Global Existence and Stability of Solution for a $p$-Kirchhoff type Hyperbolic Equation with Variable Exponents

Amar Ouaoua, Aya Khalidi and Messaoud Maouni

ABSTRACT: In this paper, we consider the following $p$-Kirchhoff type hyperbolic equation with variable exponents

$$u_{tt} - M \left( \int_{\Omega} |\nabla u|^p \, dx \right) \Delta_p u + |u_t|^{m(x)-2} u_t = |u_t|^{r(x)-2} u,$$

We prove the global existence of the solution with positive initial energy, the stability established based on Komornik’s inequality.

Key Words: Kirchhoff type hyperbolic equation, Variable exponents, Global existence.

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1. Introduction

We consider the following boundary value problem:

$$\begin{cases}
u_{tt} - M \left( \int_{\Omega} |\nabla u|^p \, dx \right) \Delta_p u + |u_t|^{m(x)-2} u_t = |u_t|^{r(x)-2} u, & (x,t) \in \Omega \times (0,T), \\
u(x,t) = 0, & (x,t) \in \partial \Omega \times (0,T), \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega,
\end{cases}$$

(1.1)

where $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n \geq 1$ with smooth boundary $\partial \Omega$ and $M(s) = a + bs$ with positive parameters $a, b$, $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$, with $p \geq 2$, $r(.)$ and $m(.)$ are given measurable functions on $\Omega$.

Equation (1.1) can be viewed as a generalization of a model introduced by Kirchhoff [6]. The following Kirchhoff type equation

$$u_{tt} - M \left( \|\nabla u\|_{L^2}^2 \right) \Delta u + g(u_t) = f(u),$$

(1.2)

have been discussed by many authors. For $g(u_t) = u_t$, the global existence and blow up results can be found in [14, 18], for $g(u_t) = |u_t|^{p-2} u_t$, $p > 2$, the main results of existence and blow up are in [4, 13].

Many authors studied the existence and nonexistence of solutions for problem with variable exponents, can refer [2, 4, 9, 10, 15, 17, 19, 20]. Messaoudi et al. [13] considered the following equation:

$$u_{tt} - \Delta u + a |u_t|^{m(x)-2} u_t = |u_t|^{p(x)-2} u, \quad \text{in } \Omega \times (0,T),$$

and used the Faedo Galerkin method to establish the existence of a unique weak local solution. They also proved that the solutions with negative initial energy blow up in finite time. Messaoudi and Talahme [11, 12], considered the following equation:

$$u_{tt} - \text{div} \left( |\nabla u|^{p(x)-2} \nabla u \right) + a |u_t|^{m(x)-2} u_t = b |u|^{p(x)-2} u, \quad \text{in } \Omega \times (0,T),$$

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where \( a, b \) is a nonnegative constant. They proved a finite-time blow-up result for the solution with negative initial energy as well as for certain solutions with positive initial energy; in the case where \( m(x) = 2 \) and under suitable conditions on the exponents, they established a blow-up result for solutions with arbitrary positive initial energy. Our objective in this paper is to study: In section 2, some notations, assumptions and preliminaries are introduced, section 3 the global existence of solution is proved and the main results of this article are shown in section 4.

2. Preliminaries

We begin this section with some notations and definitions. Denote by \( \| \cdot \|_p \), the \( L^p(\Omega) \) norm of a Lebesgue function \( u \in L^p(\Omega) \). We use \( W^{1,p}_0(\Omega) \) to the well-known sobolev space such that \( u \) and \( \nabla u \) are in \( L^p(\Omega) \) equipped with the norm \( \| u \|_{W^{1,p}_0(\Omega)} = \| \nabla u \|_p \).

Let \( q : \Omega \to [1, +\infty] \) be a measurable function, where \( \Omega \) is adomain of \( \mathbb{R}^n \). We define the Lebesgue space with a variate exponent \( q(.) \) by:

\[
L^{q(\cdot)}(\Omega) := \left\{ v : \Omega \to \mathbb{R} : \text{measurable in} \ \Omega, \ \varrho_{q(\cdot)}(\lambda v) < +\infty, \text{for some} \ \lambda > 0 \right\},
\]

where \( \varrho_{q(\cdot)}(v) = \int_{\Omega} |v(x)|^{q(x)} \, dx \).

