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A Generalized Common Fixed Point of Multi-Valued Maps in b-metric Space

Noreddine Makran, Abdelhak El Haddouchi, Brahim Marzouki

ABSTRACT: In this work we are interested to prove a general fixed point theorem for a pair of multi-valued mappings in $b-$metric spaces. The results in this paper generalize the results obtained in [19] and to obtain other particular results.

Key Words: Metric space, $b-$metric space, fixed point, $*$-continuous, multi-valued maps.

Contents

1. Introduction

Since the famous Banach fixed point theorem (1922), the study of fixed point theory in metric spaces has several applications in mathematics, especially in solving differential and functional equations. Many authors have introduced a new class of generalized metric space, in particular those called b-metric spaces, and obtained several results in fixed point theory,(see [1,2,3], [5]- [21]).

The one due to I. A. Bakhtin [4] and S. Czerwik [8], [9] who, motivated by the problem of the convergence of measurable functions with respect to measure, introduced b-metric spaces (a generalization of metric spaces) and proved the contraction principle in this framework.

Let (X,d) be a $b-$metric space. A subset $A \subset X$ is said to be closed if for every sequence $x_n \in A$ such that $\lim\limits_{n\to\infty} d(x,x_n)=0$ $(x_n \longrightarrow x)$ we have $x \in A$.

A subset $A \subset X$ is said to be bounded if $\sup\limits_{x,y\in A} d(x,y) < +\infty$.

We denote by $B(X)$ the set of nonempty closed bounded subsets of X provided with the Hausdorff-Pompeiu metric H defined by

$$H(A,B)=\max\left(\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right),$$

we define also $\delta(A,B)$ by

$$\delta(A,B)=\sup\{d(a,b), \quad a\in A \quad b\in B\},$$

it follows immediately from the definition of δ that

$$\delta(A,B)=0 \iff A=B=\{.\} \text{ and } \delta(\{.\},B)=H(\{.\},B) \quad \text{and}$$

$$d(a,b)\le \delta(A,B) \quad \forall a\in A \quad \forall b\in B.$$

given $F,G : X \longrightarrow B(X)$, for $c,d \in [0,1]$ and $x,y \in X$, we shall use the following notation:

$N_{c,d}(x,y) = max\{d(x,y), cd(x,Fx), cd(y,Gy), \frac{d}{2}(d(x,Gy)+d(y,Fx))\}$

for a sequence (x_n), of elements from X, sometimes, for the sake of brevity, we shall use the notation: $d_n = d(x_n, x_{n+1})$, where $n \in \mathbb{N}$.

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2. Preliminary

Definition 2.1 ([9]). *Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \longrightarrow \mathbb{R}^+$ is said to be a b-metric on X if the following conditions hold:*

(i) $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$,

(iii) $d(x,y) \leq s[d(x,z) + d(z,y)]$ for all $x, y, z \in X$.

Note that every metric space is a $b-$metric space with $s = 1$. But the converse need not be true as is shown in the following example.

Example 2.2 ([7]). 1)*Let $X = l_p(\mathbb{R})$ with $0 < p < 1$ and $l_p(\mathbb{R}) = \{\{x_n\} \subset \mathbb{R} : \sum\limits_{n\geq 1} \mid x_n \mid^p < \infty\}$.*
We define $d : X \times X \longrightarrow \mathbb{R}^+$ by :

$$d(x,y) = (\sum_{n\geq 1} \mid x_n - y_n \mid^p)^{\frac{1}{p}}$$

where $x = \{x_n\}, y = \{y_n\}$, then (X,d) is a b-metric space of constant $s = 2^{\frac{1}{p}-1}$.
2) *let $X = L_p[0,1]$ is a space of real functions $x(t)$, $t \in [0,1]$ such that: $\int_0^1 \mid x(t) \mid^p dt < \infty$ with $0 < p < 1$. We define $d : X \times X \longrightarrow \mathbb{R}^+$ by :*

$$d(x,y) = (\int_0^1 \mid x(t) - y(t) \mid^p dt)^{\frac{1}{p}}.$$

Then (X,d) is a b-metric space of constant $s = 2^{\frac{1}{p}-1}$.

