Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2022 (40)** : 1–9.
ISSN-0037-8712 in press
doi:10.5269/bspm.51616

A Generalized Common Fixed Point Theorem in Complex Valued $b-$Metric Spaces

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ABSTRACT: In this work we are interested in the generalization of coincidence point and fixed point theorem for a 4-tuple of mappings satisfying a new type of implicit relation in complex valued $b-$metric spaces.

Key Words: Metric space, Complex valued b-metric, Fixed point, Implicit relation, $(P_{n,m})$.

Contents

1. Introduction

The study of fixed point theory in metric spaces has done a great service in several areas of mathematics, namely, in solving differential and functional equations, in the field of approximation theory, in optimization etc. In 2011 Azam A. et al (see [3]) introduced and studied complex valued metric spaces wherein some fixed point theorems for mappings satisfying a rational inequality were established and obtained several results in fixed point theory. The concept of complex valued b-metric space as a generalization of complex valued metric space. Subsequently, many authors proved fixed and common fixed point results in complex valued b-metric spaces (for example [5], [17]).

In this work we are interested in the generalization of coincidence point and fixed point theorem for a 4-tuple of mappings satisfying a new type of implicit relation in complex valued $b-$metric spaces.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:
$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$, $Im(z_1) \leq Im(z_2)$.
Consequently, one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied:
(i) $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$,
(ii) $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) < Re(z_2)$, $Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) = Re(z_2)$, $Im(z_1) = Im(z_2)$.
In particular, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied.

Definition 1.1 ([4]). *Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \longrightarrow \mathbb{R}^+$ is said to be a b-metric on X if the following conditions hold:*
(i) $d(x,y) = 0$ *if and only if* $x = y$,
(ii) $d(x,y) = d(y,x)$ *for all* $x, y \in X$,
(iii) $d(x,y) \leq s[d(x,z) + d(z,y)]$ *for all* $x, y, z \in X$.

Definition 1.2. *[17] Let X be a nonempty set and $s \geq 1$ a given real number. A function $d : X \times X \to \mathbb{C}$, satisfies the following conditions:*
(d_1) $0 \precsim d(x,y)$, *for all* $x, y \in X$ *and* $d(x,y) = 0$ *if and only if* $x = y$,
(d_2) $d(x,y) = d(y,x)$ *for all* $x, y \in X$,
(d_3) $d(x,y) \precsim s[d(x,z) + d(z,y)]$, *for all* $x, y, z \in X$.
Then (X,d) is called a complex valued $b-$metric space.

2010 *Mathematics Subject Classification:* 54H25, 47H10.
Submitted December 24, 2019. Published April 22, 2020

Note that every complex valued metric space is a complex valued $b-$metric space with $s=1$. But the converse need not be true.

Example 1.3. *Let $X=\mathbb{C}$. Define $d: X\times X\to\mathbb{C}^+$ by $d(x,y)=((Re(x-y))^2+i\times(Im(x-y))^2$ for all $x,y\in X$. Then (X,d) is a complex valued $b-$metric space with $s=2$.*

Definition 1.4. *[16] let $f:\mathbb{C}\to\mathbb{C}$ be a given mapping, we say that f is a non-decreasing mapping with respect $\precsim$ if for every $x,y\in\mathbb{C}$, $x\precsim y$ implies $fx\precsim fy$.*

Definition 1.5. *Let (X,d) be a complex valued $b-$metric space and let*
1) $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n,x)|\to 0$ as $n\to\infty$.
2) $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n,x_{n+m})|\to 0$ as $n\to\infty$.
3) $A\subset X$ is said to be bounded is $\sup\limits_{x,y\in A}|d(x,y)|<+\infty$.

Definition 1.6. *Let $f,F: X\to X$*
1) A point $x\in X$ is said to be a coincidence point of f and F if $fx=Fx$. We denote by $C_{f,F}$ the set of all coincidence points of f and F.
2) A point $x\in X$ is a fixed point of F if $x=Fx$.

If $f=Id$ we have $C_{Id,F}$ the set of all fixed points of F.

Definition 1.7. *[2] The pair $f,F: X\longrightarrow X$ is occasionally weakly compatible (owc) if $fFx=Ffx$ for some $x\in C_{f,F}$.*

Definition 1.8. *[8] The pair $f: X\longrightarrow X$ and $F: X\longrightarrow B(X)$ satisfies $(P_{n,m})$ if $\exists\, x\in X$ such that $f^m x\in Fx$ and $f^n x\in(Ff^{n-m}x\cap Ff^m x)$, with $n,m\in\mathbb{N}$ and $n>m$. $(f^0x=x)$.*
$B(X)$ the set of all nonempty bounded subset of X.

