Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 ON LINE
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–12.
ISSN-0037-8712 IN PRESS
doi:10.5269/bspm.51652

Hausdorff Measure of Noncompactness of Matrix Mappings on Certain Difference Sequence Spaces

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ABSTRACT: The well-known difference sequence spaces were introduced by Kızmaz [16] in 1981 and have been generalized by many authors uptill now. These spaces were extended for the first time by Sarıgöl [30] to the sequence spaces $l_\infty(\Delta_q)$, $c(\Delta_q)$ and $c_0(\Delta_q)$. The aim of this paper is to establish some identities or estimates for the operator norms and the Hausdorff measures of noncompactness of certain matrix operators on the extended spaces and also to characterize some classes of compact operators by using the Hausdorff measure of noncompactness.

Key Words: Difference matrix, matrix transformation, sequence space, norm, Hausdorff measure of noncompactness.

Contents

1. Background, notation and preliminaries

Let w denotes the space of all complex valued sequences. Any vector subspace of w is called a *sequence space*. Let X, Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of complex numbers. If the series

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k \, (n \in \mathbb{N}) \, ,$$

converges for all $x = (x_k) \in X$ and the sequence $A(x) = (A_n(x))$, A-transform of a sequence x, is in Y, then we say that A defines a matrix mapping from X into Y. By (X, Y), we denote the class of all such matrices A.

Let l_∞, c_0, c and ϕ be the linear spaces of all bounded, null, convergent and finite sequences; cs, bs and $l_p, 1 \leq p < \infty$, be the linear spaces of all convergent, bounded and p-absolutely convergent series, respectively. Further, for any sequence space X, the matrix domain X_A is defined by

$$X_A = \{x \in \omega : A(x) \in X\} \, ,$$

which is also a sequence space. The new sequence space X_A generated by the limitation matrix A from the sequence space X can be the expansion or the restriction of the original space X.

The concepts of α, β, γ-duals of a sequence space play very important role in the summability theory. For the sequences spaces X, Y, the set $M(X, Y)$ defined by

$$M(X, Y) = \{y = (y_k) : \forall x \in X, (x_k y_k) \in Y\}$$

is called the multiplier space of X and Y. According to this notation, the α, β, γ-duals of a sequence space X are denoted by

$$X^\alpha = M(X, l), X^\beta = M(X, cs), X^\gamma = M(X, bs).$$

2010 *Mathematics Subject Classification:* 40C05, 46A45, 46B45, 46B50.

Submitted December 29, 2019. Published July 20, 2020

An infinite matrix $A=(a_{nk})$ is called a triangle if $a_{nn}\neq 0$ for all n and $a_{nk}=0$ for $n<k$.

A complete linear sequence space X with a norm is called a BK-space provided that the map $p_n: X\to\mathbb{C}$ defined by $p_n(x)=x_n$ is continuous for all $n\geq 0$, where $\mathbb{C}$ denotes the complex field. A sequence (b_n) in a normed space X is called a Schauder base for X if for every $x\in X$ there is a unique sequence (x_n) of scalars such that $x=\sum_{n=0}^{\infty}x_nb_n$, or,

$$\left\|x-\sum_{n=0}^{m}x_nb_n\right\|\to 0 \text{ as } m\to\infty.$$

Also a BK-space $X\supset\phi$ is said to have AK if the sequence $\left(e^{(j)}\right)$ is a Schauder base for X, where $e^{(j)}$ is the sequence whose only non-zero term is 1 in jth place for each $j\in\mathbb{N}$. For example, the sequence $(e^{(j)})$ is a Schauder base of $l_p, 1\leq p<\infty$, with respect to its natural norm, but the space l_∞ doesn't have the Schauder base [21].

The theory of BK-spaces is one of the most important tools characterizing of matrix transformations between sequence spaces. For example, matrix operators between BK-spaces are continuous and the matrix domain of a triangle A in the BK-space X is also a BK-space and its norm is given by

$$\|x\|_{X_A}=\|A(x)\|_X,$$

[32].

Let X and Y be two Banach spaces. By $\mathcal{B}(X,Y)$, we denote the set of all continuous linear operators from X into Y and write

$$\|A\|=\sup_{x\neq 0}\frac{\|A(x)\|_Y}{\|x\|_X}$$

for the operator norm of A. In the special case $Y=\mathbb{C}$, we write $X^*=\mathcal{B}(X,\mathbb{C})$, the set of all continuous linear functionals on X.

