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Periodic Solutions for a Higher-order p-Laplacian Neutral Differential Equation with Multiple Deviating Arguments

Loubna Moutaouekkil, Omar Chakrone, Zakaria El Allali and Said Taarabti.

ABSTRACT: In this article, we consider the following higher-order p-Laplacian neutral differential equation with multiple deviating arguments:

$$(\varphi_p(x(t)-cx(t-r))^{(m)}(t))^{(m)} = f(x(t))x'(t)+g(t,x(t),x(t-\tau_1(t)),...,x(t-\tau_k(t)))+e(t).$$

By applying the continuation theorem, theory of Fourier series, Bernoulli numbers theory and some analytic techniques, sufficient conditions for the existence of periodic solutions are established.

Key Words: Periodic solution, neutral equation, deviating argument, higher-order, p-Laplacian, Mawhin's continuation.

Contents

1. Introduction

In the last several years, the existence of periodic solutions for functional differential equations have been widely studied and are still being investigated due to their applications in many fields such as physics, mechanics, the engineering technique fields and so on...(see for example [1-2] and the references given therein), especially, the p-laplacian functional differential equations which arises from fluid mechanical and nonlinear elastic mechanical phenomena has received more and more attention for example in paper [3], by using Mawhin's continuation theorem, the authors have studied the existence of periodic solution for p-Laplacian neutral functional differential equation:

$$(\varphi_p(x'(t)-c(t)x'(t-r)))' = f(x(t))x'(t)+g(t,x(t),x(t-\tau_1(t)),...,x(t-\tau_m(t)))+e(t).$$

where $|c|_0 < \frac{1}{2}, \tau_i \in C(\mathbb{R},\mathbb{R})(i=1,2,...,k)$ with $\tau_i(t+T)=\tau_i(t)$.

Recently, there has been a great deal of work on the problem of the periodic solutions of higher-order differential equations. However, as far as we know, work on the existence of periodic solutions for higher-order p-Laplacian differential equations was discussed in [8-9]. For instance, Li [9] had studied the existence and uniqueness of periodic solutions for a kind of higher-order p-Laplacian differential equation as follows:

$$(\varphi_p(x^{(m)}(t)))^{(m)} + \beta(t))x'(t)+g(t,x(t)) = e(t).$$

In the present paper, motivated by [5-8-9] mentioned previously, we aim at studying the existence of periodic solutions for the following higher-order p-Laplacian neutral differential equation with multiple deviating arguments:

$$(\varphi_p(x(t)-cx(t-r))^{(m)}(t))^{(m)} = f(x(t))x'(t)+g(t,x(t),x(t-\tau_1(t)),...,x(t-\tau_k(t)))+e(t). \tag{1.1}$$

Where $p \geq 2$ is a fixed real number. The conjugate exponent of p is denoted by q, i.e $\frac{1}{p}+\frac{1}{q}=1$. Let $\varphi_p : \mathbb{R} \to \mathbb{R}$ be the mapping defined by $\varphi_p(s)=|s|^{p-2}s$ for $s \neq 0$, and $\varphi_p(0)=0$, m is a positive integer,

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c, r are constant with $|c| < 1, r \geq 0$ $f, e \in C(\mathbb{R}, \mathbb{R})$ are continuous T-periodic functions defined on $\mathbb{R}$ and $T > 0$, $g \in C(\mathbb{R}^{k+2}, \mathbb{R})$ and $g(t+T, u_0, u_1, ..., u_k) = g(t, u_0, u_1, ..., u_k), \forall (t, u_0, u_1, ..., u_k) \in \mathbb{R}^{k+2}$, $\tau_i \in C^1(\mathbb{R}, \mathbb{R})(i = 1, 2, ..., k)$ with $\tau_i(t+T) = \tau_i(t)$.Therefore, in this paper based on the Mawhin continuation theorem and some analysis skills without assumption of $\int_0^T e(t)dt = 0$, some new sufficient conditions for the existence of T-periodic solution of p-Laplacian equation (1.1) will be established. The rest of this paper is organized as follows: Section 2 is devoted to introducing some definitions and recalling some preliminary results that will be extensively used. The existence results will be obtained in Section 3. Finally, a example is given to illustrate the effectiveness of our result in Section 4. Our results are different from those of bibliographies listed in the previous texts and they are a generalization of the results of the article [3] in the case where c is constant with $|c| < 1, p \geq 2, \tau_i \in C^1(\mathbb{R}, \mathbb{R})(i = 1, 2, ..., k)$.

2. Preliminaries

For convenience, define $C_T = \{x | x \in C(\mathbb{R}, \mathbb{R}), x(t+T) = x(t)\}$ with the norm $|x|_0 = \max_{t \in [0,T]} |x(t)|$, and $C_T^1 = \{x | x \in C^1(\mathbb{R}, \mathbb{R}), x(t+T) = x(t)\}$ with the norm $\|x\| = \max_{t \in [0,T]} \{|x|_0, |x'|_0\}$. Define a linear operator $A : C_T \to C_T,\ (Ax)(t) = x(t) - cx(t-r)$.