Definition 2.3 ([6]). *Let (X,d) be a $b-$metric space, $x \in X$ and (x_n) be a sequence in X. Then*

(i) (x_n) converges to x if and only if $\lim\limits_{n\to\infty} d(x,x_n) = 0$. We denote this by $x_n \to x\,(n \to \infty)$ or $\lim\limits_{n\to\infty} x_n = x$.

(ii) (x_n) is Cauchy if and only if $\lim\limits_{n,m\to\infty} d(x_n,x_m) = 0$.

(iii) (X,d) is complete if and only if every Cauchy sequence in X is convergent.

Lemma 2.4 ([19]). *Every sequence (x_n) of elements from a b-metric space (X,d) having the property that there exists $\gamma \in [0,1)$ such that*

$$d(x_{n+1},x_n) \leq \gamma d(x_n,x_{n-1}),$$

for every $n \in \mathbb{N}$, is Cauchy.

3. Main results

Theorem 3.1. *Let (X,d) be a b-metric space of constant s and $F, G : X \longrightarrow B(X)$ having the property that there exist $c, d \in [0,1]$ and $k \in [0,1)$ such that:*

(i) $ksd < 1$,

(ii) $H(Fx,Gy) \leq kN_{c,d}(x,y)$ for all $x, y \in X$.

Then, for every $x_0 \in X$, there exist $\gamma \in [0,1)$ and a sequence (x_n) of elements from X such that:

(a) $x_{2n+1} \in Fx_{2n}$ and $x_{2n} \in Gx_{2n-1}$ for every $n \in \mathbb{N}$,

(b) $d(x_{n+1},x_n) \leq \gamma d(x_n,x_{n-1})$ for every $n \in \mathbb{N}^*$,

(c) (x_n) is Cauchy.

Proof.

Let us consider $\beta = \frac{1}{2}(min(1, \frac{1}{ds}) + k)$ and $\gamma = max\{\beta, \frac{ds\beta}{2-ds\beta}\} < 1$, $x_0 \in X$ and $x_1 \in Fx_0$, then, using (ii), we have

$$d(x_1, Gx_1) \leq H(Fx_0, Gx_1) \leq kN_{c,d}(x_0, x_1).$$

According to the characterization of inf we have for $\varepsilon = \frac{1}{2}(min(1, \frac{1}{ds}) - k)N_{c,d}(x_0, x_1)$, there exists $x_2 \in Gx_1$ such that

$$\begin{aligned} d(x_1, x_2) &\leq kN_{c,d}(x_0, x_1) + \frac{1}{2}(min(1, \frac{1}{ds}) - k)N_{c,d}(x_0, x_1) \\ &= \frac{1}{2}(min(1, \frac{1}{ds}) + k)N_{c,d}(x_0, x_1) \\ &= \beta N_{c,d}(x_0, x_1) \end{aligned}$$

Since

$$d(x_2, Fx_2) \leq H(Fx_2, Gx_1) \leq kN_{c,d}(x_2, x_1).$$

According to the characterization of inf we have for $\varepsilon = \frac{1}{2}(min(1, \frac{1}{ds}) - k)N_{c,d}(x_2, x_1)$, there exists $x_3 \in Fx_2$ such that

$$\begin{aligned} d(x_2, x_3) &\leq kN_{c,d}(x_2, x_1) + \frac{1}{2}(min(1, \frac{1}{ds}) - k)N_{c,d}(x_2, x_1) \\ &= \frac{1}{2}(min(1, \frac{1}{ds}) + k)N_{c,d}(x_2, x_1) \\ &= \beta N_{c,d}(x_2, x_1) \end{aligned}$$