If $a\in\omega$ and $X\supset\phi$ is a BK-space, then

$$\|a\|_X^*=\sup_{x\in S_X}\left|\sum_{k=0}^{\infty}a_kx_k\right|$$

provided the right hand side of the equation exists, where S_X is the unit sphere in X, and it is finite for $a\in X^\beta$.

The well known difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ were introduced by Kızmaz [16]. These spaces are generalized and studied for the first time by Sarıgöl [30] as follows:

$$l_\infty(\Delta_q)=\{x=(x_k):\Delta_qx\in l_\infty, q<1\}$$

$$c(\Delta_q)=\{x=(x_k):\Delta_qx\in c, q<1\}$$

$$c_0(\Delta_q)=\{x=(x_k):\Delta_qx\in c_0, q<1\},$$

which are also Banach spaces with respect to the norm

$$\|x\|_{\Delta_q}=|x_1|+\|\Delta_qx\|_\infty,$$

where $y=\Delta_qx=(n^q(x_n-x_{n+1}))$. According to the matrix domain, these spaces can also be redefined by $(l_\infty)_{\Delta_q}=l_\infty(\Delta_q)$, $(c)_{\Delta_q}=c(\Delta_q)$, $(c_0)_{\Delta_q}=c_0(\Delta_q)$, where the matrix $\Delta_q=(\delta_{nv}^q)$ is defined by

$$\delta_{nv}^q=\begin{cases} n^q, & v=n\\ -n^q, & v=n+1\\ 0, & \text{otherwise}.\end{cases} \tag{1.1}$$

Note that, using $\lim_n \frac{1}{n^{1-q}} \sum_{k=1}^{n} \frac{1}{k^q} = \frac{1}{1-q}$ [17], we have a constant M such that, for all x,

$$\begin{aligned} |p_n(x)| = \left| x_1 - \sum_{k=1}^{n-1} k^{-q} y_k \right| &\leq |x_1| + \sup_k |y_k| \left| \sum_{k=1}^{n} k^{-q} n^{q-1} \right| n^{1-q} \\ &\leq M n^{1-q} \|x\|_{\Delta_q}, \text{ for each } n \geq 0, \end{aligned}$$

which means that the coordinate functional p_n defined on these difference spaces is bounded. So, these are also BK- spaces.

Throughout the paper, we denote E and F by one of the spaces l_∞, c, c_0, and $E'_q = (E)_{\Delta_q}$, $F'_q = (F)_{\Delta_q}$ by one of the spaces $l_\infty(\Delta_q)$, $c(\Delta_q)$ and $c_0(\Delta_q)$. Further, define the operator $S : E'_q \to E'_q$ by $S(x) = (0, x_2, x_3, ...), q < 1$. It is clearly that the operator is a bounded linear operator with $\|S\| = 1$. Then the space

$$SE'_q = \left\{ x = (x_k) : x \in E'_q, x_0 = 0 \right\}$$

is also Banach space with the same norm [30]. Moreover, the transformation Δ_q from SE'_q to E is a linear bijection, that is, $SE'_q \cong E$.

In the literature, many difference spaces have been introduced and investigated by the authors. For example, the space bv_p, consisting of all sequences x such that $(x_v - x_{v-1})$ is in l_p, was introduced and studied by Başar & Altay [2] for $1 \leq p < \infty$, and Altay & Başar [1] for $0 < p < 1$, the sequence spaces $\hat{l}_\infty$, $\hat{l}_p$, $\hat{c}$, $\hat{c}_0$, the set of all sequences whose $B(r,s)$-transforms are in the spaces l_∞, l_p, c, c_0, respectively, were studied by Kirişçi & Başar [15]. Also, using Fibonacci band matrix, the Fibonacci difference sequence spaces $l_p(\hat{F})$ and $l_\infty(\hat{F})$ are investigated by Kara [14]. Further, in different perspective, using the absolute summability, a lot of series spaces have been given and investigated by Mohapatra, Sarıgöl, Gökçe, Güleç (see [5,6,7,8,9,10,11,24,29]).

The Kuratowsky measure of noncompactness α, the first meausure of noncompactness, was defined by Kuratowsky [18]. Then, the Hausdorff measure of noncompactness χ was introduced by Goldenstein, Gohberg and Markus [4]. By using the Hausdorff measure of noncompactness, many authors characterized the class of compact operators on the sequence spaces. For example, Mursaleen and Noman in [25,26], Malkowsky and Rakocevic in [22] have used the Hausdorff measure of noncompactness to characterize the class of compact operators on the spaces, (see also [5,12,28]).

In the present paper, we give some identities and estimates for the norms and the Hausdorff measure of noncompactness of the matrix operators on the spaces $l_\infty(\Delta_q)$, $c(\Delta_q)$ and $c_0(\Delta_q)$ and also characterize certain compact operators.

