $(E, \tau)$  is  $B - \tau_0$  space.

Conversely, Let  $(E, \tau)$  be a  $B - \tau_0$  space and let  $x, y$  be two distinct points of  $E$ . Then we have to show that  $B - (\bar{x}) = B - (\bar{y})$ . Since the space is  $B - \tau_0$ , there exists a  $B$ -open set  $N$  containing one of them, suppose containing  $x$ , but not containing  $y$ . Then  $(E - N)$  is a  $B$ -closed set which does not contain  $x$  but contains  $y$ .  $B - (\bar{y})$  is the intersection of all  $B$ -closed sets containing  $y$ . It follows that  $B - (\bar{y}) \subset (E - N)$ . Hence  $x \notin (E - N)$  implies that  $x \notin B - (\bar{y})$ . Thus  $x \in B - (\bar{x})$ , but  $x \notin B - (\bar{y})$ . It follows that  $B - (\bar{x}) \neq B - (\bar{y})$ . Thus it is shown that in a  $B - \tau_0$  space distinct points have distinct closures.  $\square$

**Definition 5.5.** A  $B$ -topology space  $(E, \tau)$  is said to be a  $B - \tau_1$  space if and only if given any pair of distinct points  $x$  and  $y$  of  $E$  there exists two  $B$ -open sets  $N_1$  and  $N_2$  such that  $x \in N_1$  but  $y \notin N_1$  and  $y \in N_2$  but  $x \notin N_2$ .

**Theorem 5.6.** Every  $B$ -subspace of a  $B - \tau_1$  space is a  $B - \tau_1$  space.

*Proof.* Let  $(E, \tau)$  be a  $B - \tau_1$  space and let  $(Y, \tau_y)$  be a  $B$ -subspace of  $(E, \tau)$ . Let  $y_1, y_2$  be two distinct points of  $Y$ . Since  $Y \subset E$ ,  $y_1, y_2$  are also distinct points of  $E$ . Since  $(E, \tau)$  is a  $B - \tau_1$  space, there exist  $B$ -open sets  $N$  and  $M$  such that  $y_1 \in N$  but  $y_2 \notin N$  and  $y_2 \in M$  but  $y_1 \notin M$ . Then  $N_1 = N \cap Y$  and  $M_1 = M \cap Y$  are  $B$ -open sets such that  $y_1 \in N_1$  but  $y_2 \notin N_1$  and  $y_2 \in M_1$  but  $y_1 \notin M_1$ . We have  $(Y, \tau_y)$  is a  $B - \tau_1$  space.  $\square$

**Example 5.7.** Every bornological topological space  $B$ -finer than a  $B - \tau_1$  on any  $B$ -topology space  $E$  is a  $B - \tau_1$  space.

**Definition 5.8.** A  $B$ -topology space  $(E, \tau)$  is said to be a  $B - \tau_2$  space iff for every pair of distinct points  $x, y$  of  $E$  there exists  $B$ -open  $N$  of  $x$  and  $M$  of  $y$  such that  $N \cap M = \phi$

**Example 5.9.** Let  $(E, \tau_0)$  is a  $B - \tau_2$  space, if  $\tau_1$  be a  $B$ -topology space on  $E$  and  $B$ -finer than  $\tau_0$  then  $(E, \tau_1)$  is also  $B - \tau_2$  space.

*Proof.* Let  $x, y$  be any two distinct points of  $E$  since  $(E, \tau_0)$  is a  $B - \tau_2$  space, there exists  $B$ -open sets in  $\tau_0$ ,  $A_0, A_1$  such that  $x \in A_0$ ,  $y \in A_1$  and  $A_0 \cap A_1 = \phi$ . Since  $\tau_1$  is  $B$ -finer than  $\tau_0$ . Then  $A_0, A_1$  are also  $B$ -open sets in  $\tau_1$  such that  $x \in A_0$ ,  $y \in A_1$  and  $A_0 \cap A_1 = \phi$ . Hence  $(E, \tau_1)$  is also  $B - \tau_2$  space.  $\square$