Lemma 2.1. *([7]) If $|c| < 1$, then A has continuous bounded inverse on C_T with the following properties:*

(1) $\|A^{-1}x\| \leq \dfrac{|x|_0}{1-|c|},\ \forall x \in \mathcal{C}_T$

(2) $\displaystyle\int_0^T |(A^{-1}x)(t)|^p dt \leq \frac{1}{(1-|c|)^p} \int_0^T |x(t)|^p dt,\ \forall x \in \mathcal{C}_T.$

Lemma 2.2. *([16]) Let $T > 0$ be constant, $x \in C^m(\mathbb{R}, \mathbb{R}), m \geq 2$ and $x(t+T) = x(t)$, $|x^{(i)}|_0 = \max_{t \in [0,T]} |x^{(i)}(t)|$ then there are $M_i(m) > 0$ independent of x such that*

$$|x^{(i)}|_0 \leq M_i(m) \int_0^T |x^{(m)}(t)| dt \ \ i = 1, 2, \ldots, m-1, \tag{2.1}$$

where, if m is an even integer

$$M_i(m) = \begin{cases} M_{2s-1}(m) = T^{m-2s}\sqrt{\dfrac{-B_{2m-4s}}{12(2m-4s)!}},\ s = 1, 2, \ldots, \dfrac{m}{2} - 1; \\[2ex] M_{2s}(m) = \dfrac{(-1)^{\frac{m-2s}{2}+1} T^{m-2s-1} B_{m-2s}}{(m-2s)!},\ s = 1, 2, \ldots, \dfrac{m}{2} - 1; \\[2ex] M_{m-1}(m) = \frac{1}{2}, \end{cases} \tag{2.2}$$

if m is an odd integer

$$M_i(m) = \begin{cases} M_{2s+1}(m) = \dfrac{(-1)^{\frac{m-2s-1}{2}+1} T^{m-2s-2} B_{m-2s-1}}{(m-2s-1)!},\ s = 1, 2, \ldots, \dfrac{m+1}{2} - 2; \\[2ex] M_{2s}(m) = T^{m-2s-1}\sqrt{\dfrac{-B_{2m-4s-2}}{12(2m-4s-2)!}},\ s = 1, 2, \ldots, \dfrac{m+1}{2} - 2; \\[2ex] M_{m-1}(m) = \frac{1}{2} \end{cases} \tag{2.3}$$

and $B_{m-2s}, B_{2m-4s}, B_{m-2s-1}, B_{2m-4s-2}$ are Bernoulli numbers, which can be calculed using the following recursion formula:

$B_0 = 1, B_p = \dfrac{-\sum_{i=0}^{p-1} C_{p+1}^i B_i}{p+1}$,
where C_{p+1}^i *is the combination number.*

Lemma 2.3. *Let* $k > 0, T > 0$ *be two constant,* $s \in C_T(\mathbb{R}, \mathbb{R})$, $\tau_i \in C_T^1(\mathbb{R}, \mathbb{R})$ *and* $|\tau_i'|_0 < 1$. *Then*

$$\int_0^T |s(t-\tau_i(t))|^k dt \le \delta_i \int_0^T |s(t)|^k dt,$$

where $\delta_i = \frac{1}{1-|\tau_i'|_0}, |\tau_i'|_0 = \max_{t\in[0,T]} |\tau_i'(t)|$.

Proof. It is easy to see that

$$\int_0^T |s(t-\tau_i(t))|^k dt = \int_0^T |s(t-\tau_i(t))|^k d(t-\tau_i(t)) + \int_0^T \tau_i'(t)|s(t-\tau_i(t))|^k dt$$

i.e.

$$(1-|\tau_i'|_0)\int_0^T |s(t-\tau_i(t))|^k dt \le \int_0^T |s(t)|^k dt$$

and thus

$$\int_0^T |s(t-\tau_i(t)|^k dt \le \frac{1}{1-|\tau_i'|_0}\int_0^T |s(t)|^k dt.$$

This completes the proof. □

Lemma 2.4. *(Borsuk [14]).* $\Omega \subset \mathbb{R}^n$ *is an open bounded set, and symmetric with respect to* $0 \in \Omega$. *If* $f \in C(\overline{\Omega}, \mathbb{R}^n)$ *and* $f(x) \neq \mu f(-x), \forall x \in \partial\Omega, \forall \mu \in [0,1]$, *then* $deg(f, \Omega, 0)$ *is an odd number.*

Now, we recall Mawhin's continuation theorem which our study is based upon.
Let X and Y be real Banach spaces and $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero. Here $D(L)$ denotes the domain of L. This means that ImL is closed in Y and $\dim KerL = \dim(Y/ImL) < +\infty$. Consider the supplementary subspaces X_1 and Y_1 and such that $X = KerL \oplus X_1$ and $Y = ImL \oplus Y_1$ and let $P : X \to KerL$ and $Q : Y \to Y_1$ be natural projections. Clearly, $KerL \cap (D(L) \cap X_1) = \{0\}$, thus the restriction $L_p := L|_{D(L)\cap X_1}$ is invertible. Denote the inverse of L_p by K.
Now, let Ω be an open bounded subset of X with $D(L)\cap\Omega \neq \emptyset$, a map $N : \overline{\Omega} \to Y$ is said to be L-compact on $\overline{\Omega}$, if $QN(\overline{\Omega})$ is bounded and the operator $K(I-Q)N : \overline{\Omega} \to Y$ is compact.