The set \( L^{q(\cdot)}(\Omega) \) equipped with the norm ( Luxemburg’s norm)

\[
\| v \|_{q(\cdot)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \frac{|v(x)|^{q(x)}}{\lambda} \, dx \leq 1 \right\},
\]

\( L^{q(\cdot)}(\Omega) \) is a Banach space [8].

Next, we define the variable-exponent Sobolev space \( W^{1,q(\cdot)}(\Omega) \) as follows:

\[
W^{1,q(\cdot)}(\Omega) := \left\{ v \in L^{q(\cdot)}(\Omega) \text{ such that } \nabla v \text{ exists and } |\nabla v| \in L^{q(\cdot)}(\Omega) \right\}.
\]

This is a Banach space with respect to the norm \( \| v \|_{W^{1,q(\cdot)}(\Omega)} = \| v \|_{q(\cdot)} + \| \nabla v \|_{q(\cdot)} \).

Furthermore, we set \( W^{1,q(\cdot)}_0(\Omega) \) to be the closure of \( C^\infty_0(\Omega) \) in the space \( W^{1,q(\cdot)}(\Omega) \). Let us note that the space \( W^{1,q(\cdot)}(\Omega) \) has a different definition in the case of variable exponents.

However, under the log-Hölder continuity condition, both definitions are equivalent [8]. The space \( W^{-1,q(\cdot)}(\Omega) \), dual of \( W^{1,q(\cdot)}(\Omega) \), is defined in the same way as the classical Sobolev spaces, where

\[
\frac{1}{q(\cdot)} + \frac{1}{q(\cdot)} = 1.
\]

Lemma 2.1 If

\[
1 \leq q_1 := \operatorname{ess inf}_{x \in \Omega} q(x) \leq q(x) \leq q_2 := \operatorname{ess sup}_{x \in \Omega} q(x) < \infty,
\]

then we have

\[
\min \left\{ \| u \|_{q_1(\cdot)}^{q_1}, \| u \|_{q_2(\cdot)}^{q_2} \right\} \leq \varrho_{q(\cdot)}(u) \leq \max \left\{ \| u \|_{q_1(\cdot)}^{q_1}, \| u \|_{q_2(\cdot)}^{q_2} \right\},
\]

for any \( u \in L^{q(\cdot)}(\Omega) \).

Lemma 2.2 (Hölder’s Inequality) Suppose that \( p, q, s \geq 1 \) are measurable functions defined on \( \Omega \) such that

\[
\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \text{ for a.e. } y \in \Omega.
\]

For the existence of the local solution of problem (1.1), we refer the reader to [13]. Their result is given in the following theorem:
Theorem 2.1 Suppose that \( r, m \in C(\overline{\Omega}) \) with
\[
2 \leq r_1 \leq r(x) \leq r_2 < \frac{2n-1}{n-2}, \quad \text{if} \ n \geq 3,
\]
\[
r(x) \geq 2, \quad \text{if} \ n = 1, \ 2,
\]
and
\[
2 \leq m_1 \leq m(x) \leq m_2 < \frac{2n}{n-2}, \quad \text{if} \ n \geq 3,
\]
\[
m(x) \geq 2, \quad \text{if} \ n = 1, \ 2,
\]
\[
r_1 := \text{ess inf}_{x \in \Omega} r(x), \quad r_2 := \text{ess sup}_{x \in \Omega} r(x),
\]
\[
m_1 := \text{ess inf}_{x \in \Omega} m(x), \quad m_2 := \text{ess sup}_{x \in \Omega} m(x).
\]

We also assume that \( m(.) \) and \( r(.) \) satisfy the log-Hölder continuity condition:
\[
|q(x) - q(y)| \leq -\frac{A}{\log |x-y|}, \quad \text{for a.e.} \ x, y \in \Omega, \ \text{with} \ |x-y| < \delta, \quad (2.1)
\]
\( A > 0, \ 0 < \delta < 1. \)

Then, for any \((u_0, u_1) \in W^{1,p}_0(\Omega) \times L^2(\Omega), \) problem (1.1) has a unique weak local solution
\[
u \in L^\infty((0,T), W^{1,p}_0(\Omega)),
\]
\[
u_t \in L^\infty((0,T), L^2(\Omega) \cap L^{m(\cdot)}(\Omega \times (0,T))),
\]
\[
u_{tt} \in L^2((0,T), W^{-1,p'}(\Omega)).
\]