**Theorem 5.10.** Every  $B$ -subspace of  $B - \tau_2$  space is  $B - \tau_2$  space.

*Proof.* Let  $(E, \tau)$  be a  $B - \tau_2$  space and  $Y$  be a non-empty subset of  $E$ . Let  $x, y$  be two distinct points of  $Y$  then  $x, y$  are also distinct points of  $E$ . Since  $E$  is  $B - \tau_2$  space there exists disjoint  $B$ -open sets  $N, M$  in  $E$  of  $x$  and  $y$  respectively.  $N \cap Y$  and  $M \cap Y$  are  $B$ -open sets in  $Y$ . Also  $x \in A$  and  $x \in Y$  then  $x \in N \cap Y$  and  $y \in B$  and  $y \in Y$ . Then  $y \in M \cap Y$  and  $N \cap M = \phi$ , we have  $(Y \cap N) \cap (Y \cap M) = Y \cap (N \cap M) = Y \cap \phi = \phi$ . Thus,  $N \cap Y$  and  $M \cap Y$  are disjoint  $B$ -open set in  $Y$  of  $x$  and  $y$  respectively. Hence  $(Y, \tau_y)$  is  $B - \tau_2$  space.  $\square$

**Theorem 5.11.** Every  $B - \tau_2$  space is a  $B - \tau_1$  space.

*Proof.* Let  $(E, \tau)$  be a  $B - \tau_2$  space and let  $x, y$  be any two distinct points of  $E$ . Since the space is  $B - \tau_2$ , there exists a  $B$ -open set  $N$  of  $x$  and a  $B$ -open  $M$  of  $y$  such that  $N \cap M = \phi$ . Hence the space is  $B - \tau_1$  space.  $\square$

**Theorem 5.12.** *Let  $(E, \tau^*)$  be a  $B - \tau_2$  space. Let  $f$  be a one-to-one, onto and  $(B\tau O)$  mapping from  $(E, \tau^*)$  to  $(F, \tau^{**})$  then  $(F, \tau^{**})$  is  $B - \tau_2$  space.*

*Proof.* Let  $y_1, y_2$  be two distinct points of  $F$ .  $f$  is one-to-one, on-to map, there exists distinct point  $x_1, x_2$  of  $E$  such that  $f(x_1) = y_1$  and  $f(x_2) = y_2$ .  $(E, \tau^*)$  is  $B - \tau_2$  space, there exist  $B$ -open sets in  $E$   $N$  and  $M$  such that  $x_1 \in N, x_2 \in M$  and  $N \cap M = \phi$ . Now since  $f$  is a  $(B\tau O)$ ,  $f(N)$  and  $f(M)$  are  $B$ -open sets in  $F$  such that:

$y_1 = f(x_1) \in f(N), y_2 = f(x_2) \in f(M)$  and  $f(N) \cap f(M) = f(N \cap M) = f(\phi) = \phi$  (since  $f$  is one-to-one). Then  $(F, \tau^{**})$  is  $B - \tau_2$  space.  $\square$

**Theorem 5.13.** *Let  $(E, \tau^{**})$  be  $B$ -topology space and  $(F, \tau^{**})$  be a  $B - \tau_2$  space. If  $f: E \rightarrow F$  be a one-to-one and  $(B\tau C)$  then  $(E, \tau^{**})$  is a  $B - \tau_2$  space..*

*Proof.* Let  $x_1, x_2$  be any two distinct points of  $E$ . Since  $f$  is one-to-one, if  $x_1 \neq x_2$  then  $f(x_1) \neq f(x_2)$ . Let  $y_1 = f(x_1), y_2 = f(x_2)$  so that  $x_1 = f^{-1}(y_1)$  and  $x_2 = f^{-1}(y_2)$ . Then  $y_1, y_2 \in F$  such that  $y_1 \neq y_2$ . Since  $(F, \tau^{**})$  is a  $B - \tau_2$  space there exists  $B$ -open sets in  $F$ ,  $N$  and  $M$  such that  $y_1 \in N$  and  $y_2 \in M$  and  $N \cap M = \phi$ . Since  $f$  a  $(B\tau C)$ .  $f^{-1}(N)$  and  $f^{-1}(M)$  are  $B$ -open sets in  $E$ .  $f^{-1}(N) \cap f^{-1}(M) = f^{-1}(N \cap M) = f^{-1}(\phi) = \phi$  and  $y_1 \in N, f^{-1}(y_1) \in f^{-1}(N)$ , that is  $x_1 \in f^{-1}(N), y_2 \in M, f^{-1}(y_2) \in f^{-1}(M), x_2 \in f^{-1}(M)$ . Thus for every pair of distinct points  $x_1, x_2$  of  $E$  there exist disjoint  $B$ -open sets if  $f^{-1}(N)$  and  $f^{-1}(M)$  such that  $x_1 \in f^{-1}(N)$  and  $x_2 \in f^{-1}(M)$ , we have  $(E, \tau^{**})$  is  $B - \tau_2$  space.  $\square$