Lemma 2.5. *(Mawhin [12]). Suppose that* X *and* Y *are two Banach spaces, and* $L : D(L) \subset X \to Y$ *is a Fredholm operator with index zero. Furthemore,* $\Omega \subset X$ *is an open bounded set and* $N : \overline{\Omega} \to Y$ *is L-compact on* $\overline{\Omega}$. *If all of the following conditions hold:*

(1) $Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L), \lambda \in]0,1[$;

(2) $Nx \notin ImL, \forall x \in \partial\Omega \cap KerL$;

(3) $deg\{JQN, \Omega \cap KerL, 0\} \neq 0$, *where* $J : ImQ \to KerL$ *is an isomorphism.*

Then the equation $Lx = Nx$ *has at least one solution on* $\overline{\Omega} \cap D(L)$.

In order to use Mawhin's continuation theorem to study the existence of T-periodic solution for equation (1.1), we rewrite equation (1.1) in the following system

$$\begin{cases} x_1^{(m)}(t) = [A^{-1}\varphi_q(x_2)](t), \\ x_2^{(m)}(t) = f(x_1(t))x_1'(t) + g(t, x_1(t), x_1(t-\tau_1(t)), ..., x_1(t-\tau_m(t))) + e(t). \end{cases} \tag{2.4}$$

Where $q \geq 2$ is constant with $\frac{1}{p}+\frac{1}{q}=1$. Clearly, if $x(t)=(x_1(t),x_2(t))^\top$ is a T-periodic solution to equation set (2.4), then $x_1(t)$ must be a T-periodic solution to equation (1.1). Thus, in order to prove that equation (1.1) has a T-periodic solution, it suffices to show that equation set (2.4) has a T-periodic solution.
$X=\{x=(x_1(t),x_2(t))^\top \in C^1(\mathbb{R},\mathbb{R}^2) : x(t+T)=x(t)\}$ with the norm $\|x\|_X = max\{\|x_1\|,\|x_2\|\}$, $Y=\{x=(x_1(t),x_2(t))^\top \in C(\mathbb{R},\mathbb{R}^2) : x(t+T)=x(t)\}$ with the norm $\|x\|_Y = max\{|x_1|_0,|x_2|_0\}$. Obviously, X and Y are two Banach spaces. Meanwhile, let

$$L : D(L) \subset X \to Y,\ Lx = x^{(m)} = \begin{pmatrix} x_1^{(m)} \\ x_2^{(m)} \end{pmatrix}. \tag{2.5}$$

$$N : X \to Y,$$
$$[Nx](t) = \begin{pmatrix} [A^{-1}\varphi_q(x_2)](t) \\ f(x_1(t))x_1'(t)+g(t,x_1(t),x_1(t-\tau_1(t)),\ldots,x_1(t-\tau_m(t)))+e(t), \end{pmatrix} \tag{2.6}$$

where $D(L)=\{x=(x_1(t),x_2(t))^\top \in C^m(\mathbb{R},\mathbb{R}^2) : x(t+T)=x(t)\}$. It is easy to see that equation set (2.4) can be converted to the abstract equation $Lx=Nx$. Moreover, from the definition of L, we see that $KerL=\mathbb{R}^2,\ ImL=\{y : y\in Y, \int_0^T y(s)ds=0\}$. So L is a Fredholm operator with index zero.
Let projectors $P : X \to KerL$ and $Q : Y \to ImQ$ be defined by

$$Px = x(0),\ Qy = \frac{1}{T}\int_0^T y(s)ds$$

and let K represent the inverse of $L|_{KerP\cap D(L)}$. Clearly, $KerL = ImQ = \mathbb{R}^2$ and

$$[Ky](t) = \sum_{i=1}^{m-1} \frac{1}{i!}x^{(i)}(0)t^i + \frac{1}{(m-1)!}\int_0^t (t-s)^{m-1}y(s)ds, \tag{2.7}$$

where $x^{(i)}(0)$ $(i=1,2,\ldots,m-1)$ are defined by the equation $AX=D$,
where

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ c_1 & 1 & 0 & \cdots & 0 & 0 \\ c_2 & c_1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{m-3} & c_{m-4} & c_{m-5} & \cdots & 1 & 0 \\ c_{m-2} & c_{m-3} & c_{m-4} & \cdots & c_1 & 1 \end{pmatrix}$$

$$X = (x^{(m-1)}(0), x^{(m-2)}(0), \ldots, x''(0), x'(0))^\top$$
$$D = (d_1, d_2, \ldots, d_{m-2}, d_{m-1})^\top$$

$$d_i = -\frac{1}{i!T}\int_0^T (T-s)^i y(s)ds \qquad i=1,2,\ldots,m-1$$

and

$$c_j = \frac{T^j}{(j+1)!} \qquad j=1,2,\ldots,m-2.$$

From (2.6)and (2.7), it isn't hard to find that N is L-compact on $\overline{\Omega}$, where Ω is an arbitrary open bounded subset of X.
For the sake of convenience, we list the following assumptions which will be used by us in studing the existence of $T-$ periodic solution to equation (1.1).