3. Exponential growth

In the order to state and prove our result, we define the potential energy functional and the Nehari’s functional, respectively, by the following
\[
E(t) = E(u(t)) = \frac{1}{2} \|u_t(t)\|_2^2 + \frac{a}{p} \|\nabla u(t)\|_p^p + \frac{b}{2p} \|\nabla u(t)\|_{2p}^{2p} - \int_\Omega \frac{1}{r(x)} |u(t)|^{r(x)} \, dx. \quad (3.1)
\]
\[
I(t) = I(u(t)) = a \|\nabla u(t)\|_p^p + b \|\nabla u(t)\|_{2p}^{2p} - \int_\Omega |u(t)|^{r(x)} \, dx. \quad (3.2)
\]

We can considering \( a = b = 1, \) and this does not change the general result.

Lemma 3.1 Under the assumptions of theorem 2.1, we have
\[
E'(t) = -\int_\Omega |u_t(t)|^{m(x)} \, dx \leq 0, \ \quad t \in [0, T]. \quad (3.3)
\]

and
\[
E(t) \leq E(0).
\]

Proof: We multiply the first equation of (1.1) by \( u_t \) and integrating over the domain \( \Omega, \) we get
\[
\frac{d}{dt} \left( \frac{1}{2} \|u_t\|_2^2 + \frac{1}{p} \int_\Omega |\nabla u(t)|^p \, dx + \frac{1}{2p} \left( \int_\Omega |\nabla u(t)|^p \, dx \right)^2 - \int_\Omega \frac{1}{r(x)} |u(t)|^{r(x)} \, dx \right) = -\int_\Omega |u_t(t)|^{m(x)} \, dx,
\]
then

$$E'(t) = - \int_{\Omega} |u_t(t)|^{m(x)} \, dx \leq 0.$$  

Integrating (3.3) over \((0, t)\), we obtain

$$E(t) \leq E(0).$$

**Lemma 3.2** Assume that the assumptions of theorem 2.1 and \(r_1 > 2p\), hold,

$$I(0) > 0,$$

and

$$\beta_1 + \beta_2 < 1,$$

where

$$\beta_1 = \max \left\{ \alpha c_* \left( \frac{pr_1}{r_1 - p} E(0) \right)^{\frac{r_1 - p}{2p}}, \alpha c_* \left( \frac{pr_1}{r_1 - p} E(0) \right)^{\frac{r_2 - p}{2p}} \right\},$$

$$\beta_2 = \max \left\{ (1 - \alpha) c_* \left( \frac{2pr_1}{r_1 - 2p} E(0) \right)^{\frac{r_2 - 2p}{2p}}, (1 - \alpha) c_* \left( \frac{2pr_1}{r_1 - 2p} E(0) \right)^{\frac{r_2 - 2p}{2p}} \right\},$$

with \(0 < \alpha < 1\), \(c_*\) is the best embedding constant of \(W_0^{1,p}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)\), then \(I(t) > 0\), for all \(t \in [0, T]\).

**Proof:** By continuity, there exists \(T_*\), such that

$$I(t) \geq 0, \quad \text{for all } t \in [0, T_*].$$

Now, we have for all \(t \in [0, T]\):

$$J(t) = J(u(t)) = \frac{1}{p} \left\| \nabla u(t) \right\|_p^p + \frac{1}{2p} \left\| \nabla u(t) \right\|_p^{2p} - \int_{\Omega} \frac{1}{r(\cdot)} |u(t)|^{r(\cdot)} \, dx$$

$$\geq \frac{1}{p} \left\| \nabla u(t) \right\|_p^p + \frac{1}{2p} \left\| \nabla u(t) \right\|_p^{2p} - \frac{1}{r_1} \left( \left\| \nabla u(t) \right\|_p^p + \left\| \nabla u(t) \right\|_p^{2p} - I(t) \right)$$

$$\geq \frac{r_1 - p}{pr_1} \left\| \nabla u(t) \right\|_p^p + \frac{r_1 - 2p}{2pr_1} \left\| \nabla u(t) \right\|_p^{2p} + \frac{1}{r_1} I(t)$$

using (3.5), we obtain

$$\frac{r_1 - p}{pr_1} \left\| \nabla u(t) \right\|_p^p + \frac{r_1 - 2p}{2pr_1} \left\| \nabla u(t) \right\|_p^{2p} \leq J(t), \quad \text{for all } t \in [0, T_*].$$