**Theorem 5.14.** *Let  $(E, \tau^*)$  be a  $B$ -topology space and let  $(F, \tau^{**})$  be a  $B - \tau_2$  space. If  $f, g$  are  $B$ -continuous function of  $E$  into  $F$  then  $N = \{x \in E: f(x) = g(x)\}$  is a  $B$ -closed subset of  $E$ .*

*Proof.*  $E - N = \{x \in E: f(x) \neq g(x)\}$ ..... (1)

Let  $y \in E - N$  then  $y_1 = f(y), y_2 = g(y)$ . By (1) we have  $y_1 \neq y_2$ . Thus,  $y_1, y_2$  are two distinct points of  $F$ . Since  $(F, \tau^{**})$  is a  $B - \tau_2$  space there exist  $B$ -open sets  $N, M$  in  $F$  such that  $y_1 = f(y) \in N, y_2 = g(y) \in M$  and  $N \cap M = \phi$ . Then  $y \in f^{-1}(N), y \in g^{-1}(M)$ , so that  $y \in f^{-1}(N) \cap g^{-1}(M)$ . Since  $f, g$  are  $B$ -continuous functions,  $f^{-1}(N)$  and  $f^{-1}(M)$  are  $B$ -open sets in  $E$  and  $f^{-1}(N) \cap f^{-1}(M)$  is a  $B$ -open set in  $E$  containing  $y$ . Let  $y_0 \in f^{-1}(N) \cap f^{-1}(M)$ . Then  $y_0 \in f^{-1}(N)$  and  $y_0 \in f^{-1}(M)$ . We have  $f(y_0) \in N$  and  $f(y_0) \in M$ .....(2)

Since  $N \cap M = \phi$  it follows from (2) that  $f(y_0) \notin g(y_0)$  and so by (1),  $y_0 \in E - N$ . Thus we have shown that  $y_0 \in f^{-1}(N) \cap f^{-1}(M), y_0 \in E - N$ . Therefore  $f^{-1}(N) \cap f^{-1}(M) \subset E - N$ . Hence  $E - N$  contains a  $B$ -open set of each of its points and so  $E - N$  is  $B$ -open set in  $E$ . It follows that  $N$  is a  $B$ -closed subset of  $E$ .  $\square$

**Corollary 5.15.** *Let  $(E, \tau)$  be a  $B - \tau_2$  space and let  $f$  be a  $B$ -continuous function of  $E$  into itself. Then  $N = \{x \in E: f(x) = x\}$  is  $B$ -closed set of  $E$ .*

**Definition 5.16.** *A  $B$ -topological space  $(E, \tau)$  is said to be  $B - \tau_3$  space if and only if for every  $B$ -closed set  $F_1$  in  $E$  and every point  $a \notin F_1$ , there exists  $B$ -open sets  $N$  and  $M$  in  $E$  such that  $F_1 \subset N, a \in M$  and  $N \cap M = \phi$ .*

**Theorem 5.17.** *A  $B$ -topology space  $E$  is  $B - \tau_3$  space if for every point  $x \in E$  and for all  $B$ -open set  $N$  such that  $x \in N$ , there exists a  $B$ -open set  $M$  containing  $x$  such that  $B - \bar{M} \subset N$ .*

*Proof.* Let  $N$  be any  $B$ -open set of  $x$ . Then there exists  $B$ -open set  $N_1$  such that  $x \in N_1 \subset N$ , Since  $N_1^c$  is  $B$ -closed and  $x \notin N_1^c$ . There exists  $B$ -open sets  $N_2, M$  such that  $N_1^c \subset N_2, x \in M$  and  $N_2 \cap M = \phi$  so that  $M \subset N_2^c, (B - M)\bar{M} \subset B - \bar{N}_2^c$ ..... (1)

$N_1^c \subset N_2, N_2^c \subset N_1 \subset N$ ..... (2)

From (1) and (2), we have  $B - \bar{M} \subset N$ .