Remark 1.9. *[8] If f and F are owc, then (f,F) satisfies $(P_{2,1})$.*

Example 1.10. *[8] Let $f:[0,1]\longrightarrow[0,1]$ and $F:[0,1]\longrightarrow B([0,1])$, such that*
$$f(x)=\begin{cases}1 \text{ if } x\in\{0,1\}\\ 0 \text{ else}\end{cases}\quad\text{and } Fx=\begin{cases}]0,1] \text{ if } x\in\{0,1\}\\ 0 \text{ else}\end{cases}$$
then $f(0)\in F0$ and $f^3(0)\in(Ff^2(0))\cap(Ff(0))$, so (f,F) satisfies $(P_{3,1})$.

Example 1.11. *Let $f:[0,1]\longrightarrow[0,1]$ and $F:[0,1]\longrightarrow[0,1]$, such that*
$$f(x)=\begin{cases}\frac{1}{2} \text{ if } x=0\\ 1 \text{ if } x=\frac{1}{2}\\ 0 \text{ else}\end{cases}\quad\text{and } Fx=\begin{cases}0 \text{ if } x\in\{\frac{1}{2},1\}\\ \\ \frac{1}{2} \text{ else}\end{cases}$$
then $f(0)=F0$ and $f^3(0)=Ff^2(0)=Ff(0)$, so (f,F) satisfies $(P_{3,1})$.

Definition 1.12. *[7][Altering Distance Function] A function $\psi:[0,1)\longrightarrow[0,1)$ is called an altering distance function if the following properties are satisfied:*
(i) is continuous and strictly increasing,
(ii) $\psi(t)=0$ if and only if $t=0$.

Notations(see [12])

$$\Psi=\{\psi:[0,1)\longrightarrow[0,1)|\psi \text{ is an altering distance function }\},$$

$$\Phi_1=\left\{\varphi:[0,\infty)\longrightarrow[0,\infty),\ \varphi \text{ is continuous, } \varphi(t)=0\Leftrightarrow t=0, \text{ and } \varphi(\liminf_{n\to\infty}a_n)\le\liminf_{n\to\infty}\varphi(a_n)\right\}.$$

$$\Phi_2=\left\{\begin{array}{l}\varphi:[0,\infty)\times[0,\infty)\longrightarrow[0,\infty),\ \varphi \text{ is continuous, } \varphi(x,y)=0\Leftrightarrow x=y=0,\\ \text{and } \varphi(\liminf\limits_{n\to\infty}a_n,\liminf\limits_{n\to\infty}b_n)\le\liminf\limits_{n\to\infty}\varphi(a_n,b_n)\end{array}\right\}.$$

Theorem 1.13 (theorem 4 [18]). *Let (X,d) be a complete b-metric space with constant $s \geq 1$ and let $T: X \longrightarrow X$ be such that*

$$d(T(x),T(y)) \leq \alpha d(x,y) + \beta d(x,T(x)) + \gamma d(y,T(y))$$

for every $x,y \in X$, where $\alpha,\beta,\gamma \geq 0$ with $\alpha+\beta+\gamma < \frac{1}{s}$. Then T has a unique fixed point in X.

Theorem 1.14 (theorem 2.1 [15]). *If S and T are self-mappings defined on a complete complex valued metric space (X,d) satisfying the condition*

$$d(Sx,Ty) \precsim \lambda d(x,y) + \frac{\mu d(x,Sx)d(y,Ty) + \gamma d(y,Sx)d(x,Ty)}{1+d(x,y)}$$

for all $x,y \in X$ where λ,μ,γ are nonnegative reals with $\lambda+\mu+\gamma < 1$, then S and T have a unique common fixed point.

Theorem 1.15 (theorem 3.1 [5]). *Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and $x_0 \in X$. Let $0 \prec r \in \mathbb{C}$ and A,B,C,D and E are nonnegative reals such that $A+B+C+2sD+2sE < 1$. Let $S,T: X \longrightarrow X$ are mappings satisfying:*

$$d(Sx,Ty) \precsim Ad(x,y) + B\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)} + C\frac{d(y,Sx)d(x,Ty)}{1+d(x,y)} + D\frac{d(x,Sx)d(x,Ty)}{1+d(x,y)} + E\frac{d(y,Sx)d(y,Ty)}{1+d(x,y)}$$

for all $x,y \in \overline{B(x_0,r)}$. If $|d(x_0,Sx_0)| \leq (1-\lambda)|r|$ where $\lambda = \max\{\frac{A+sD}{1-B-sD}, \frac{A+sE}{1-B-sE}\}$, then there exists a unique point $u \in \overline{B(x_0,r)}$ such that $u = Su = Tu$.