By Lemma 3.1, we get

$$\left\| \nabla u(t) \right\|_p^p \leq \frac{pr_1}{r_1 - p} E(t) \leq \frac{pr_1}{r_1 - p} E(0)$$

and

$$\left\| \nabla u(t) \right\|_p^{2p} \leq \frac{2pr_1}{r_1 - 2p} E(t) \leq \frac{2pr_1}{r_1 - 2p} E(0)$$

On the other hand, by Lemma 2.1, we have

$$\int_{\Omega} \left| u(t) \right|^{r(\cdot)} \, dx \leq \text{Max} \left\{ \left\| u(t) \right\|_{r(\cdot)}^{r_1(\cdot)}, \left\| u(t) \right\|_{r(\cdot)}^{r_2(\cdot)} \right\}$$

$$= \alpha \text{Max} \left\{ \left\| u(t) \right\|_{r(\cdot)}^{r_1(\cdot)}, \left\| u(t) \right\|_{r(\cdot)}^{r_2(\cdot)} \right\}$$

$$+ (1 - \alpha) \text{Max} \left\{ \left\| u(t) \right\|_{r(\cdot)}^{r_1(\cdot)}, \left\| u(t) \right\|_{r(\cdot)}^{r_2(\cdot)} \right\}. $$
Proof: We have
\[
\int_{\Omega} |u(t)|^{r(x)} dx \leq \alpha \text{ Max} \left\{ c^r_1 \|\nabla u(t)\|_{p_r}^{r_1}, \ c^r_2 \|\nabla u(t)\|_{p_r}^{r_2} \right\} \\
+ (1 - \alpha) \text{ Max} \left\{ c^r_1 \|\nabla u(t)\|_{p_r}^{r_1}, \ c^r_2 \|\nabla u(t)\|_{p_r}^{r_2} \right\} \\
\leq \alpha \text{ Max} \left\{ c^r_1 \|\nabla u(t)\|_{p_r}^{r_1-p}, \ c^r_2 \|\nabla u(t)\|_{p_r}^{r_2-p} \right\} \times \|\nabla u(t)\|_{p_r}^p \\
+ (1 - \alpha) \text{ Max} \left\{ c^r_1 \|\nabla u(t)\|_{p_r}^{r_1-2p}, \ c^r_2 \|\nabla u(t)\|_{p_r}^{r_2-2p} \right\} \times \|\nabla u(t)\|_{p_r}^p
\]
By (3.7) and (3.8), we get
\[
\int_{\Omega} |u(t)|^{r(x)} dx \leq \beta_1 \|\nabla u(t)\|_p^p + \beta_2 \|\nabla u(t)\|_{2p}^p, \quad \text{for all } t \in [0, T_*]. \tag{3.9}
\]
Since \(\beta_1 + \beta_2 < 1\), then
\[
\int_{\Omega} |u(t)|^{r(x)} dx < \|\nabla u(t)\|_p^p + \|\nabla u(t)\|_{2p}^p, \quad \text{for all } t \in [0, T_*].
\]
This implies that
\[
I(t) > 0, \quad \text{for all } t \in [0, T_*].
\]
By repeating the above procedure, we can extend \(T_*\) to \(T\).

**Theorem 3.1** Under the assumptions of lemma 3.2, the local solution of (1.1) is global.

**Proof:** We have
\[
E(u(t)) = \frac{1}{2} \|u_t(t)\|_2^2 + \frac{1}{p} \|\nabla u(t)\|_p^p + \frac{1}{2p} \|\nabla u(t)\|_{2p}^p - \int_{\Omega} \frac{1}{r(x)} |u(t)|^{r(x)} dx.
\]
\[
\geq \frac{1}{2} \|u_t(t)\|_2^2 + \frac{r_1 - p}{pr_1} \|\nabla u(t)\|_p^p + \frac{r_1 - 2p}{2pr_1} \|\nabla u(t)\|_{2p}^p.
\]
So that
\[
\|u_t(t)\|_2^2 + \|\nabla u(t)\|_p^p \leq CE(t). \tag{3.10}
\]
By Lemma 3.1, we obtain
\[
\|u_t(t)\|_2^2 + \|\nabla u(t)\|_p^p \leq CE(0). \tag{3.11}
\]
This implies that the local solution is global in time.