Conversely, Let  $M$  be any  $B$ -closed set and let  $x \notin M$ . Then  $x \in M^c$ .  $M^c$  is a  $B$ -open set containing  $x$  by given there exists a  $B$ -open set  $N$  such that  $x \in N$  and  $B - N \subset M^c$ , therefore  $M \subset (B - \bar{N})^c$ . Then  $(B - \bar{N})^c$  is a  $B$ -open set containing  $M$ . Also  $N \cap N^c = \phi, N \cap (B - \bar{N}_2^c)^c = \phi$ . Hence  $E$  is  $B - \tau_3$  space.  $\square$

**Theorem 5.18.** *Every  $B - \tau_3$  space is a  $B - \tau_2$  space.*

*Proof.* Let  $(E, \tau)$  be a  $B - \tau_3$  space and  $x, y$  be two distinct points of  $E$ . Since  $x \in E$  then every  $B$ -open set  $N$  containing  $x$  and  $y \notin N$  there exists a  $B$ -open set  $M$  containing  $x$  such that  $B - \bar{M} \subset N$  (Theorem (4-14)). Thus  $x \in B - \bar{M}$  and  $x \in N_0$  since  $(B - \bar{M})$  closed set containing  $x$  not  $y$ . Then there exists  $B$ -open set  $N_1, y \in N_1$  and  $N \cap N_1 = \phi$  ( $E$  is  $B - \tau_3$ )space. Hence  $E$  is  $B - \tau_2$  space  $\square$

**Theorem 5.19.** *Let  $(E, \tau^*)$  be a  $B - \tau_3$  space if  $f\{:\}(E, \tau^*) \longrightarrow (F, \tau^{**})$  be a  $B$ -homomorphism then  $(F, \tau^{**})$  be a  $B - \tau_3$  space.*

*Proof.* Let  $M$  be a  $B$ -closed subset of  $F$  and let  $y$  be a point of  $F$  such that  $y \in M$ . Since  $f$  is a one-one, on to function, there exists  $x \in E$  such that  $f(x) = y$  then  $f^{-1}(y) = x$ . Since  $f$  is  $B$ -continuous function  $f^{-1}(M)$  is  $B$ -closed set in  $E$ . Also,  $y \notin M$ , then  $f^{-1}(y) \notin f^{-1}(M)$  that is  $x \notin f^{-1}(M)$ . Thus  $f^{-1}(M)$  is a  $B$ -closed set in  $E$  and  $x$  is a point of  $E$  such that  $x \notin f^{-1}(M)$ .  $(E, \tau)$  is  $B - \tau_3$  space, there exists  $B$ -open sets  $N_1$  and  $N_2$  such that  $x \in N_1, f^{-1}(M) \subset N_2$  and  $N_1 \cap N_2 = \phi$ .  $x \in N_1$ . Then  $f(x) \in f(N_1)$  that is  $y \in f(N_1), f^{-1}(M) \subset N_2$  therefore  $f(f^{-1}(M)) \subset f(N_2)$ , implies  $B \subset f(N_2)$  and  $N \cap N_2 = \phi$ .  $f(N_1 \cap N_2) = f(\phi), f(N_1) \cap f(N_2) = \phi$  (since  $f$  is one-to-one). Also since  $f$  is a  $B$ -open function,  $N_3 = f(N_1)$  and  $N_4 = f(N_2)$  are  $B$ -open sets in  $F$ . Thus there exists  $B$ -open sets  $N_3$  and  $N_4$  such that  $y \in N_3, M \subset N_4 = \phi$  and  $N_3 \cap N_4 = \phi$ . Then  $(F, \tau^{**})$  is also  $B - \tau_3$  space.  $\square$

**Theorem 5.20.** *Let  $(E, \tau)$  be a  $B - \tau_3$  space and  $(Y, \tau_Y)$  be a  $B$ -subspace of  $(E, \tau)$  then  $(Y, \tau_Y)$  is  $B - \tau_3$  space.*