(H_1) There is a constant $d > 0$ such that:

(1) $g(t, u_0, u_1, \ldots, u_k) > |e|_0, \forall (t, u_0, u_1, \ldots, u_k) \in [0, T] \times \mathbb{R}^{k+1}$ with $u_i > d$ $(i = 0, 1, \ldots, k)$.

(2) $g(t, u_0, u_1, \ldots, u_k) < -|e|_0, \forall (t, u_0, u_1, \ldots, u_k) \in [0, T] \times \mathbb{R}^{k+1}$ with $u_i < -d$ $(i = 0, 1, \ldots, k)$.

(H_2) $|g(t, u_0, u_1, \ldots, u_k)| \leq \sum_{i=0}^{k} \alpha_i |u_i|^{p-1} + \beta$,
where $\alpha_i (i = 0, \ldots, k), \beta$ are positive constants.

(H_3) There exist positive constants l, δ

$$|f(x)| \leq l|x|^{p-2} + \delta.$$

3. Main results

Lemma 3.1. *Suppose that (H_1) hold, if $x \in D(L)$ is an arbitrary solution of the equation $Lx = \lambda N x, \lambda \in]0, 1[$,where L and N are defined by (2.5) and (2.6), respectively, then there must be a point $t^* \in [0, T]$ such that*

$$|x_1(t^*)| \leq d. \tag{3.1}$$

Proof. Suppose $x \in D(L)$ is an arbitrary solution of the equation $Lx = \lambda N x$, for some $\lambda \in]0, 1[$, then

$$\begin{cases} x_1^{(m)}(t) = \lambda [A^{-1} \varphi_q(x_2)](t), \\ x_2^{(m)}(t) = \lambda f(x_1(t)) x_1'(t) + \lambda g(t, x_1(t), x_1(t - \tau_1(t)), ..., x_1(t - \tau_k(t))) + \lambda e(t). \end{cases} \tag{3.2}$$

From the first equation of (3.2), we have $x_2(t) = \lambda^{-1} \varphi_p[(Ax_1)^{(m)})(t)]$ and then by substituting it into the second equation of (3.2),we have

$$(\varphi_p(Ax_1)^{(m)}(t)))^{(m)} = \lambda^p f(x_1(t)) x_1'(t) + \lambda^p g(t, x_1(t), x_1(t - \tau_1(t)), ..., x_1(t - \tau_k(t))) + \lambda^p e(t). \tag{3.3}$$

Integrating both sides of equation(3.3) on the interval $[0, T]$, we have
$\int_0^T g(t, x_1(t), x_1(t - \tau_1(t)), ..., x_1(t - \tau_k(t))) + \int_0^T e(t) = 0.$
By the integral mean value theorem, there is a constant $t_0 \in [0, T]$ such that

$$g(t, x_1(t_0), x_1(t_0 - \tau_1(t_0)), ..., x_1(t_0 - \tau_k(t_0))) = -\frac{1}{T} \int_0^T e(t) dt. \tag{3.4}$$

Case 1 If $|x_1(t_0)| \leq d$, then taking $t^* = t_0$ such that $|x_1(t^*)| \leq d$.
Case 2 If $|x_1(t_0)| > d$, in this case we need to prove that there exist $\xi \in \mathbb{R}$ such that
$|x_1(\xi)| \leq d$.
By (3.4), we can get
$g(t, x_1(t_0), x_1(t_0 - \tau_1(t_0)), ..., x_1(t_0 - \tau_k(t_0))) = -\frac{1}{T} \int_0^T e(t) dt \leq |e|_0.$
From assumption $(H_1)(1)$, we see that there exist $r \in \{1, 2, ..., k\}$ such that
$x_1(t_0 - \tau_r(t_0)) \leq d$.
On the other hand, we have
$g(t, x_1(t_0), x_1(t_0 - \tau_1(t_0)), ..., x_1(t_0 - \tau_k(t_0))) = -\frac{1}{T} \int_0^T e(t) dt \geq -|e|_0.$
From $(H_1)(2)$ there exist $l \in \{1, 2, ..., k\}$ such that $x_1(t_0 - \tau_l(t_0)) \geq -d$.
In this case we consider the following two other cases

- If $l = r$,we get $|x_1(t_0 - \tau_l(t_0))| \leq d$, then taking $\xi = x_1(t_0 - \tau_l(t_0))$ such that $|x_1(\xi)| \leq d$.
- If $l \neq r$ we consider three other cases:

– If $x_1(t_0-\tau_l(t_0)) \leq x_1(t_0-\tau_r(t_0))$, which yield $|x_1(t_0-\tau_l(t_0))| \leq d$ and $|x_1(t_0-\tau_r(t_0))| \leq d$, let $\xi = x_1(t_0-\tau_l(t_0))$ or $\xi = x_1(t_0-\tau_r(t_0))$ obviously $|x_1(\xi)| \leq d$.