In the same there exists $x_4 \in Gx_3$ such that

$$d(x_3, x_4) \leq \beta N_{c,d}(x_2, x_3).$$

By recurrence, we construct a sequence (x_n) such that $x_{2n+1} \in Fx_{2n}$, and $x_{2n} \in Gx_{2n-1}$ which satisfies :

$$d(x_{2n}, x_{2n+1}) \leq \beta N_{c,d}(x_{2n}, x_{2n-1}) \qquad \text{and} \quad d(x_{2n-1}, x_{2n}) \leq \beta N_{c,d}(x_{2n-2}, x_{2n-1}) \;\; n = 1, 2, 3, ... \tag{3.1}$$

According to (3.1) we have:

$$\begin{aligned} d_{2n} &\leq \beta N_{c,d}(x_{2n}, x_{2n-1}) \\ &= \beta max\{d(x_{2n}, x_{2n-1}), cd(x_{2n}, Fx_{2n}), cd(x_{2n-1}, Gx_{2n-1}), \frac{d}{2}(d(x_{2n}, Gx_{2n-1}) + d(x_{2n-1}, Fx_{2n})\} \\ &\leq \beta max\{d_{2n-1}, cd_{2n}, cd_{2n-1}, \frac{d}{2}d(x_{2n-1}, x_{2n+1})\} \\ &\leq \beta max\{d_{2n-1}, cd_{2n}, cd_{2n-1}, \frac{ds}{2}(d_{2n-1} + d_{2n})\} \\ &\leq \beta max\{d_{2n-1}, \frac{ds}{2}(d_{2n-1} + d_{2n})\}, \end{aligned}$$

for every $n \in \mathbb{N}^*$, where the justification of the last inequality is as follow :
if $max\{d_{2n-1}, cd_{2n}, cd_{2n-1}, \frac{ds}{2}(d_{2n-1} + d_{2n})\} = cd_{2n}$, then we get that $d_{2n} \leq \beta cd_{2n} \leq \beta d_{2n} < d_{2n}$,which is a contradiction.

Consequently, $d_{2n} \leq \beta d_{2n-1}$ or $d_{2n} \leq \beta \frac{ds}{2}(d_{2n-1} + d_{2n})$, i.e $d_{2n} \leq \beta d_{2n-1}$ or $d_{2n} \leq \frac{ds\beta}{2-ds\beta} d_{2n-1}$ for every $n \in \mathbb{N}^*$, thus $d_{2n} \leq max\{\beta, \frac{ds\beta}{2-ds\beta}\}d_{2n-1}$, i.e

$$d(x_{2n+1}, x_{2n}) \leq \gamma d(x_{2n}, x_{2n-1}) \quad \forall n \in \mathbb{N}^*. \tag{3.2}$$

According to (3.1) in the same way we have:

$$\begin{aligned} d_{2n-1} &\leq \beta N_{c,d}(x_{2n-2}, x_{2n-1}) \\ &= \beta max\{d(x_{2n-2}, x_{2n-1}), cd(x_{2n-2}, Fx_{2n-2}), cd(x_{2n-1}, Gx_{2n-1}), \frac{d}{2}(d(x_{2n-2}, Gx_{2n-1}) \\ &\quad + d(x_{2n-1}, Fx_{2n-2})\} \\ &\leq \beta max\{d_{2n-2}, cd_{2n-2}, cd_{2n-1}, \frac{d}{2} d(x_{2n-2}, x_{2n})\} \\ &\leq \beta max\{d_{2n-2}, cd_{2n-2}, cd_{2n-1}, \frac{ds}{2}(d_{2n-2} + d_{2n-1})\} \\ &\leq \beta max\{d_{2n-2}, \frac{ds}{2}(d_{2n-2} + d_{2n-1})\}, \end{aligned}$$

for every $n \in \mathbb{N}^*$, where the justification of the last inequality is as follow :
if, $max\{d_{2n-2}, cd_{2n-2}, cd_{2n-1}, \frac{ds}{2}(d_{2n-2} + d_{2n-1})\} = cd_{2n-1}$, then we get that $d_{2n-1} \leq \beta cd_{2n-1} \leq \beta d_{2n-1} < d_{2n-1}$,which is a contradiction.