### 4. Stability solution

In this section our main result is based a Komornik’s inequality [7], as in [5]. For this, we need the following Lemma:

**Lemma 4.1** Suppose that the assumptions of Lemma 3.2 and \(m_1 > p\), hold, then there exists a positive constant \(c\) such that
\[
\int_{\Omega} |u(t)|^{m(x)} dx \leq cE(t). \tag{4.1}
\]

**Proof:** We have
\[
\int_{\Omega} |u(t)|^{m(x)} dx = \text{ max} \left\{ \|u(t)\|_{m_1(\cdot)}, \|u(t)\|_{m_2(\cdot)} \right\} \\
\leq \text{ max} \left\{ c^m_1 \|\nabla u(t)\|_{p_{m_1}}^{m_1}, \ c^m_2 \|\nabla u(t)\|_{p_{m_2}}^{m_2} \right\} \\
\leq \text{ max} \left\{ c^m_1 \|\nabla u(t)\|_{p_{m_1}}^{m_1-p}, \ c^m_2 \|\nabla u(t)\|_{p_{m_2}}^{m_2-p} \right\} \times \|\nabla u(t)\|_p^p
\]
By using (3.7), we obtain
\[ \int_{\Omega} |u(t)|^{m(x)} \, dx \leq cE(t). \]

Now, we state our main result:

**Theorem 4.1** Let the assumptions of Lemma 3.2, then, there exists constants \( C, \zeta > 0 \), such that
\[ E(t) \leq \frac{C}{(1 + t)^{\frac{m_2 - 2}{2}}}, \quad \text{for all } t \geq 0 \text{ if } m_2 > 2. \]
\[ E(t) \leq Ce^{-\zeta t}, \quad \text{for all } t \geq 0 \text{ if } m_2 = 2. \]

**Proof:** Multiplying first equation of (1.1) by \( u(t)E^q(t) (q > 0) \) and integrating over \( \Omega \times (S, T) \), we obtain
\[
\int_{S}^{T} \int_{\Omega} E^q(t) \left[ u(t)u_{tt}(t) - u(t) \left( \int_{\Omega} |\nabla u|^p \, dx \right) \Delta_p u + |u_t|^{m(x) - 2} u_t \right] \, dx \, dt
\]
\[
= \int_{S}^{T} \int_{\Omega} E^q(t) \left| u(t) \right|^{r(x)} \, dx \, dt
\]
So that
\[
\int_{S}^{T} \int_{\Omega} E^q(t) \left[ (u(t)u_t(t))_t - |u_t(t)|^2 + |\nabla u(t)|^p + \|\nabla u(t)\|^p_p |\nabla u(t)|^p + u(t)|u_t|^{m(x) - 2} u_t \right] \, dx \, dt = \int_{S}^{T} \int_{\Omega} E^q(t) \left| u(t) \right|^{r(x)} \, dx \, dt
\]
We add and subtract the term
\[
\int_{S}^{T} \int_{\Omega} E^q(t) \left[ \beta_1 |\nabla u(t)|^p + \beta_2 \|\nabla u(t)\|^p_p |\nabla u(t)|^p + (2 + \beta_1 + \beta_2) |u_t(t)|^2 \right] \, dx \, dt
\]
and use (3.9), to get
\[
(1 - \beta_1) \int_{S}^{T} \int_{\Omega} \left[ |\nabla u(t)|^p + |u_t(t)|^2 \right] \, dx \, dt
\]
\[
+ (1 - \beta_2) \int_{S}^{T} \int_{\Omega} \left[ \|\nabla u(t)\|^p_p |\nabla u(t)|^p + |u_t(t)|^2 \right] \, dx \, dt
\]
\[
+ \int_{S}^{T} \int_{\Omega} E^q(t) \left[ (u(t)u_t(t))_t - (3 - \beta_1 - \beta_2) |u_t(t)|^2 \right] \, dx \, dt
\]
\[
+ \int_{S}^{T} \int_{\Omega} E^q(t) \left[ u(t)u_t(t) |u_t(t)|^{m(x) - 2} \right] \, dx \, dt
\]
\[
= - \int_{S}^{T} \int_{\Omega} E^q(t) \left[ \beta_1 |\nabla u(t)|^p + \beta_2 \|\nabla u(t)\|^p_p |\nabla u(t)|^p - \left| u(t) \right|^{r(x)} \right] \, dx \, dt \leq 0
\] (4.2)
It is clear that