2. Main Results

Definition 2.1. *Let $s \geq 1$ and $\mathcal{F}_s$ be the set of all functions $\phi(t_1,t_2,...,t_6): \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ satisfying the following conditions:*

(ϕ_1) *ϕ continuous on $\mathbb{C}_+^6$,*

(ϕ_2) *$\exists \alpha,\beta \in \mathbb{R}_+$ such that $\alpha+2s\beta < 1$, $\forall u,v,w \in \mathbb{C}_+$:*

$$\phi(u,v,u,v,0,w) \precsim 0 \text{ or } \phi(u,v,v,u,w,0) \precsim 0 \Rightarrow |u| \leq \alpha|v| + \beta|w|,$$

(ϕ_3) *$\exists \gamma,\mu \in \mathbb{R}_+$ such that $s\gamma + s^2\mu < 1$, $\forall u,v,w \in \mathbb{C}_+$:*

$$\phi(u,0,v,0,0,w) \precsim 0 \Rightarrow |u| \leq \gamma|v| + \mu|w|,$$

(ϕ_4) *$\phi(u,0,u,0,0,u) \precsim 0$ or $\phi(u,u,0,0,u,u) \precsim 0 \Rightarrow u = 0$.*

Example 2.2. $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = \eta t_1 - (\alpha t_2 + \beta t_3 + \gamma t_4)$.
Where $\eta,\alpha,\beta,\gamma \in \mathbb{C}_+$, with $s(\alpha+\beta+\gamma) \prec \eta$.

Example 2.3. $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = at_1 - rt_2$.
Where $r,a \in \mathbb{C}_+$, with $sr \prec a$.

Example 2.4.

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = \eta t_1 - (\alpha t_2 + \beta t_3 + \gamma t_4 + \mu[t_5+t_6]).$$

Where $\mu \in \mathbb{R}_+$, $\eta,\alpha,\beta,\gamma \in \mathbb{C}_+$, with $s(\alpha+\beta) + \gamma + (s^2+s)\mu \precsim \eta$.

Example 2.5.

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - r\max\left\{t_2,t_3,t_4,\frac{t_5+t_6}{2s}\right\}.$$

Where $0 \leq r < 1$, with $rs < 1$.

Example 2.6. $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - r\max\{t_2,t_3,t_4\}$.
With $0 \leq r < \frac{1}{s}$.

Example 2.7. $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \mu[t_3+t_4]$.
With $\mu < \min\{\frac{1}{2},\frac{1}{s}\}$,

Example 2.8. $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - (\lambda t_2 + \frac{\mu t_3 t_4 + \gamma t_5 t_6}{1+t_2})$. *Where* λ,μ,γ *are nonnegative reals with* $\lambda+\mu+\gamma<1$,

Example 2.9. $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - (At_2 + B\frac{t_3t_4}{1+t_2} + C\frac{t_5t_6}{1+t_2} + D\frac{t_3t_5}{1+t_2} + E\frac{t_4t_6}{1+t_2})$. *Where* A,B,C,D,E *are nonnegative reals with* $A+B+C+2sD+2sE<1$,

Example 2.10. $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \frac{t_2}{s^3}$.

Example 2.11. $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \left(\frac{t_5}{s+1} + \frac{t_6}{s^4(s+1)}\right)$.

Theorem 2.12. *Let* (X,d) *be a complex valued* $b-$*metric space with constant* s, f,g,F *and* $G: X \longrightarrow X$ *satisfying* $GX \subseteq fX$, $FX \subseteq gX$, *and*

$$\phi\left(d(Fx,Gy), d(fx,gy), d(fx,Fx), d(gy,Gy), d(fx,Gy), d(Fx,gy)\right) \precsim 0, \tag{2.1}$$

for all $x,y \in X$, *where* $\phi \in \mathfrak{F}_s$, *if one of* FX, GX, fX *or* gX *is a complete subspace of* X,
then $C_{f,F} \neq \emptyset$, $C_{g,G} \neq \emptyset$ *and* $f(C_{f,F}) = F(C_{f,F}) = g(C_{g,G}) = G(C_{g,G}) = \{fx\} = \{gy\} = \{.\}$, *for all* $x \in C_{f,F}$, $y \in C_{g,G}$.