We require the following theorems given by Sarıgöl [30].

Theorem A Let $q < 1$. Then,

$$\left\{ E'_q \right\}^\alpha = \left\{ a = (a_k) : \sum_{k=1}^{\infty} k^{1-q} |a_k| < \infty \right\} = D_1,$$

$$\left\{ E'_q \right\}^\beta = \left\{ a = (a_k) : \sum_{k=1}^{\infty} k^{1-q} a_k \text{ is convergent and } \sum_{k=1}^{\infty} k^{-q} |R_k| < \infty \right\} = D_2,$$

$$\left\{ E'_q \right\}^\gamma = \left\{ a = (a_k) : \sup_n \left| \sum_{k=1}^{\infty} k^{1-q} a_k \right| < \infty, \sum_{k=1}^{\infty} k^{-q} |R_k| < \infty \right\} = D_3,$$

where

$$R_k = \sum_{v=k+1}^{\infty} a_v.$$

Theorem B Let $q < 1$. $A \in (E'_q, F)$ if and only if

(i) $(a_{n1}) \in F$ and $(A_n(k^{1-q})) \in F$

(ii) $R_q = (k^{-q} r_{nk}) \in (E, F)$,

where

$$A_n(k^{1-q}) = \sum_{k=1}^{\infty} k^{1-q} a_{nk}, r_{nk} = \sum_{v=k+1}^{\infty} a_{nv}.$$

In the rest of the paper, we define the terms of the matrix $\tilde{A} = (\tilde{a}_{nk})$ as follows

$$\tilde{a}_{nk} = k^{-q} \sum_{v=k+1}^{\infty} a_{nv}$$

and let's denote p^* as the conjugate of p such that $p^* = p/(p-1)$ for $p > 1$, $1/p^* = 0$ for $p_n = 1$. Also, we use the following conditions:

$$(i) \quad \sup_n \sum_k |a_{nk}| < \infty.$$

$$(ii) \quad \lim_{n\to\infty} a_{nk} = 0 \text{ for each } k \in \mathbb{N}.$$

$$(iii) \text{ There is some } a_k \text{ such that } \lim_{n\to\infty} a_{nk} = a_k \in \mathbb{R}, \text{ for each } k \in \mathbb{N}.$$

$$(iv) \quad \lim_{n\to\infty} \sum_k a_{nk} = 0.$$

$$(v) \text{ There is some } a \text{ such that } \lim_{n\to\infty} \sum_k a_{nk} = a \in \mathbb{R}.$$

$$(vi) \quad \sum_k |a_{nk}| \text{ converges uniformly in } n.$$

$$(vii) \lim_{n\to\infty} \sum_k |a_{nk}| = 0.$$

$$(viii) \quad \sum_n \left| \sum_k a_{nk} \right|^p < \infty.$$

Lemma 1.1. *[31]*
(a) $A \in (c_0, c_0)$ *if and only if (i) and (ii) hold.*
(b) $A \in (c_0, c)$ *if and only if (i) and (iii) hold.*
(c) $A \in (c, c_0)$ *if and only if (i), (ii) and (iv) hold.*
(d) $A \in (c, c)$ *if and only if (i), (iii) and (v) hold.*
(e) $A \in (c_0, \ell_\infty) = (c, \ell_\infty) = (\ell_\infty, \ell_\infty)$ *if and only if the condition (i) holds.*
(f) $A \in (\ell_\infty, c)$ *if and only if the conditions (iii) and (vi) hold.*
(g) $A \in (\ell_\infty, c_0)$ *if and only if the condition (vii) holds.*
(h) $A \in (c_0, \ell_p) = (c, \ell_p) = (l_\infty, \ell_p), 1 \le p < \infty$ *if and only if the condition (viii) holds.*

Lemma 1.2. *[21] Let T be a triangle. Then, we have*

(a) For arbitrary subsets X and Y of ω, $A \in (X, Y_T)$ if and only if $B = TA \in (X, Y)$.

(b) Further, if X and Y are BK-spaces and $A \in (X, Y_T)$, then $\|L_A\| = \|L_B\|$.

We note that the following results are immediate from Lemma 1.1, Lemma 1.2 and Theorem B.

Theorem 1.3. *Let $q<1$. $A\in(E,F_q')$ if and only if $B=(b_{nk})\in(E,F)$ where $b_{nk}=n^q(a_{nk}-a_{n+1,k})$.*

Theorem 1.4. *Let $q<1$, the matrix $B=(b_{nk})$ be as in Theorem 1.3. Then, $A\in(E_q',F_q')$ if and only if*

(i) $(b_{n1})\in F$, $\big(B_n(k^{1-q})\big)\in F$

(ii) $\tilde{B}\in(E,F)$.