– If $x_1(t_0-\tau_r(t_0)) \leq x_1(t_0-\tau_l(t_0))$ and one of the following assumption hold $x_1(t_0-\tau_r(t_0)) \geq -d$ or $x_1(t_0-\tau_l(t_0)) \leq d$, we assume $\xi = x_1(t_0-\tau_l(t_0))$ or $\xi = x_1(t_0-\tau_r(t_0))$, we can obtain $|x_1(\xi)| \leq d$.

– If $x_1(t_0-\tau_r(t_0)) \leq x_1(t_0-\tau_l(t_0))$, $x_1(t_0-\tau_r(t_0)) < -d$ and $x_1(t_0-\tau_l(t_0)) > d$. By intermediate value theorem there exist t_1 such that $x_1(t_1) = 0$, then taking $\xi = t_1$, we have $|x_1(\xi)| \leq d$.

Let $k' = \left[\frac{\xi}{T}\right]$, where $\left[\frac{\xi}{T}\right]$ is integer part of the number $\frac{\xi}{T}$, then taking $t^* = \xi - k'T$. Furthermore, $|x_1(t^*)| \leq d$ with $t^* \in [0,T]$.
□

Theorem 3.2. *Suppose* $|\tau_i'|_0 < 1,\ (i = 1, \cdots, k)$ *and assumption* $(H_1)-(H_3)$ *hold. Then equation (1.1) has at one least one* $T-$*periodic solution,*
if $\dfrac{(1+|c|)M_1^p(m)T^{2p-1}}{2^{p-1}(1-|c|)^p}\left[l + \dfrac{T}{2}(\alpha_0 + \sum_{i=1}^{k}\alpha_i\delta_i)\right] < 1,$ *where* $M_1(m)$ *and* δ_i *are defined in Lemma 2.2, Lemma 2.3.*

Proof. Let $\Omega_1 = \{x \in X : Lx = \lambda Nx, \lambda \in]0,1[\}$ if $x(.) = (x_1(.), x_2(.))^\top \in \Omega_1$, then from (2.5) and (2.6), we have

$$\begin{cases} x_1^{(m)}(t) = \lambda[A^{-1}\varphi_q(x_2)](t), \\ x_2^{(m)}(t) = \lambda f(x_1(t))x_1'(t) + \lambda g(t, x_1(t), x_1(t-\tau_1(t)), ..., x_1(t-\tau_m(t))) + \lambda e(t). \end{cases} \tag{3.5}$$

From Lemma 3.1, we have

$$|x_1(t)| = |x_1(t^*) + \int_{t^*}^{t} x_1'(s)ds| \leq d + \int_{t^*}^{t} |x_1'(s)|ds, \quad t \in [t^*, t^*+T],$$

and

$$|x_1(t)| = |x_1(t-T)| = |x(t^*) - \int_{t-T}^{t^*} x_1'(s)ds| \leq d + \int_{t^*-T}^{t^*} |x_1'(s)|ds, \quad t \in [t^*, t^*+T].$$

Combining the above two inequalities, we obtain

$$\begin{aligned} |x_1|_0 = \max_{t\in[0,T]} |x_1(t)| = \max_{t\in[t^*,t^*+T]} |x_1(t)| &\leq \max_{t\in[t^*,t^*+T]} \left\{ d + \frac{1}{2}\left(\int_{t^*}^{t} |x_1'(s)|ds + \int_{t-T}^{t^*} |x_1'(s)|ds \right) \right\} \\ &\leq d + \frac{1}{2}\int_0^T |x_1'(s)|ds. \end{aligned} \tag{3.6}$$

On the hand, multiplying both sides of equation (3.3) by $[Ax_1](t)$ and integrating it from 0 to T, we obtain

$$\begin{aligned} \int_0^T (\varphi_p(Ax_1^{(m)})(t))^{(m)}(Ax_1)(t)dt \leq{}& (1+|c|)|x_1|_0 \int_0^T |f(x_1(t))||x_1'(t)|dt \\ &+ (1+|c|)|x_1|_0 \int_0^T |g(t, x_1(t), x_1(t-\tau_1(t)), ..., x_1(t-\tau_k(t)))|dt \\ &+ (1+|c|)|x_1|_0 \int_0^T |e(t)|dt. \end{aligned}$$

Case 1. If m is even,we obtain

$$\int_0^T (\varphi_p(Ax_1^{(m)})(t))^{(m)}(Ax_1)(t)dt = (-1)^m \int_0^T |(Ax_1)^{(m)}(t)|^p dt = \int_0^T |(Ax_1)^{(m)}(t)|^p dt.$$

In view of assumption $(H_2)-(H_3)$ we have

$$\begin{aligned}\int_0^T |(Ax_1)^{(m)}(t)|^p dt \le & (1+|c|)|x_1|_0 \int_0^T (l|x_1(t)|^{p-2}+\delta)|x_1'(t)|dt \\ & + (1+|c|)|x_1|_0 \int_0^T \alpha_0|x_1(t)|^{p-1} + \sum_{i=1}^k \alpha_i|x_1(t-\tau_i(t))|^{p-1}dt \\ & + (1+|c|)|x_1|_0 T(|e|_0+\beta).\end{aligned} \tag{3.7}$$