*Proof.* Let  $M$  be a  $B$ -closed subset of  $y$  and  $x$  be a point of  $Y$  such that  $x \notin M$ . Then  $(B - \bar{B})_Y = (B - \bar{M}_E) \cap Y$ . Since  $M$  is  $B$ -closed set in  $Y$ , we have  $(B - \bar{M})_Y = M, M = (B - \bar{M}) \cap Y \dots \dots (1)$   
Therefore  $x \notin M, \notin (B - \bar{M})_E \cap Y$  that is  $x \notin (B - \bar{M})_E$ . Thus  $(B - \bar{M})_E$  is a  $B$ -closed subset of  $E$  such that  $x \notin (B - \bar{M})_E$ . Since  $(E, \tau)$  is a  $B - \tau_3$ , there exists  $B$ -open set  $N_1$  and  $N_2$  such that  $x \in N_1, (B - \bar{M})_E \subset N_2$  and  $N_1 \cap N_2 = \phi$ . Since  $x \in N_1, x \in Y$  that is  $x \in N_1 \cap Y, (B - \bar{M})_E \subset N_2$ . Then  $(B - \bar{M})_E \cap Y \subset N_2 \cap Y, M \subset N_2 \cap Y$  by (1). Also  $(N_1 \cap Y) \cap (N_2 \cap Y) = (N_1 \cap N_2) \cap Y = \phi \cap Y = \phi$ . Thus  $(Y, \tau_Y)$  is a  $B - \tau_3$  space.  $\square$

**Definition 5.21.** *A  $B$ -topology space  $(E, \tau)$  is said to be  $B - \tau_4$  if and only if for every pair  $M_1, M_2$  of disjoint  $B$ -closed sets of  $E$  there exists  $B$ -open sets  $N_1, N_2$  such that  $M_1 \subset N_1, M_2 \subset N_2$  and  $N_1 \cap N_2 = \phi$ .*

**Theorem 5.22.** *A  $B$ -topology space  $E$  is  $B - \tau_4$  if and only if for any  $B$ -closed  $M$  and  $B$ -open set  $N_1 \supset M$  there exists a  $B$ -open set  $N_2$  such that  $M \subset N_2$  and  $B - N_2 \subset N_1$ .*

*Proof.* Let  $E$  be a  $B - \tau_4$  space and  $M$  be  $B$ -closed set and  $N_1$  a  $B$ -open set such that  $M \subset N_1$ . Then  $N_1^c$  is a  $B$ -closed set such that  $M \cap N_1^c = \phi$ . Thus,  $N_1^c$  and  $M$  are disjoint  $B$ -closed subset of  $E$ . Since the space is  $B - \tau_4$ , there exists  $B$ -open sets  $N_2$  and  $N_3$  such that  $N_1^c \subset N_2, M \subset N_3$  and  $N_2 \cap N_3 = \phi$  so that  $N_3 \subset N_2^c$ . But  $N_3 \subset N_2^c, B - \bar{N}_3 \subset B - (N_2^c) = N_2^c$  ( $N_2^c$  is closed)  $\dots \dots (1)$

Also,  $N_1^c \subset N_2, N_2^c \subset N_1 \dots \dots (2)$ .

from (1) and (2), we get  $B - N_3 \subset N_1$ . Then there exists a  $B$ -open set  $N_3$  such that  $M \subset N_3$  and  $B - N_3 \subset N_1$ .

Conversely, Let  $M_1$  and  $M_2$  be  $B$ -closed subsets of  $E$  such that  $M_1 \cap M_2 = \phi$  so that  $M_1 \subset M_2^c$ . Thus the  $B$ -closed set  $M_1$  is contained in the  $B$ -open set  $M_2^c$ . Then there exists a  $B$ -open set  $N_1$  such that  $M_1 \subset N_1$  and  $B - \bar{N}_1 \subset M_2^c$  which implies  $M_2 \subset (B - N_1)^c$ .