2. The Hausdorff Measure of Noncompactness

Let S and H are subsets of a metric space (X,d). Then S is called an ε-net of H, if, for every $h\in H$, there exists an $s\in S$ such that $d(h,s)<\varepsilon$; if S is finite, then the ε-net S of H is called a finite ε-net of H. Let X and Y be Banach spaces. A linear operator $L:X\to Y$ is called compact if its domain is all of X and, for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a convergent subsequence in Y. We denote the class of such operators by $C(X,Y)$. If Q is a bounded subset of the metric space X, then the Hausdorff measure of noncompactness of Q is defined by

$$\chi(Q)=\inf\{\varepsilon>0: Q \text{ has a finite } \varepsilon-\text{net in } X\},$$

and χ is called the Hausdorff measure of noncompactness.

Lemma 2.1. *[19] Let X be a Banach space with a Schauder basis (b_n), Q be a bounded subset of the space X and $P_r:X\to X$ is a projector onto the linear span of $\{b_0,b_1,...,b_r\}$. Then, we have*

$$\frac{1}{a}\limsup_{r\to\infty}\left\{\sup_{x\in Q}\|(I-P_r)(x)\|_X\right\}\le\chi(Q)\le\limsup_{r\to\infty}\left\{\sup_{x\in Q}\|(I-P_r)(x)\|_X\right\},$$

where $a=\limsup_{r\to\infty}\|I-P_r\|_X$.

Lemma 2.2. *[27] Let Q be a bounded subset of the normed space X where $X=c_0$ or $X=\ell_p$ for $1\le p<\infty$. If $P_n:X\to X$ is the operator defined by $P_r(x)=(x_0,x_1,...,x_r,0,...)$ for all $x\in X$, then*

$$\chi(Q)=\lim_{r\to\infty}\left\{\sup_{x\in Q}\|(I-P_r)(x)\|_X\right\},$$

where I is the identity operator on X.

Let X and Y be Banach spaces and χ_1 and χ_2 be Hausdorff measures on X and Y, the linear operator $L:X\to Y$ is said to be (χ_1,χ_2)- bounded if $L(Q)$ is a bounded subset of Y and there exists a positive constant M such $\chi_2(L(Q))\le M\chi_1(L(Q))$ for every bounded subset Q of X. If an operator L is (χ_1,χ_2)-bounded, then the number

$$\|L\|_{(\chi_1,\chi_2)}=\inf\{M>0:\chi_2(L(Q))\le M\chi_1(L(Q))\,\text{for all bounded set } Q\subset X\}$$

is called the (χ_1,χ_2)-measure noncompactness of L. In particular, if $\chi_1=\chi_2=\chi$ then $\|L\|_{(\chi,\chi)}=\|L\|_\chi$.

Lemma 2.3. *[13] Let X and Y be Banach spaces, S_X be the unit sphere in the space X and $L\in\mathcal{B}(X,Y)$. Then, the Hausdorff measure of noncompactness of L, denoted by $\|L\|_\chi$, is defined by*

$$\|L\|_\chi=\chi(L(S_X)),$$

and

$$L \text{ is compact iff } \|L\|_\chi=0.$$

Lemma 2.4. *[13] Let X be a normed sequence space, $T=(t_{nv})$ be a triangle matrix, χ_T and χ denote the Hausdorff measures of noncompactness on M_{X_T} and M_X, the collections of all bounded sets in X_T and X, respectively. Then, $\chi_T(Q)=\chi(T(Q))$ for all $Q\in M_{X_T}$.*

Also, the Hausdorff measure of noncompactness χ has the following basic properties: Let Q,Q' be bounded subsets of the metric linear space X, $x\in X$ and $a\in\mathbb{C}$, then

$$\chi(Q)=0 \Leftrightarrow Q \text{ is totally bounded},$$

$$Q\subset Q' \Rightarrow \chi(Q)\le\chi(Q'),$$
$$\chi(Q+Q')\le\chi(Q)+\chi(Q'),$$
$$\chi(aQ)=|a|\,\chi(Q),$$
$$\chi(x+Q)=\chi(Q).$$

3. Compact operators on the space E_q' and applications

In this section, we determine the norms and the Hausdorff measures of noncompactness of some matrix operators related to the space E_q', and also characterize compact operators using Hausdorff measure of noncompactness.