Proof.
Let x_0 be an arbitrary point in X. Since $FX \subseteq gX$, we find a point x_1 in X such that $Fx_0 = gx_1$. Also, since $GX \subseteq fX$, we choose a point x_2 with $Gx_1 = fx_2$. Thus in general for the point x_{2n-2} one find a point x_{2n-1} such that $Fx_{2n-2} = gx_{2n-1}$ and then a point x_{2n} with $Gx_{2n-1} = fx_{2n}$ for $n = 1,2,......$
Repeating such arguments one can construct sequences x_n and y_n in X such that,

$$y_{2n-1} = Fx_{2n-2} = gx_{2n-1}, y_{2n} = Gx_{2n-1} = fx_{2n}, n = 1,2,..... \tag{2.2}$$

For $x = x_{2n}$ and $y = x_{2n+1}$ By the inequality (2.1) we have :

$$\phi\left(\begin{array}{l} d(Fx_{2n},Gx_{2n+1}), d(fx_{2n},gx_{2n+1}), d(fx_{2n},Fx_{2n}) \\ , d(gx_{2n+1},Gx_{2n+1}), d(fx_{2n},Gx_{2n+1}), d(gx_{2n+1},Fx_{2n}) \end{array}\right) \precsim 0.$$

Implies

$$\phi\left(d(y_{2n+1},y_{2n+2}), d(y_{2n},y_{2n+1}), d(y_{2n},y_{2n+1}), d(y_{2n+1},y_{2n+2}), d(y_{2n},y_{2n+2}), 0\right) \precsim 0.$$

So, by (ϕ_2) we have

$$\begin{array}{rcl} |d(y_{2n+1},y_{2n+2})| & \leq & \alpha|d(y_{2n},y_{2n+1})| + \beta|d(y_{2n},y_{2n+2})| \\ & \leq & \alpha|d(y_{2n},y_{2n+1})| + \beta s[|d(y_{2n},y_{2n+1})| + |d(y_{2n+1},y_{2n+2})|]. \end{array}$$

So

$$|d(y_{2n+1},y_{2n+2})| \leq h|d(y_{2n},y_{2n+1})| \text{ with } h = \frac{\alpha+s\beta}{1-s\beta} < 1. \tag{2.3}$$

For $x = x_{2n+2}$ and $y = x_{2n+1}$, by the inequality (2.1) we have :

$$\phi\left(\begin{array}{l} d(Fx_{2n+2},Gx_{2n+1}), d(fx_{2n+2},gx_{2n+1}), d(fx_{2n+2},Fx_{2n+2}) \\ , d(gx_{2n+1},Gx_{2n+1}), d(fx_{2n+2},Gx_{2n+1}), d(gx_{2n+1},Fx_{2n+2}) \end{array}\right) \precsim 0.$$

Implies

$$\phi\left(d(y_{2n+3},y_{2n+2}), d(y_{2n+2},y_{2n+1}), d(y_{2n+3},y_{2n+2}), d(y_{2n+2},y_{2n+1}), 0, d(y_{2n+1},y_{2n+3})\right) \precsim 0.$$

So, by (ϕ_2) we have

$$\begin{aligned}|d(y_{2n+3},y_{2n+2})| &\leq \alpha|d(y_{2n+2},y_{2n+1})|+\beta|d(y_{2n+3},y_{2n+1})| \\ &\leq \alpha|d(y_{2n+2},y_{2n+1})|+s\beta[|d(y_{2n+3},y_{2n+2})|+|d(y_{2n+2},y_{2n+1})|].\end{aligned}$$

$$|d(y_{2n+3},y_{2n+2})|\leq h|d(y_{2n+2},y_{2n+1})|. \tag{2.4}$$

By (2.3) and (2.4) we have

$$|d(y_{n+1},y_n)|\leq h^{n-1}|d(y_1,y_2)|,\ n=2,3,.....$$

Therefore, for any $n,m\in\mathbb{N}^*$ with $n\geq 2$, we have

$$\begin{aligned}|d(y_n,y_{n+m})| &\leq s|d(y_n,y_{n+1})|+s^2|d(y_{n+1},y_{n+2})|+s^3|d(y_{n+2},y_{n+3})|+ \\ &\quad ...+s^{m-1}|d(y_{n+m-2},y_{n+m-1})|+s^{m-1}|d(y_{n+m-1},y_{n+m})|.\end{aligned}$$