\[
\gamma \int_s^T E^q (t) \int_\Omega \left[ \frac{1}{p} |\nabla u(t)|^p + \frac{1}{2p} \|\nabla u(t)\|_p^p |\nabla u(t)|^p + \frac{|u_t(t)|^2}{2} - \frac{|u(t)|^{r(x)}}{r(x)} \right] dx dt
\]

\[
\leq (1 - \beta_1) \int_s^T E^q (t) \int_\Omega \left[ \frac{1}{p} |\nabla u(t)|^p + \frac{|u_t(t)|^2}{2} \right] dx dt
\]

\[
+ (1 - \beta_2) \int_s^T E^q (t) \int_\Omega \left[ \frac{1}{2p} \|\nabla u(t)\|_p^p |\nabla u(t)|^p + \frac{|u_t(t)|^2}{2} \right] dx dt
\]

(4.3)

where \( \gamma = \text{Min} ((1 - \beta_1), (1 - \beta_2)) \). By (4.2), (4.3) and definition of \( E(t) \), we get

\[
\gamma \int_s^T E^{q+1} (t) dt \leq - \int_s^T E^q (t) \int_\Omega (u(t) u_t(t))_t dx dt
\]

\[
+ (3 - \beta_1 - \beta_2) \int_s^T E^q (t) \int_\Omega |u_t(t)|^2 dx dt
\]

\[
- \int_s^T E^q (t) \int_\Omega (u(t) u_t(t))_t dx dt
\]

(4.4)

Using the definition of \( E(t) \) and the following expression

\[
\frac{d}{dt} \left( E^q (t) \int_\Omega u(t) u_t(t) dx \right) = qE^{q-1} (t) \frac{d}{dt} E(t) \int_\Omega u(t) u_t(t) dx
\]

\[
+ E^q (t) \int_\Omega (u(t) u_t(t))_t dx.
\]

Inequality (4.4), becomes

\[
\gamma \int_s^T E^{q+1} (t) dt \leq q \int_s^T E^{q-1} (t) \frac{d}{dt} E(t) \int_\Omega u(t) u_t(t) dx
\]

\[
- \int_s^T \frac{d}{dt} \left( E^q (t) \int_\Omega u(t) u_t(t) dx \right) dt
\]

\[
- \int_s^T E^q (t) \int_\Omega (u(t) u_t(t))_t dx dt
\]

\[
+ (3 - \beta_1 - \beta_2) \int_s^T E^q (t) \int_\Omega |u_t(t)|^2 dx dt.
\]

(4.5)

We denote by \( c \) the various constants.

We estimate the terms in the right-hand side of (4.5) as follow:
By (3.3) and Young’s inequality, we obtain
\[
q \int_s^T E^{-1} q(t) \frac{d}{dt} E(t) \int_\Omega u(t) u_t(t) \, dx \\
\leq q \int_s^T E^{-1} q(t) \left( -E'(t) \right) \int_\Omega \left[ \frac{1}{p} |u(t)|^p + \frac{p-1}{p} |u_t(t)|^\frac{p}{p-1} \right] \, dx dt
\] (4.6)

Since, \( 1 \leq \frac{p}{p-1} < 2 \), by the embedding of \( L^2(\Omega) \hookrightarrow L^\frac{2p}{p-1}(\Omega) \), we have
\[
q \int_s^T E^{-1} q(t) \frac{d}{dt} E(t) \int_\Omega u(t) u_t(t) \, dx \\
\leq q \int_s^T E^{-1} q(t) \left( -E'(t) \right) \int_\Omega \left[ \frac{1}{p} |u(t)|^p + \frac{p-1}{p} |u_t(t)|^2 \right] \, dx dt
\]

Thus, by (3.10), we find
\[
q \int_s^T E^{-1} q(t) \frac{d}{dt} E(t) \int_\Omega u(t) u_t(t) \, dx \\
\leq c \int_s^T E^q(t) \left( -E'(t) \right) \, dt \\
\leq cE^{q+1}(S) - cE^{q+1}(T) \\
\leq cE^q(0) E(S) \leq cE(S). \tag{4.7}
\]