We first note that if $x\in E_q'$ and $a=(a_k)\in\left\{E_q'\right\}^\beta$, then $\tilde{a}=(\tilde{a}_k)=(k^{-q}\sum\limits_{v=k+1}^{\infty}a_v)\in\ell$ and

$$\sum_{k=1}^{\infty}a_kx_k=a_1x_1-\sum_{k=1}^{\infty}\tilde{a}_ky_k, \tag{3.1}$$

where x and y are connected with $x_n=-\sum\limits_{k=1}^{n-1}k^{-q}y_k$ for all $n>1$, [30].

Lemma 3.1. *[21] Let $1<p<\infty$ and p^* denotes the conjugate of p. Then, we have $\ell_p^\beta=\ell_{p^*}$ and $\ell_\infty^\beta=c^\beta=c_0^\beta=\ell_1$, $\ell_1^\beta=\ell_\infty$. Also, let X denotes any of the spaces ℓ_∞,c,c_0,ℓ_1 and ℓ_p. Then, we have*

$$\|a\|_X^*=\|a\|_{X^\beta}$$

for all $a\in X^\beta$, where $\|.\|_{X^\beta}$ is the natural norm on the dual space X^β.

Lemma 3.2. *[20] Let X and Y be BK-spaces. Then, we have*

(a) $(X,Y)\subset\mathcal{B}(X,Y)$, that is, every matrix $A\in(X,Y)$ defines an operator $L_A\in\mathcal{B}(X,Y)$ by $L_A(x)=A(x)$ for all $x\in X$.

(b) If X has AK, then $\mathcal{B}(X,Y)\subset(X,Y)$, that is, for every operator $L\in\mathcal{B}(X,Y)$ there exists a matrix $A\in(X,Y)$ such that by $L(x)=A(x)$ for all $x\in X$.

Lemma 3.3. *[3] Let $X\supset\phi$ be a BK-space and Y be any of the spaces ℓ_∞,c,c_0. If $A\in(X,Y)$, then*

$$\|L_A\|=\|A\|_{(X,\ell_\infty)}=\sup_n\|A_n\|_X^*<\infty.$$

Theorem 3.4. *Let Y be any of the spaces ℓ_∞,c,c_0. If $A\in\left(E_q',Y\right)$, then A defines a bounded linear operator L_A such that $L_A(x)=A(x)$ and*

$$\|L_A\|=\|A\|_{(E_q',\ell_\infty)}\le\sup_n\left\{\sum_{k=1}^{\infty}|\tilde{a}_{nk}|+|a_{n1}|\right\}.$$

Proof: Since E_q' and Y are BK-spaces, by Theorem 4.2.8 of Wilansky [32], it is clear that L_A is a bounded linear operator. To calculate the norm, let $S_{E_q'}$ be a unit sphere and $x \in E_q'$. Note that $\|L_A\| = \|A\|_{(E_q',\ell_\infty)} = \sup_n \|A_n\|^*_{E_q'}$. Further,

$$\begin{aligned}\|A_n\|^*_{E_q'} &= \sup_{x\in S_{E_q'}} \left|\sum_{k=1}^{\infty} a_{nk}x_k\right| \\ &\le \sup_{x\in S_{E_q'}} \left|\sum_{k=2}^{\infty} a_{nk}x_k\right| + |a_{n1}| \\ &= \left\|\tilde{A}_n\right\|^*_E + |a_{n1}|.\end{aligned}$$

Since $E \in \{\ell_\infty, c, c_0\}$, it is clear that $E^\beta = \ell_1$ and so

$$\left\|\tilde{A}_n\right\|^*_E = \left\|\tilde{A}_n\right\|_{E^\beta} = \left\|\tilde{A}_n\right\|_\ell = \sum_{k=1}^{\infty} |\tilde{a}_{nk}|.$$

This gives that $\|L_A\| = \sup_n \|A_n\|^*_{E_q'} \le \sup_n \left\{\sum_{k=1}^{\infty} |\tilde{a}_{nk}| + |a_{n1}|\right\}$. □

Theorem 3.5. *Let $p \ge 1$. If $A \in \left(E_q', \ell_p\right)$, then L_A is a bounded linear operator and*

$$\|L_A\| = \|A\|_{(E_q',\ell_p)} \le \sum_{n=1}^{\infty}\left|\sum_{k=1}^{\infty} \tilde{a}_{nk}\right|^p + \left(\sum_{n=1}^{\infty} |a_{n1}|^p\right)^{1/p}.$$

Proof: The first part is as in Theorem 3.4. Using the operator norm, by Minkowski inequality and the equation (3.1), we get

$$\begin{aligned}\|A\|_{(E_q',\ell_p)} &\le \|\tilde{A}\|_{(E,\ell_p)} + \left(\sum_{n=1}^{\infty} |a_{n1}|^p\right)^{1/p} \\ &= \sum_{n=1}^{\infty}\left|\sum_{k=1}^{\infty} \tilde{a}_{nk}\right|^p + \left(\sum_{n=1}^{\infty} |a_{n1}|^p\right)^{1/p}.\end{aligned}$$

□

If we take zero of the first term of a sequence $x \in E_q'$, Theorem 3.4 and Theorem 3.5 are reduced to the following results.