On the other hand we have :

$$\begin{aligned}|d(y_n,y_{n+m})| &\leq \left(sh^{n-1}|d(y_1,y_2)|+...+s^{m-1}h^{n+m-3}|d(y_1,y_2)|+s^{m-1}h^{n+m-2}|d(y_1,y_2)|\right) \\ &\leq sh^{n-1}\left(1+(sh)+(sh)^2+...+(sh)^{m-2}+s^{m-2}h^{m-1}\right)|d(y_1,y_2)| \\ &= sh^{n-1}\left(\frac{1-(sh)^{m-1}}{1-sh}+s^{m-2}h^{m-1}\right)|d(y_1,y_2)| \\ &\leq h^{n-1}\left(\frac{s}{1-sh}+(sh)^{m-1}\right)|d(y_1,y_2)|,\end{aligned}$$

from where $\lim_{n\to\infty} d(y_n,y_{n+m})=0$ for $m\in\mathbb{N}^*$. By definition 1.5 then (y_n) is a Cauchy sequence in (X,d).

If fX is a complete subspace of X, there exists $u\in fX$ such that $\lim_{n\to\infty} d(y_{2n},u)=0$. Then we can find $v\in X$ such that

$$fv=u \tag{2.5}$$

We claim that $u=Fv$.

$$\begin{aligned}|d(Fv,y_{2n})| &\leq s|d(Fv,y_{2n+1})|+s|d(y_{2n+1},y_{2n})| \\ &\leq s^2[|d(Fv,u)|+|d(u,y_{2n+1})|]+s|d(y_{2n+1},y_{2n})|\end{aligned}$$

we deduce that the sequence $(d(Fv,y_{2n}))$ is bounded, similarly, we obtain $(d(Fv,y_{2n-1}))$ is bounded.

Then there exists a strictly increasing application $\theta:\mathbb{N}\longrightarrow\mathbb{N}$ such that $\left(d\left(Fv,y_{2\theta(n)-1}\right)\right)$ and $\left(d\left(Fv,y_{2\theta(n)}\right)\right)$ are convergent.

Using inequality (2.1) and (2.5), we have

$$\phi\left(\begin{array}{l} d\left(Fv,Gx_{2\theta(n)-1}\right),d\left(fv,gx_{2\theta(n)-1}\right),d(fv,Fv) \\ ,d\left(gx_{2\theta(n)-1},Gx_{2\theta(n)-1}\right),d\left(fv,Gx_{2\theta(n)-1}\right),d\left(Fv,gx_{2\theta(n)-1}\right)\end{array}\right)\precsim 0.$$

We have successively

$$\phi\left(d\left(Fv,y_{2\theta(n)}\right),d\left(u,y_{2\theta(n)-1}\right),d(u,Fv),d\left(y_{2\theta(n)-1},y_{2\theta(n)}\right),d\left(u,y_{2\theta(n)}\right),d\left(Fv,y_{2\theta(n)-1}\right)\right)\precsim 0.$$

letting $n\to\infty$ by (ϕ_1) we obtain

$$\phi\left(\lim_{n\to+\infty} d\left(Fv,y_{2\theta(n)}\right),0,d(u,Fv),0,0,\lim_{n\to+\infty} d\left(Fv,y_{2\theta(n)-1}\right)\right)\precsim 0.$$

Then by (ϕ_3), we have

$$\begin{aligned} |\lim_{n\to+\infty} d\left(Fv, y_{2\theta(n)}\right)| &\leq \gamma|d(u,Fv)| + \mu|\lim_{n\to+\infty} d\left(Fv, y_{2\theta(n)-1}\right)| \\ &\leq \gamma|d(u,Fv)| + s\mu|\lim_{n\to+\infty}[d(Fv,u) + d\left(u, y_{2\theta(n)-1}\right)]|, \end{aligned}$$

so

$$|\lim_{n\to+\infty} d\left(Fv, y_{2\theta(n)}\right)| \leq (\gamma + s\mu)|d(Fv,u)|. \tag{2.6}$$

On the other hand we have

$$|d(u,Fv)| \leq s[|d\left(u, y_{2\theta(n)}\right)| + |d\left(y_{2\theta(n)}, Fv\right)|].$$

By (2.6) we have

$$\begin{aligned} |d(u,Fv)| &\leq \lim_{n\to+\infty} s[|d\left(u, y_{2\theta(n)}\right)| + |d\left(y_{2\theta(n)}, Fv\right)|] \\ &= s\lim_{n\to+\infty}|d\left(y_{2\theta(n)}, Fv\right)| \\ &\leq (s\gamma + s^2\mu)|d(u,Fv)| \\ &< |d(u,Fv)|, \end{aligned}$$

so $d(Fv,u) = 0$, that is $u = fv = Fv$.