For the second term, we have
\[
- \int_s^T \frac{d}{dt} \left( E^q(t) \int_\Omega u(t) u_t(t) \, dx \right) \, dx dt \\
\leq \left| E^q(t) \int_\Omega u(S) u_t(S) \, dx - E^q(t) \int_\Omega u(T) u_t(T) \, dx \right| \\
\leq E^q(t) \left| \int_\Omega u(x, S) u_t(x, S) \, dx \right| + E^q(t) \left| \int_\Omega u(x, T) u_t(x, T) \, dx \right| \\
\leq cE^{q+1}(S) + cE^{q+1}(T) \\
\leq cE^q(0) E(S) \leq cE(S). \tag{4.8}
\]

For the third term, we use the following Young inequality:
\[
XY \leq \frac{\varepsilon}{\lambda_1} X^{\lambda_1} + \frac{1}{\lambda_2 \varepsilon \lambda_1^2} Y^{\lambda_2}, \quad X, Y \geq 0, \ \varepsilon > 0 \text{ and } \frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1,
\]
with \( \lambda_1(x) = m(x), \ \lambda_2(x) = \frac{m(x)}{m(x)-1} \).
By (3.3) and Lemma 4.1, we have

\[
- \int_{S}^{T} E^{q}(t) \int_{\Omega} u(t) u_{t}(t) |u(t)|^{m(x)-2} \, dx \, dt \leq \int_{S}^{T} E^{q}(t) \left( \varepsilon c \int_{\Omega} |u(t)|^{m(x)} \, dx + c_{\varepsilon} \int_{\Omega} |u(t)|^{m(x)} \, dx \right) \, dt
\]

\[
\leq \varepsilon c \int_{S}^{T} E^{q}(t) \int_{\Omega} |u(t)|^{m(x)} \, dx \, dt + c_{\varepsilon} \int_{S}^{T} E^{q}(t) (-E'(t)) \, dt
\]

\[
\leq \varepsilon c \int_{S}^{T} E^{q+1}(t) \, dt + c_{\varepsilon} E(S). \tag{4.9}
\]

For the last term of (4.5), we have

\[
(3 - \beta_{1} - \beta_{2}) \int_{S}^{T} E^{q}(t) \int_{\Omega} |u_{t}(t)|^{2} \, dx \, dt \leq (3 - \beta_{1} - \beta_{2}) \int_{S}^{T} E^{q}(t) \left[ \int_{\Omega^{-}} |u_{t}(t)|^{2} \, dx + \int_{\Omega^{+}} |u_{t}(t)|^{2} \, dx \right] \, dt
\]

\[
\leq c \int_{S}^{T} E^{q}(t) \left[ \left( \int_{\Omega^{-}} |u(t)|^{m_{2}} \, dx \right)^{\frac{2}{m_{2}}} + \left( \int_{\Omega^{+}} |u(t)|^{m_{1}} \, dx \right)^{\frac{2}{m_{1}}} \right] \, dt
\]

\[
\leq c \int_{S}^{T} E^{q}(t) \left[ \left( \int_{\Omega} |u(t)|^{m(x)} \, dx \right)^{\frac{2}{m(x)}} + \left( \int_{\Omega} |u(t)|^{m(x)} \, dx \right)^{\frac{2}{m(x)}} \right] \, dt.
\]

This implies

\[
(3 - \beta_{1} - \beta_{2}) \int_{S}^{T} E^{q}(t) \int_{\Omega} |u_{t}(t)|^{2} \, dx \, dt \leq c \int_{S}^{T} E^{q}(t) (-E'(t))^{\frac{2}{m_{2}}} \, dt + c \int_{S}^{T} E^{q}(t) (-E'(t))^{\frac{2}{m_{1}}} \, dt. \tag{4.10}
\]

First, if we use Young’s inequality with \( \lambda_{1} = (q+1)/q \) and \( \lambda_{2} = q+1 \), we have

\[
\int_{S}^{T} E^{q}(t) (-E'(t))^{\frac{2}{m_{2}}} \, dt \leq \varepsilon c \int_{S}^{T} E^{q+1}(t) \, dt + c_{\varepsilon} \int_{S}^{T} (-E'(t))^{\frac{2(q+1)}{m_{2}}} \, dt.
\]

We take \( q = \frac{m_{2}}{2} - 1 \) to find

\[
\int_{S}^{T} E^{q}(t) (-E'(t))^{\frac{2}{m_{2}}} \, dt \leq \varepsilon c \int_{S}^{T} E^{q+1}(t) \, dt + c_{\varepsilon} \int_{S}^{T} (-E'(t)) \, dt.
\]
This implies
\[
\int_{S}^{T} E^{q} (t) \left( -E' (t) \right) \frac{2}{m_{1}^{2}} dt \leq \varepsilon c \int_{S}^{T} E^{q+1} (t) dt + c_{\varepsilon} E (S).
\]  
(4.11)