Corollary 3.6. *Let Y be any of the spaces ℓ_∞, c, c_0. If $A \in \left(SE_q', Y\right)$, then L_A is a bounded linear operator and*

$$\|L_A\| = \|A\|_{(SE_q',\ell_\infty)} = \sup_n \sum_{k=1}^{\infty} |\tilde{a}_{nk}|.$$

Corollary 3.7. *Let $p \ge 1$. If $A \in \left(SE_q', \ell_p\right)$, then L_A is a bounded linear operator and*

$$\|L_A\| = \|A\|_{(SE_q',\ell_p)} = \sum_{n=1}^{\infty}\left|\sum_{k=1}^{\infty} \tilde{a}_{nk}\right|^p.$$

Theorem 3.8. *Let $q < 1$.*

a-) If $A \in \left(E_q', c_0\right)$, *then*

$$\left\|L_A\right\|_\chi \leq \limsup_{n\to\infty} \left(\sum_{k=1}^{\infty} \left|\tilde{a}_{nk}\right| + \left|a_{n1}\right|\right).$$

If $\limsup_{n\to\infty} \left(\sum_{k=1}^{\infty} \left|\tilde{a}_{nk}\right| + \left|a_{n1}\right|\right) = 0$, *then* L_A *is compact.*

b-) If $A \in \left(E_q', \ell_\infty\right)$, *then*

$$0 \leq \left\|L_A\right\|_\chi \leq \limsup_{n\to\infty} \left(\sum_{k=1}^{\infty} \left|\tilde{a}_{nk}\right| + \left|a_{n1}\right|\right).$$

If $\limsup_{n\to\infty} \left(\sum_{k=1}^{\infty} \left|\tilde{a}_{nk}\right| + \left|a_{n1}\right|\right) = 0$, *then* L_A *is compact.*

Proof: *a*) Let $S_{E_q'}$ be the unit sphere in the space E_q'. From Lemma 2.3, we write

$$\left\|L_A\right\|_\chi = \chi(AS_{E_q'}). \tag{3.2}$$

Since $A \in \left(E_q', c_0\right)$, it is clear that $AS_{E_q'} \in M_{c_0}$. So, it follows from Lemma 2.2 that

$$\chi(AS_{E_q'}) = \lim_{r\to\infty} \left\{ \sup_{x\in S_{E_q'}} \left\|(I - P_r)\left(A(x)\right)\right\|_\infty \right\},$$

where the operator $P_r : c_0 \to c_0$ is defined by $P_r(x) = (x_0, x_1, ...x_r, ...)$ for all $x \in c_0$ and $r \in \mathbb{N}$. It is obvious that, for all $x \in E_q'$,

$$\left\|(I - P_r)(A(x))\right\|_\infty = \sup_{n>r} \left|A_n(x)\right|,$$

which implies

$$\sup_{x\in S_{E_q'}} \left\|(I - P_r)(A(x))\right\|_\infty = \sup_{n>r} \left\|A_n\right\|_{E_q'}^* \leq \sup_{n>r} \left(\sum_{k=1}^{\infty} \left|\tilde{a}_{nk}\right| + \left|a_{n1}\right|\right).$$

The last inequality and (3.2) yield

$$\left\|L_A\right\|_\chi \leq \lim_{r\to\infty} \sup_{n>r} \left(\sum_{k=1}^{\infty} \left|\tilde{a}_{nk}\right| + \left|a_{n1}\right|\right).$$

Finally, the compactness of A is immediately obtained by Lemma 2.3, which completes the proof of (a).