By $FX \subset gX$ we have $w \in X$ such that $gw = u$. Then we have also $w \in C_{g,G} \neq \emptyset$, and $f(C_{f,F}) \cap g(C_{g,G}) \neq \emptyset$.

For $x = v \in C_{f,F}$ and $y = w \in C_{g,G}$ by (2.1) we have successively

$$\phi\left(d(Fv,Gw), d(fv,gw), d(fv,Fv), d(gw,Gw), d(fv,Gw), d(Fv,gw)\right) \precsim 0,$$

so

$$\phi\left(d(fv,Gw), d(fv,Gw), 0, 0, d(fv,Gw), d(fv,Gw)\right) \precsim 0,$$

then by (ϕ_4), we have $d(fv,Gw) = 0$, there is $g(C_{g,G}) = G(C_{g,G}) = gw = fv = Fv$. Similarly, we have $f(C_{f,F}) = F(C_{f,F}) = g(C_{g,G}) = G(C_{g,G}) = gw = fv$, for all $v \in C_{f,F}$, $w \in C_{g,G}$.

If GX is a complete subspace of X, there exists $u \in X$ such that $\lim\limits_{n\to\infty} d(y_{2n}, u) = 0$. Then we can find $w \in X$ such that

$$Gw = u.$$

And like $GX \subset fX$, there exists $v \in X$ such that $fv = u$. In the same previous way we find $u = Fv$ and there exists $w' \in X$ such that $gw' = Gw' = u$.

If FX or gX is complete, then by permuting the roles of f with g and F with G, we find the proof.

Corollary 2.13. *Let (X,d) be a complex valued $b-$metric space with constant s, let $F, G : X \longrightarrow X$ satisfying*

$$\phi\left(d(Fx,Gy), d(x,y), d(x,Fx), d(y,Gy), d(x,Gy), d(Fx,y)\right) \precsim 0, \tag{2.7}$$

for all $x, y \in X$, where $\phi \in \mathfrak{F}_s$, if one of FX, GX, or X is a complete subspace of X, then F and G have a unique common fixed point.

Proof. Suppose $f = g = Id$, so (2.7) $\Rightarrow$ (2.1), by theorem 2.12 we have $C_{Id,F} = C_{Id,G} \neq \emptyset$ and $C_{Id,F} = F(C_{Id,F}) = C_{Id,G} = G(C_{Id,G}) = \{x\} = \{y\} = \{.\}$, for all $x \in C_{Id,F}$, $y \in C_{Id,G}$.

Theorem 2.14. *Let (X,d) be a complex valued $b-$metric space with constant s, let $f,g,F,G : X \longrightarrow X$ satisfying $GX \subseteq f^{m_1}X$, $FX \subseteq g^{m_2}X$, $m_1, m_2 \in \mathbb{N}$ and*

$$\phi\left(d\left(Fx,Gy\right),d\left(f^{m_1}x,g^{m_2}y\right),d\left(f^{m_1}x,Fx\right),d\left(g^{m_2}y,Gy\right),d\left(f^{m_1}x,Gy\right),d\left(Fx,g^{m_2}y\right)\right) \precsim 0, \tag{2.8}$$

for all $x,y \in X$, where $\phi \in \mathfrak{F}_s$, if one of FX, GX, $f^{m_1}X$ or $g^{m_2}X$ is a complete subspace of X. Then

(i) $C_{f^{m_1},F} \neq \emptyset$, $C_{g^{m_2},G} \neq \emptyset$ and $f^{m_1}(C_{f^{m_1},F}) = F(C_{f^{m_1},F}) = g^{m_2}(C_{g^{m_2},G}) = G(C_{g^{m_2},G}) = \{.\}$.

(ii) If the pair (F,f) satisfies (P_{n_1,m_1}), and (G,g) satisfies (P_{n_2,m_2}), then F, G, $f^{n_1-m_1}$ and $g^{n_2-m_2}$ have common fixed point $u \in X$.