Consequently, $d_{2n-1} \leq \beta d_{2n-2}$ or $d_{2n-1} \leq \beta \frac{ds}{2}(d_{2n-2} + d_{2n-1})$,
i.e $d_{2n-1} \leq \beta d_{2n-2}$ or $d_{2n-1} \leq \frac{ds\beta}{2-ds\beta} d_{2n-2}$ for every $n \in \mathbb{N}^*$, thus $d_{2n-1} \leq max\{\beta, \frac{ds\beta}{2-ds\beta}\} d_{2n-2}$, i.e

$$d(x_{2n}, x_{2n-1}) \leq \gamma d(x_{2n-2}, x_{2n-1}) \quad \forall n \in \mathbb{N}^*. \tag{3.3}$$

According to (3.2) and (3.3) we have for every $n \in \mathbb{N}^*$ $\quad d(x_{n+1}, x_n) \leq \gamma d(x_n, x_{n-1})$.

Hence the sequence (x_n) satisfies (a) and (b). From Lemma 2.4 we deduce that (c) this also satisfied.

Definition 3.2. *A function $F : X \longrightarrow B(X)$, where (X, d) is a b-metric space, is called closed if for all sequences (x_n) and (y_n) of elements from X and $x, y \in X$ such that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ and $y_n \in F(x_n)$ for every $n \in \mathbb{N}$, we have $y \in F(x)$.*

Theorem 3.3. *Let (X, d) be a complete $b-$metric space with constant $s \geq 1$ and $F, G : X \longrightarrow B(X)$, satisfying the following conditions:*

(i) *F and G are closed,*

(ii) *there exist $c, d \in [0, 1]$ and $k \in [0, 1)$ such that*
$H(Fx, Gy) \leq kN_{c,d}(x, y)$ for all $x, y \in X$,

(iii) *$ksd < 1$.*

Then F and G have a common fixed point $x \in X$.
Moreover, if x is absolutely fixed for F or G (which means that $F(x) = \{x\}$ or $G(x) = \{x\}$), then the fixed point is unique.

proof.

Existence.

Based on (ii) and (iii), according to Theorem 3.1, there exists a Cauchy sequence (x_n) of elements of X such that:

$$x_{2n+1} \in Fx_{2n} \ \ and \ \ x_{2n} \in Gx_{2n-1} \qquad for\ every\ n \in \mathbb{N}. \tag{3.4}$$

As the b-metric space (X, d) is complete, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. We combine (i) with (3.4) to see that $x \in Fx$ and $x \in Gx$, i.e F and G have a common fixed point $x \in X$.

Unicity.
Suppose that $F(x) = \{x\}$ and $y \in X$ is another common fixed point of F and G, then by (ii) we have

$$\begin{aligned} d(x,y) \leq H(Fx,Gy) &\leq kN_{c,d}(x,y) \\ &= k\ max\{d(x,y), cd(x,Fx), cd(y,Gy), \frac{d}{2}(d(x,Gy)+d(y,Fx))\} \\ &= k\ max\{d(x,y), \frac{d}{2}(d(x,Gy)+d(y,Fx))\} \\ &\leq k\ max\{d(x,y), \frac{d}{2}(s(d(x,y)+d(y,Gy))+s(d(y,x)+d(x,Fx)))\} \\ &\leq k\ max\{d(x,y), ds\ d(x,y)\} < d(x,y), \quad because\ (kds<1). \end{aligned}$$

which is a contradiction. Hence $d(x,y) = 0$ then $x = y$.

So x is the unique common fixed point of F and G.