On the other hand, we have
\[
\int_{S}^{T} E^{q} (t) \left( -E' (t) \right) \frac{2}{m_{1}} dt \leq \varepsilon c \int_{S}^{T} E^{q+1} (t) dt + c_{\varepsilon} E (S).
\]  
(4.12)

Indeed,

- if \( m_{1} = 2 \) then
  \[
  \int_{S}^{T} E^{q} (t) \left( -E' (t) \right) \frac{2}{m_{1}} dt \leq cE (S) \leq \varepsilon c \int_{S}^{T} E^{q+1} (t) dt + c_{\varepsilon} E (S).
  \]

- if \( m_{1} > 2 \), we use the Young’s inequality \( \lambda_{1} = \frac{m_{1}}{m_{1} - 2} \) and \( \lambda_{2} = \frac{m_{1}}{2} \), to obtain
  \[
  \int_{S}^{T} E^{q} (t) \left( -E' (t) \right) \frac{2}{m_{1}} dt \leq \varepsilon c \int_{S}^{T} E^{q \frac{m_{1}}{m_{1} - 2}} (t) dt + c_{\varepsilon} \int_{S}^{T} \left( -E' (t) \right) dt \\
  \leq \varepsilon c \int_{S}^{T} E^{q \frac{m_{1}}{m_{1} - 2}} (t) dt + c_{\varepsilon} E (S).
  \]

We notice that \( q \frac{m_{1}}{m_{1} - 2} = q + 1 + \frac{m_{1} - m_{2}}{m_{1} - 2} \), then
\[
\int_{S}^{T} E^{q} (t) \left( -E' (t) \right) \frac{2}{m_{1}} dt \leq \varepsilon c \left( E (S) \right)^{\frac{m_{1} - m_{2}}{m_{1} - 2}} \int_{S}^{T} E^{q+1} (t) dt + c_{\varepsilon} E (S) \\
\leq \varepsilon c \left( E (0) \right)^{\frac{m_{1} - m_{2}}{m_{1} - 2}} \int_{S}^{T} E^{q+1} (t) dt + c_{\varepsilon} E (S) \\
\leq \varepsilon c \int_{S}^{T} E^{q+1} (t) dt + c_{\varepsilon} E (S).
\]

We substituting (4.11) and (4.12) in (4.10), we obtain
\[
(3 - \beta_{1} - \beta_{2}) \int_{S}^{T} E^{q} (t) \int_{\Omega} \left| u_{t} (t) \right|^{2} dx dt \leq \varepsilon c \int_{S}^{T} E^{q+1} (t) dt + c_{\varepsilon} E (S).
\]  
(4.13)

By insert (4.7), (4.8), (4.9) and (4.13) in (4.5), we arrive at
\[
\gamma \int_{S}^{T} E^{\frac{m_{2}}{2}} (t) dt \leq \varepsilon c \int_{S}^{T} E^{\frac{m_{2}}{2}} (t) dt + c_{\varepsilon} E (S).
\]

Choosing \( \varepsilon \) small enough for that
\[
\int_{S}^{T} E^{\frac{m_{2}}{2}} (t) dt \leq cE (S).
\]
Global Existence and Stability of Solution

By taking $T$ goes to $\infty$, we get

$$\int_{S}^{\infty} E \left( \frac{m}{2} (t) \right) dt \leq cE(S).$$

By Komornik’s integral inequality yields the result.

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**References**


Amar Ouaoua,
Department of Mathematics,
Laboratory of Applied Mathematics and History and Didactics of Mathematics (LAMAHIS)
University of 20 August 1955, Skikda,
Algeria.
E-mail address: ouaouaam21@gmail.com

and

Aya Khaldi
Department of Mathematics,
Laboratory of Applied Mathematics and History and Didactics of Mathematics (LAMAHIS)
University of 20 August 1955, Skikda,
Algeria.
E-mail address: ayakhaldi21@gmail.com

and

Messaoud Maouni,
Department of Mathematics,
Laboratory of Applied Mathematics and History and Didactics of Mathematics (LAMAHIS)
University of 20 August 1955, Skikda,
Algeria.
E-mail address: m.maouni@univ-skikda.dz