Moreover, if $n_1 = 2m_1$ or $n_2 = 2m_2$, then u is unique.

Proof. (i) For $f = f^{m_1}$ and $g = g^{m_2}$ we have (2.8) $\Rightarrow$ (2.1), so by theorem 2.12, $C_{f^{m_1},F} \neq \emptyset$, $C_{g^{m_2},G} \neq \emptyset$ and $f^{m_1}(C_{f^{m_1},F}) = F(C_{f^{m_1},F}) = g^{m_2}(C_{g^{m_2},G}) = G(C_{g^{m_2},G}) = \{.\}$.

(ii) Now, we prove that $F, G, f^{n_1-m_1}$ and $g^{n_2-m_2}$, have a common fixed point. Since (F,f) satisfies (P_{n_1,m_1}), and (G,g) satisfies (P_{n_2,m_2}), there exist $v,w \in X$ such that $f^{m_1}v = Fv$, $f^{n_1}v = Ff^{m_1}v = Ff^{n_1-m_1}v$, $g^{m_2}w = Gw$ and $g^{n_2}w = Gg^{m_2}w = Gg^{n_2-m_2}w$, then $v \in C_{f^{m_1},F}$, $w \in C_{g^{m_2},G}$ and we have (i). So $u = f^{m_1}v = Fv = g^{m_2}w = Gw$.

For $x = f^{n_1-m_1}v$, $y = w$, by (2.1) we have successively :

$$\phi\left(\begin{array}{l} d\left(Ff^{n_1-m_1}v,Gw\right),d\left(f^{n_1}v,g^{m_2}w\right),d\left(f^{n_1}v,Ff^{n_1-m_1}v\right) \\ ,d\left(g^{m_2}w,Gw\right),d\left(f^{n_1}v,Gw\right),d\left(Ff^{n_1-m_1}v,g^{m_2}w\right) \end{array}\right) \precsim 0,$$

$$\phi\left(d\left(Ff^{n_1-m_1}v,Gw\right),d\left(Ff^{n_1-m_1}v,Gw\right),0,0,d\left(Ff^{n_1-m_1}v,Gw\right),d\left(Ff^{n_1-m_1}v,Gw\right)\right) \precsim 0,$$

by (ϕ_3), we have $d\left(Ff^{n_1-m_1}v,Gw\right) = 0$, this implies that $Ff^{n_1-m_1}v = Gw = u$. $f^{n_1-m_1}u = f^{n_1}v = Ff^{m_1}v = Ff^{n_1-m_1}v = Fu = u$. Similarly, we have $u = g^{n_2-m_2}u = Gu$.

Suppose that $n_1 = 2m_1$ and u' is an other common fixed point of $f^{n_1-m_1}, g^{n_2-m_2}, F$ and G.

Then $u' = f^{n_1-m_1}u' = f^{m_1}u' = Fu'$, so $u' \in C_{f^{m_1},F}$ and we have $Fu = u = f^{n_1-m_1}u = f^{m_1}u$ by theorem 2.12 we have $f^{m_1}(C_{f^{m_1},F}) = F(C_{f^{m_1},F}) = g^{m_2}(C_{g^{m_2},G}) = G(C_{g^{m_2},G}) = \{g^{m_2}u\} = \{f^{m_1}u'\}$, hence $u = u'$.

Note that if (F,f), (G,g) are owc, then (F,f), (G,g) satisfies $(P_{2,1})$, so by theorem 2.14 we obtain :

Corollary 2.15. *Let (X,d) be a complex valued $b-$metric space with constant s, let $f,g,F,G : X \longrightarrow X$ satisfying $GX \subseteq fX$, $FX \subseteq gX$ and*

$$\phi\left(d\left(Fx,Gy\right),d\left(fx,gy\right),d\left(fx,Fx\right),d\left(gy,Gy\right),d\left(fx,Gy\right),d\left(Fx,gy\right)\right) \precsim 0, \tag{2.9}$$

for all $x,y \in X$, where $\phi \in \mathfrak{F}_s$, if one of FX, GX, fX or gX is a complete subspace of X. Then

(i) $C_{f,F} \neq \emptyset$, $C_{g,G} \neq \emptyset$ and $f(C_{f,F}) = F(C_{f,F}) = g(C_{g,G}) = G(C_{g,G}) = \{.\}$.