Now, consider the mapping P_r on ℓ_∞ and let $Q \in M_{\ell_\infty}$. Then, the elementary properties of the function χ,

$$AQ \subset P_r(AQ) + (I - P_r)(AQ)$$

implies that

$$0 \leq \chi(AQ) \leq \chi((I - P_r)(AQ)). \tag{3.3}$$

After pointing out the inequality (3.3), to avoid repetition we leave the proof of the other part of the theorem to the reader. □

Lemma 3.9. *[26] Let $X \supset \phi$ be a BK-space with AK or $X = \ell_\infty$. If $A \in (X, c)$, then we have*

$$\lim_{n\to\infty} a_{nk} = \alpha_k \quad \text{exists for all } k,$$

$$\alpha = (\alpha_k) \in X^\beta,$$

$$\sup_n \left\|A_n - \alpha\right\|_X^* < \infty,$$

$$\lim_{n\to\infty} A_n(x) = \sum_{k=0}^{\infty} \alpha_k x_k \text{ for every } x = (x_k) \in X.$$

Theorem 3.10. *Let $q \leq 1$.*

a-) If $A \in \left(SE'_q, \ell_\infty\right)$, then

$$0 \leq \|L_A\|_\chi \leq \limsup_{n\to\infty}\left(\sum_k^\infty |\tilde{a}_{nk}|\right).$$

If $\limsup_{n\to\infty}\left(\sum_k^\infty |\tilde{a}_{nk}|\right) = 0$, then L_A is compact.

b-) If $A \in \left(SE'_q, c_0\right)$, then

$$\|L_A\|_\chi = \limsup_{n\to\infty}\left(\sum_k^\infty |\tilde{a}_{nk}|\right).$$

L_A is compact if and only if $\limsup_{n\to\infty}\left(\sum_k^\infty |\tilde{a}_{nk}|\right) = 0$.

c-) If $A \in \left(SE'_q, \ell_p\right)$, $p \geq 1$, then

$$\|L_A\|_\chi = \lim_{r\to\infty}\sum_n\left|\sum_k a_{nk}^{(r)}\right|^p,$$

where the matrix $A^{(r)} = \left(a_{nk}^{(r)}\right)$ is defined by

$$a_{nk}^{(r)} = \begin{cases} \tilde{a}_{nk}, & n > r \\ 0, & n \leq r. \end{cases}$$

L_A is compact if and only if $\lim_{r\to\infty}\sum_n\left|\sum_k a_{nk}^{(r)}\right|^p = 0$.

d-) If $A \in \left(SE'_q, c\right)$, then

$$\frac{1}{2}\limsup_{n\to\infty}\left(\sum_k^\infty |\tilde{a}_{nk} - \tilde{\alpha}_k|\right) \leq \|L_A\|_\chi \leq \limsup_{n\to\infty}\left(\sum_k^\infty |\tilde{a}_{nk} - \tilde{\alpha}_k|\right),$$

where $\tilde{\alpha} = (\tilde{\alpha}_k)$ with $\tilde{\alpha}_k = \lim_{n\to\infty}\tilde{a}_{nk}$ for all $k \in \mathbb{N}$.

L_A is compact if and only if $\limsup_{n\to\infty}\left(\sum_k^\infty |\tilde{a}_{nk} - \tilde{\alpha}_k|\right) = 0$.

Proof: The proofs of (a) and (b) are simple, and so they are omitted. We begin with the proof of the part (c). Assume that $S_{SE'_q}$ is the unit sphere in the space SE'_q. To determine Hausdorff measure of noncompactness of L_A, consider $SE'_q \cong E$. Using Lemma 2.3, Lemma 2.2 and Lemma 1.1, we get immediately that

$$\begin{aligned} \|L_A\|_\chi = \chi(AS_{SE'_q}) &= \chi(\tilde{A}\Delta_q S_{SE'_q}) \\ &= \lim_{r\to\infty}\sup_{y\in\Delta_q S_{SE'_q}}\left\|(I - P_r)\left(\tilde{A}(y)\right)\right\|_{\ell_p} \\ &= \lim_{r\to\infty}\left\|A^{(r)}\right\|_{(E,\ell_p)} \\ &= \lim_{r\to\infty}\sum_{n=1}^\infty\left|\sum_{k=1}^\infty a_{nk}^{(r)}\right|^p. \end{aligned}$$

d-) To estimate Hausdorff measure of noncompactness of L_A, take $AS_{SE'_q} \in M_c$. Then, it is written from Lemma 2.3 that

$$\|L_A\|_\chi = \chi(AS_{SE'_q}).$$

On the other hand, since $SE'_q \cong E$, it follows that $A \in (SE'_q, c)$ iff $\tilde{A} \in (E, c)$, and also

$$\|L_A\|_\chi = \chi(AS_{SE'_q}) = \chi(\tilde{A}\Delta_q S_{SE'_q}).$$

Since every $z = (z_n) \in c$ has a unique representation $z = \bar{z}e + \sum_{n=0}^{\infty}(z_n - \bar{z})e^{(n)}$, we have that, for all $r \in \mathbb{N}$ and $z \in c$,