By Lemma 2.3 and (3.7), we obtain

$$\begin{aligned}\int_0^T |(Ax_1)^{(m)}(t)|^p dt \le & (1+|c|)(l|x_1|_0^{p-1}+\delta|x_1|_0)\int_0^T |x_1'(t)|dt + (1+|c|)T(\alpha_0+\sum_{i=1}^k \alpha_i\delta_i)|x_1|_0^p \\ & + (1+|c|)T(|e|_0+\beta)|x_1|_0.\end{aligned} \tag{3.8}$$

By Lemma 2.2, (3.6) and (3.8), we obtain

$$\begin{aligned}\int_0^T |(Ax_1)^{(m)}(t)|^p dt \le & (1+|c|)Tl\left(d+\frac{1}{2}TM_1(m)\int_0^T |x_1^{(m)}(t)|dt\right)^{p-1} \times M_1(m)\int_0^T |x_1^{(m)}(t)|dt \\ & + \delta(1+|c|)T\left(d+\frac{1}{2}TM_1(m)\int_0^T |x_1^{(m)}(t)|dt\right) \times M_1(m)\int_0^T |x_1^{(m)}(t)|dt \\ & + (1+|c|)T(\alpha_0+\sum_{i=1}^k \alpha_i\delta_i)\left(d+\frac{1}{2}TM_1(m)\int_0^T |x_1^{(m)}(t)|dt\right)^p \\ & + (1+|c|)T(|e|_0+\beta)\left(d+\frac{1}{2}TM_1(m)\int_0^T |x_1^{(m)}(t)|dt\right).\end{aligned} \tag{3.9}$$

By applying Jensen inequality,we can see that

$$\begin{aligned}\int_0^T |(Ax_1)^{(m)}(t)|^p dt \le & (1+|c|)Tl\left[d^{p-1}M_1(m)\int_0^T |x_1^{(m)}(t)|dt + \frac{1}{2^{p-1}}T^{p-1}M_1^p(m)\left(\int_0^T |x_1^{(m)}(t)|dt\right)^p\right] \\ & + \delta(1+|c|)T\left[dM_1(m)\int_0^T |x_1^{(m)}(t)|dt + \frac{1}{2}TM_1^2(m)\left(\int_0^T |x_1^{(m)}(t)|dt\right)^2\right] \\ & + (1+|c|)T(\alpha_0+\sum_{i=1}^k \alpha_i\delta_i)\left[d^p+\frac{1}{2^p}T^pM_1^p(m)\left(\int_0^T |x_1^{(m)}(t)|dt\right)^p\right] \\ & + (1+|c|)T(|e|_0+\beta)\left(d+\frac{1}{2}TM_1(m)\int_0^T |x_1^{(m)}(t)|dt\right).\end{aligned} \tag{3.10}$$

Furthermore

$$\begin{aligned}\int_0^T |(Ax_1)^{(m)}(t)|^p dt \le & (1+|c|)\frac{M_1^p(m)}{2^{p-1}}T^p\left[l+\frac{T}{2}(\alpha_0+\sum_{i=1}^k \alpha_i\delta_i)\right]\left(\int_0^T |x_1^{(m)}(t)|dt\right)^p \\ & + (1+|c|)\frac{M_1^2(m)}{2}T^2\delta\left(\int_0^T |x_1^{(m)}(t)|dt\right)^2 \\ & + (1+|c|)TM_1(m)\left[\delta d + ld^{p-1}+\frac{1}{2}T(|e|_0+\beta)\right]\int_0^T |x_1^{(m)}(t)|dt \\ & + (1+|c|)Td\left[(\alpha_0+\sum_{i=1}^k \alpha_i\delta_i)d^{p-1}+(|e|_0+\beta)\right].\end{aligned} \tag{3.11}$$

From which by applying Holder inequality, we have

$$\begin{aligned}\int_0^T |(Ax_1)^{(m)}(t)|^p dt \le{}& \frac{(1+|c|)M_1^p(m)T^{2p-1}}{2^{p-1}}\left[l+\frac{T}{2}(\alpha_0+\sum_{i=1}^k\alpha_i\delta_i)\right]\int_0^T |x_1^{(m)}(t)|^p dt\\ &+\frac{(1+|c|)\delta M_1^2(m)T^{2+\frac{2}{q}}}{2}\left(\int_0^T |x_1^{(m)}(t)|^p dt\right)^{\frac{2}{p}}\\ &+(1+|c|)T^{1+\frac{1}{q}}M_1(m)\left[\delta d+ld^{p-1}+\frac{1}{2}T(|e|_0+\beta)\right]\left(\int_0^T |x_1^{(m)}(t)|^p dt\right)^{\frac{1}{p}}\\ &+(1+|c|)Td\left[(\alpha_0+\sum_{i=1}^k\alpha_i\delta_i)d^{p-1}+(|e|_0+\beta)\right].\end{aligned} \tag{3.12}$$