(ii) If the pair (F,f), (G,g) are occasionally weakly compatible (owc). Then F, G, f and g have a unique common fixed point.

Proof.

(F,f), (G,g) are owc, then (F,f), (G,g) are satisfies $(P_{2,1})$. So all conditions of theorem 2.14 are satisfied with $m_1 = m_2 = 1$ and $n_1 = n_2 = 2$, then F, G, $f = f^{2-1}$ and $g = g^{2-1}$ have a unique common fixed point.

3. Consequences

By corollary 2.13 and example 2.10 we obtain:

Theorem 3.1. *Let (X,d) be a complex valued $b-$metric space with constant s, let $F,G : X \longrightarrow X$ satisfying*

$$d\left(Fx,Gy\right) \precsim \frac{d\left(x,y\right)}{s^3} \tag{3.1}$$

for all $x,y \in X$, if one of FX, GX, or X is a complete subspace of X, then F and G have a unique common fixed point.

Corollary 3.2 (theorem 2.1 [6]). *Let (X,d) be a complet $b-$metric space with constant s, let $T : X \longrightarrow X$ be a self-mapping satisfying the $(\psi,\varphi)-$weakly contractive condition*

$$\psi(sd\,(Tx,Ty)) \leq \psi(\frac{d\,(x,y)}{s^2}) - \varphi(d(x,y)), \tag{3.2}$$

for all $x,y \in X$, where $\psi \in \Psi$, $\varphi \in \Phi_1$. Then T has a unique fixed point.

Proof. we have

$$\psi(sd\,(Tx,Ty)) \leq \psi(\frac{d\,(x,y)}{s^2}) - \varphi(d(x,y)) \leq \psi(\frac{d\,(x,y)}{s^2})$$

implies

$$sd\,(Tx,Ty) \leq \frac{d\,(x,y)}{s^2}$$

then (3.2) $\Rightarrow$ (3.1).

By corollary 2.13 and example 2.11 we obtain:

Theorem 3.3. *Let (X,d) be a complex valued $b-$metric space with constant s, let $F,G : X \longrightarrow X$ satisfying*

$$d\,(Fx,Gy) \precsim \frac{s^3 d\,(x,Gy) + d\,(y,Fx)}{s^4(s+1)} \tag{3.3}$$

for all $x,y \in X$, if one of FX, GX, or X is a complete subspace of X, then F and G have a unique common fixed point.

Corollary 3.4 (theorem 3.1 [6]). *Let (X,d) be a complet $b-$metric space with constant s, let $F,G : X \longrightarrow X$ be a self-mapping satisfying the $(\psi,\varphi)-$generalized Chatterajea-type contractive condition*

$$\psi(sd\,(Fx,Gy)) \leq \psi(\frac{s^3 d\,(x,Gy) + d\,(y,Fx)}{s^3(s+1)}) - \varphi(d\,(x,Gy)\,, d\,(y,Fx)), \tag{3.4}$$

for all $x,y \in X$, where $\psi \in \Psi$, $\varphi \in \Phi_2$. Then F and G have a unique common fixed point.

Proof. we have

$$\psi(sd\,(Fx,Gy)) \leq \psi(\frac{s^3 d\,(x,Gy) + d\,(y,Fx)}{s^3(s+1)}) - \varphi(d\,(x,Gy)\,, d\,(y,Fx)) \leq \psi(\frac{s^3 d\,(x,Gy) + d\,(y,Fx)}{s^3(s+1)})$$

implies

$$d\,(Fx,Gy) \leq \frac{s^3 d\,(x,Gy) + d\,(y,Fx)}{s^4(s+1)}$$

then (3.4) $\Rightarrow$ (3.3).

By corollary 2.13 and example 2.2 with $F = G$ we obtain theorem 1.13

By corollary 2.13 and example 2.3 with $F = G$ we obtain theorem 1 [13]

By corollary 2.13 and example 2.4 with $F = G$ we obtain theorem 3.1.2 [14]

By corollary 2.13 and example 2.5 with $F = G$ we obtain theorem 3.1.8 [14]

By theorem 2.14 and example 2.5 with $r = \frac{1}{s+a}$ we obtain corollary 2.3 [19]

By theorem 2.14 and example 2.5 with $r = \frac{1}{s^2}$ we obtain corollary 2.4 [19]

By corollary 2.13 and example 2.8 we obtain theorem 1.14

By corollary 2.13 and example 2.9 we obtain theorem 1.15

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