$$(I - P_r)(z) = \sum_{n=r+1}^{\infty}(z_n - \bar{z})e^{(n)}$$

where P_r is the projector on c and $\bar{z} = \lim_{n\to\infty} z_n$. Thus, for all $z \in c$,

$$\|(I - P_r)(z)\|_\infty = \sup_{n>r}|z_n - \bar{z}|.$$

which gives $\|(I - P_r)(z)\|_\infty \leq 2\|z\|_\infty$, *i.e.*, $\|(I - P_r)\|_\infty \leq 2$. Moreover, for the sequence $z^{(r)} = (z_n^{(r)}) \in c$ given by

$$z_n^{(r)} = \begin{cases} -1, & n = r+1 \\ 1, & n \neq r+1, \end{cases}$$

$\|(I - P_r)\|_\infty \geq \left\|(I - P_r)(z^{(r)})\right\|_\infty = 2$ and so we get $\|I - P_r\| = 2$.

Further, since $\tilde{A} \in (E, c)$, it follows from Lemma 3.9 that $\tilde{\alpha}_k = \lim_{n\to\infty}\tilde{a}_{nk}$ exists for all $k \in \mathbb{N}$, and for every $y \in E$

$$\lim_{n\to\infty}\tilde{A}_n(y) = \sum_{k=0}^{\infty}\bar{\alpha}_k y_k.$$

So,

$$\begin{aligned}
\sup_{x\in S_{SE'_q}}\|(I - P_r)(A(x))\|_\infty &= \sup_{y\in S_c}\left\|(I - P_r)(\tilde{A}(y))\right\|_\infty \\
&= \sup_{y\in S_c}\sup_{n>r}\left|\tilde{A}_n(y) - \sum_{k=0}^{\infty}\bar{\alpha}_k y_k\right| \\
&= \sup_{y\in S_c}\sup_{n>r}\left|\sum_{k=0}^{\infty}(\tilde{a}_{nk} - \bar{\alpha}_k)\, y_k\right| \\
&= \sup_{n>r}\left\|\tilde{A}_n - \tilde{\alpha}\right\|_c^* \\
&= \sup_{n>r}\left\|\tilde{A}_n - \tilde{\alpha}\right\|_\ell.
\end{aligned}$$

This completes the proof of (d) with together Lemma 2.1. □

If we take the matrices $L_1 = (l_{nk}^1)$ and $L_2 = (l_{nk}^2)$ as

$$l_{nk}^1 = \begin{cases} 1, & 0 \leq k \leq n \\ 0, & k > n \end{cases}$$

and

$$l_{nk}^2 = \begin{cases} 1, & n = k \\ -1, & n = k+1 \\ 0, & otherwise, \end{cases}$$

we have $bs = \{\ell_\infty\}_{L_1}$, $cs = \{c\}_{L_1}$ and $bv_p = \{l_p\}_{L_2}$. Hence, the following result is deduced from Theorem 3.4, Theorem 3.5 and Lemma 1.2.

Corollary 3.11. *For an infinite matrix A,*

a) if $A \in (E'_q, bs)$ *or* $A \in (E'_q, cs)$, *then*

$$\|L_A\| \leq \sup_n\left\{\sum_{k=1}^{\infty}|\tilde{a}_{nk}| + |a_{n1}|\right\},$$

b) if $A \in \left(E'_q, bv_p\right)$, $1 \le p < \infty$, *then*

$$\|L_A\| \le \sum_{n=1}^{\infty}\left|\sum_{k=1}^{\infty}\tilde{a}_{nk}\right|^p + \left(\sum_{n=1}^{\infty}|a_{n1}|^p\right)^{1/p}.$$

Also, the choice of the matrix $C = T.A$ leads us the following result on the Hausdorff measures of noncompactness, where

$$c_{nk} = \sum_{v=0}^{n} t_{nv}a_{vk}, n, k \in \mathbb{N}.$$

Corollary 3.12. *Let A be an infinite matrix and T be a triangle. If* $A \in \left(E'_q, (\ell_\infty)_T\right)$ *or* $A \in \left(E'_q, (c_0)_T\right)$, *then*

$$\|L_A\|_\chi \le \limsup_{n\to\infty}\left(\sum_{k=1}^{\infty}|\tilde{c}_{nk}| + |a_{n1}|\right).$$

Also, A is a compact if $\limsup_{n\to\infty}\left(\sum_{k=1}^{\infty}|\tilde{c}_{nk}| + |a_{n1}|\right) = 0.$

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