It follows from conclusion (2) of Lemma 2.1 that
$\int_0^T |x_1^{(m)}(t)|^p dt = \int_0^T |(A^{-1}(Ax_1)^{(m)})(t)|^p dt \le \frac{\int_0^T |(Ax_1)^{(m)}(t)|^p dt}{(1-|c|)^p}$,
which together with (3.12)yields

$$\begin{aligned}\int_0^T |x_1^{(m)}(t)|^p dt \le{}& \frac{(1+|c|)M_1^p(m)T^{2p-1}}{2^{p-1}(1-|c|)^p}\left[l+\frac{T}{2}(\alpha_0+\sum_{i=1}^k\alpha_i\delta_i)\right]\int_0^T |x_1^{(m)}(t)|^p dt\\ &+\frac{(1+|c|)\delta M_1^2(m)T^{2+\frac{2}{q}}}{2(1-|c|)^p}\left(\int_0^T |x_1^{(m)}(t)|^p dt\right)^{\frac{2}{p}}\\ &+\frac{(1+|c|)T^{1+\frac{1}{q}}M_1(m)}{(1-|c|)^p}\left[\delta d+ld^{p-1}+\frac{1}{2}T(|e|_0+\beta)\right]\left(\int_0^T |x_1^{(m)}(t)|^p dt\right)^{\frac{1}{p}}\\ &+\frac{(1+|c|)Td}{(1-|c|)^p}\left[(\alpha_0+\sum_{i=1}^k\alpha_i\delta_i)d^{p-1}+(|e|_0+\beta)\right].\end{aligned} \tag{3.13}$$

In view of $p \ge 2$ and $\frac{(1+|c|)M_1^p(m)T^{2p-1}}{2^{p-1}(1-|c|)^p}\left[l+\frac{T}{2}(\alpha_0+\sum_{i=1}^k\alpha_i\delta_i)\right] < 1$, from(3.13) we see that there is a constant M_0 independent of λ such that

$$\int_0^T |x_1^{(m)}(t)|^p dt \le M_0. \tag{3.14}$$

So it follows Lemma 2.2 and (3.14) that we have

$$|x_1'|_0 \le M_1(m)\int_0^T |x_1^{(m)}(t)| dt \le M_1(m)T^{\frac{1}{q}}M_0^{\frac{1}{p}} := M_{11}. \tag{3.15}$$

By (3.6) and (3.15), we have

$$|x_1|_0 \le d+\frac{1}{2}TM_{11} := M_{12}. \tag{3.16}$$

Let $M_f = \max_{|u|\le M_{12}} |f(u)|$, $M_g = \max_{t\in[0,T],|u_0|\le M_{12},\ldots,|u_k|\le M_{12}} |g(t,u_0,\ldots,u_k)|$ and from the second equation of (3.5), we have

$$\begin{aligned}\int_0^T |x_2^{(m)}(t)|dt &\le \int_0^T |f(x_1(t))x_1'(t)|dt + \int_0^T |g(t, x_1(t), x_1(t-\tau_1(t)), \dots, x_1(t-\tau_k(t)))|dt + \int_0^T |e(t)| \\ &\le M_f \int_0^T |x_1'(t)|dt + T(M_g + |e|_0) \\ &\le M_f T|x_1'|_0 + T(M_g + |e|_0) \\ &\le M_f T M_{11} + T(M_g + |e|_0) := \overline{M_0}.\end{aligned} \tag{3.17}$$

Again from Lemma 2.2, we have

$$|x_2'|_0 \le M_1(m)\int_0^T |x_2^{(m)}(t)|dt \le M_1(m)\overline{M_0} := M_{21}.$$

Integrating the first equation of (3.5), we have $\int_0^T |x_2(t)|^{q-2}x_2(t)dt = 0$, which implies that there is a constant $\eta \in [0, T]$ such that $x_2(\eta) = 0$, thus

$$|x_2(t)| = |\int_\eta^t x_2'(s)ds + x_2(\eta)| \le \int_0^T |x_2'(s)|ds.$$

Then we can get

$$|x_2|_0 \le \int_0^T |x_2'(t)|dt \le TM_{21} := M_{22}. \tag{3.18}$$

Let $\Omega_2 = \{x|x \in KerL, QNx = 0\}$ if $x \in \Omega_2$ then $x \in \mathbb{R}^2$ is a constant vector with

$$\begin{cases} \frac{1}{T}\int_0^T [A^{-1}\varphi_q(x_2)](t)dt = 0, \\ \frac{1}{T}\int_0^T [f(x_1(t))(A^{-1}\varphi_q(x_2))(t) + g(t, x_1(t), x_1(t-\tau_1(t)), \dots, x_1(t-\tau_k(t))) + e(t)]dt = 0. \end{cases} \tag{3.19}$$

By the first formula of (3.19) , we have $x_2 = 0$. Which together with the second equation of (3.19) yields

$\frac{1}{T}\int_0^T [g(t, x_1, x_1, \dots, x_1) + e(t)]dt = 0$. In view of (H_1), we see that $|x_1| \le d$.
Now, Let $M_1 = \max\{M_{11}, M_{12}\}, M_2 = \max\{M_{21}, M_{22}\}$, then $\|x_1\| \le M_1, \|x_2\| \le M_2$. Taking $\Omega = \{x|x = (x_1, x_2)^\top \in X, \|x_1\| < M_1 + d, \|x_2\| < M_2 + d\}$, then $\Omega_1 \cup \Omega_2 \subset \Omega$. So from (3.16) and (3.18), it is easy to see that conditions (1) and (2) of Lemma 2.5 are satisfied.
Next, we verify the condition (3) of Lemma 2.5. To do this, we define the isomorphism