Also,  $N_1 \cap (B - N_1)^c = \phi$ . Thus  $N_1$  and  $(B - N_1)^c$  are two disjoint open set such that  $M_1 \subset N_1$  and  $M_2 \subset (B - N_1)^c$  Then the space is  $B - \tau_4$ .  $\square$

**Theorem 5.23.** *Let  $(E, \tau_1)$  be a  $(B - \tau_4)$  space and  $f:(E, \tau_1) \longrightarrow (F, \tau_2)$  is a  $B$ -homomorphism then  $(F, \tau_2)$  is a  $B - \tau_4$  space.*

*Proof.* Let  $M_1, M_2$  be a disjoint  $B$ -closed subset of  $F$ . Since  $f$  is a  $B$ -continuous function  $f^{-1}(M_1)$  and  $f^{-1}(M_2)$  are  $B$ -closed subset of  $E$ .  $f^{-1}(M_1) \cap f^{-1}(M_2) = f^{-1}(M_1 \cap M_2) = f^{-1}(\phi) = \phi$ . Thus  $f^{-1}(M_1), f^{-1}(M_2)$  is a disjoint of  $B$ -closed subsets of  $E$ . Since the space  $(E, \tau_1)$  is  $B - \tau_4$  space, there exists  $B$ -open sets  $N_1$  and  $N_2$  of  $E$  such that  $f^{-1}(M_1) \subset N_1, f^{-1}(M_2) \subset N_2$  and  $N_1 \cap N_2 = \phi$ . But  $f^{-1}(M_1) \subset N_1$ . Then  $f(f^{-1}(M_1)) \subset f(N_1)$ , therefore  $M_1 \subset f(N_1)$  and we have  $M_2 \subset f(N_2)$ . since  $f$  is a  $B$ -open function,  $f(N_1)$  and  $f(N_2)$  are  $B$ -open subsets of  $F$ , such that  $f(N_1) \cap f(N_2) = f(N_1 \cap N_2) = f(\phi) = \phi$  (since  $f$  is one-to-one). There exists  $B$ -open subsets  $N_3 = f(N_1)$  and  $N_4 = f(N_2)$  of  $F$  such that  $M_1 \subset N_3, M_2 \subset N_4$  and  $N_3 \cap N_4 = \phi$ . Then  $(F, \tau_2)$  is also a  $B - \tau_4$  space.  $\square$

**Theorem 5.24.** *Every  $B - \tau_4$  space is also a  $B - \tau_3$  space.*

*Proof.* Let  $(E, \tau)$  be a  $B - \tau_4$  space and  $(Y, \tau_y)$  any  $B$ -closed  $B$ -subspace of  $E$ , then  $(Y, \tau_y)$  is  $B - \tau_4$  space. Let  $B_1^*, B_2^*$  be disjoint  $B$ -closed subsets of  $Y$ . Then there exists  $B$ -closed subsets  $M_1, M_2$  of  $E$  such that  $M_1^* = M_1 \cap Y$  and  $M_2^* = M_2 \cap Y$ . Since  $Y$  is  $B$ -closed it follows that  $M_1^*, M_2^*$  are disjoint  $B$ -closed subsets of  $E$ . Then there exists  $B$ -open subsets  $N_1, N_2$  of  $E$  such that  $M_1^* \subset N_1, M_2^* \subset N_2$  and  $N_1 \cap N_2 = \phi$ .  $(E, \tau)$  be a  $B - \tau_4$  space. Since  $M_1^* \subset Y$  and  $M_2^* \subset Y$ , these relations imply that  $M_1^* \subset N_1 \cap Y = N_1^*, M_2^* \subset N_2 \cap Y = N_2^*$  and  $N_1^* \cap N_2^* = \phi$ . Then  $N_1^*, N_2^*$  are  $B$ -open subsets of  $Y$ . Such that  $M_1^* \subset N_1^*, N_2^* \subset N_2^*$  and  $N_1^* \cap N_2^* = \phi$ . We have  $(Y, \tau_Y)$  is a  $B - \tau_4$  space.  $\square$

## 6. Conclusion

In this paper we find new structure by use bornological space, we introduce the structure of bornological topological space. we define bornological topological open function, bornological topological closed function and bornological topological homeomorphism function, bornological topological continuous and investigate some new properties of them. Separation axioms have been studied in bornological topological space like  $B - \tau_0, B - \tau_1, B - \tau_2, B - \tau_3, B - \tau_4$  and the relationships among them, also we study topological properties and heredity property under exist this axioms.

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