$J : ImQ \to KerL,\ J(x_1, x_2)^\top = (x_1, x_2)^\top,$

then

$$JQN(x) = \begin{pmatrix} \frac{1}{T}\int_0^T [A^{-1}\varphi_q(x_2)](t)dt \\ \frac{1}{T}\int_0^T [f(x_1(t)(A^{-1}\varphi_q(x_2))(t) + g(t, x_1, x_1, \dots, x_1) + e(t)]dt \end{pmatrix}.$$

By Lemma 2.4 , we need to prove that

$$JQN(x) \ne \mu(JQN(-x)),\ \forall x \in \partial\Omega \cap KerL\ ,\ \mu \in [0, 1]$$

Case1. If $x = (x_1, x_2)^\top \in \partial\Omega \cap KerL \backslash \{(M_1 + d, 0)^\top, (-M_1 - d, 0)^\top\}$, then $x_2 \neq 0$ which, gives us
$\frac{1}{T}\int_0^T [A^{-1}\varphi_q(x_2)](t)dt \neq 0$

$$\left(\frac{1}{T}\int_0^T [A^{-1}\varphi_q(x_2)](t)dt\right)\left(\frac{1}{T}\int_0^T [A^{-1}\varphi_q(-x_2)](t)dt\right) < 0,$$

obviously, $\forall \mu \in [0,1]\ JQN(x) \neq \mu(JQN(-x))$.
Case2. If $x = (M_1 + d, 0)^\top$ or $x = (-M_1 - d, 0)^\top$, then

$$JQN(x) = \begin{pmatrix} 0 \\ \frac{1}{T}\int_0^T [g(t, x_1, x_1, \ldots, x_1) + e(t)]dt \end{pmatrix},$$

which, together with (H_1), yields $\forall \mu \in [0,1],\ JQN(x) \neq \mu(JQN(-x))$.
Thus, the condition (3) of Lemma 2.5 is also satisfied. Therefore, by applying Lemma 2.5, we conclude that the equation $Lx = Nx$ has at least one T-periodic solution on $\overline{\Omega}$, so equation(1.1) has at least one T-periodic solution.
The case m is odd can be treated similarly. □

4. Example

In this section, we provide an example to illustrate effectiveness of Theorem 3.2.
Let us consider the following equation

$$(\varphi_3(x(t) - \frac{1}{10}(x - \frac{\pi}{8}))^{(8)}(t))^{(8)} = f(x(t))x'(t) + g(t, x(t), x(t - \frac{\cos 20\pi t}{90}), x(t - \frac{\sin 20\pi t}{100})) + e(t), \quad (4.1)$$

where $p = 3,\ m = 8,\ T = \frac{1}{10},\ c = \frac{1}{10},\ f(u) = \frac{u^2}{6 + |u|} + 3,\ l = \frac{1}{6},\ \tau_1(t) = \frac{\cos 20\pi t}{90},\ \tau_2(t) = \frac{\sin 20\pi t}{100}$,
$e(t) = \frac{6}{225}\cos 20\pi t + \frac{1}{2},\ g(t, u, v, w) = sgn(u)u^2(2 + \sin 20\pi t) + \frac{3}{225}\left(sgn(v)v^2 + sgn(w)w^2\right)|\cos 20\pi t|$.
Therefore we can choose $d = 1,\ \alpha_1 = \alpha_2 = 0,014,\ M_1(8) = \left(\frac{1}{10}\right)^6 \sqrt{\frac{691}{2730 \times 12 \times 12!}}$.
We can easily check the condition $(H_1), (H_2)$ of Theorem 3.2 hold. We can compute
$\frac{(1 + |c|)M_1^p(m)T^{2p-1}}{2^{p-1}(1 - |c|)^p}\left[l + \frac{T}{2}(\alpha_0 + \sum_{i=1}^{k}\alpha_i\delta_i)\right] < 1.$
By Theorem 3.2, equation (4.1) has at least one $\frac{1}{10}$ -periodic solution.

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Loubna Moutaouekkil,
Department of Mathematics, Multidisciplinary Faculty of Nador,
University Mohamed First, Oujda, Morocco.
E-mail address: loubna_anits@yahoo.fr

and

Omar Chakrone,
Department of Mathematics, Faculty of Sciences Oujda,
University Mohammed First, Oujda, Morocco.
E-mail address: chakrone@yahoo.fr

and

Zakaria El Allali ,
Department of Mathematics, Multidisciplinary Faculty of Nador,
University Mohamed First, Oujda, Morocco.
E-mail address: z.elallali@ump.ma

and

Said Taarabti,
Department of Mathematics and Computer, National School of Applied Sciences Agadir,
University Ibn Zohr, Agadir, Morocco.
E-mail address: taarabti@gmail.com