Remark 3.4. *In Theorem 3.3, if we replace $H(Fx,Gy)$ by $\delta(Fx,Gy)$, then F and G have a unique common fixed point because $H(Fx,Gy) \leq \delta(Fx,Gy)$, then by Theorem 3.3 F and G have a common fixed point. On the other hand the unicity becomes from the fact that*

$$d(a,b) \leq \delta(Fx,Gy) \quad \forall a \in Fx \quad \forall b \in Gy.$$

Definition 3.5. *Given a b-metric space (X,d), the b-metric d is called $*$-continnuous if for every $A \in B(X)$, every $x \in X$ and every sequence (x_n) of elements from X such that $\lim_{n\to\infty} x_n = x$, we have $\lim_{n\to\infty} d(x_n, A) = d(x,A)$.*

Theorem 3.6. *Let (X,d) be a complete $b-$metric space with constant $s \geq 1$ and $F,G : X \longrightarrow B(X)$, satisfying the following conditions:*

(i) d is $*$-continuous,

(ii) there exist $c,d \in [0,1]$ and $k \in [0,1)$ such that
$H(Fx,Gy) \leq kN_{c,d}(x,y)$ for all $x,y \in X$,

(iii) $ksd < 1$.

Then F and G have a common fixed point $x \in X$.
Moreover, if x is absolutely fixed for F or G (which means that $F(x) = \{x\}$ or $G(x) = \{x\}$), then the fixed point is unique.

proof.
Existence.
Based on (ii) and (iii), according to Theorem 3.1, there exists a Cauchy sequence (x_n) of elements of X such that:

$$x_{2n+1} \in Fx_{2n} \ \ and \ \ x_{2n} \in Gx_{2n-1} \qquad for\ every\ n \in \mathbb{N}. \tag{3.5}$$

As the b-metric space (X,d) is complete, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

Then, using (ii) and (3.5), with the notation $d(x_n,x) = \delta_n$, we have

$$\begin{aligned} d(x_{2n+1},Gx) &\leq H(Fx_{2n},Gx) \leq kN_{c,d}(x_{2n},x) \\ &= k\ max\{\delta_{2n}, cd(x_{2n},Fx_{2n}), cd(x,Gx), \frac{d}{2}(d(x_{2n},Gx)+d(x,Fx_{2n}))\} \\ &\leq k\ max\{\delta_{2n}, cd_{2n}, cd(x,Gx), \frac{d}{2}(s(\delta_{2n}+d(x,Gx)+\delta_{2n+1}+d(x_{2n+1},Fx_{2n})))\}, \end{aligned}$$

$$= k\ max\{\delta_{2n}, cd_{2n}, cd(x,Gx), \frac{d}{2}(s(\delta_{2n}+d(x,Gx)+\delta_{2n+1}))\}, \tag{3.6}$$

for every $n \in \mathbb{N}$.

Since $\lim_{n\to\infty} \delta_{2n+1} = \lim_{n\to\infty} \delta_{2n} = \lim_{n\to\infty} d_{2n} = 0$ and $\lim_{n\to\infty} d(x_{2n+1}, Gx) = d(x, Gx)$ (as d is $*$-continuous and $d_{2n} \leq s(\delta_{2n} + \delta_{2n+1})$ and $\lim_{n\to\infty} x_n = x$), letting $n \longrightarrow \infty$ in (3.6), we get $d(x, Gx) = 0$, because if $d(x, Gx) > 0$, then

$$\begin{aligned} d(x,Gx) &\leq k\ max\{cd(x,Gx), \frac{d}{2}sd(x,Gx)\} \\ &\leq max\{kc, \frac{kds}{2}\}d(x,Gx) \\ &< d(x,Gx) \quad because \quad max\{kc, \frac{kds}{2}\} < 1, \end{aligned}$$

which is a contradiction, hence $x \in Gx$ and G has a fixed point.
In the same way we have:

$$d(x_{2n}, Fx) \leq H(Fx, Gx_{2n-1}) \leq kN_{c,d}(x, x_{2n-1})$$

$$\leq k\ max\{\delta_{2n-1}, cd(x,Fx), cd_{2n-1}, \frac{d}{2}(s(\delta_{2n} + \delta_{2n-1} + d(x,Fx)))\}, \tag{3.7}$$

when $n \longrightarrow \infty$ in (3.7), we get $d(x, Fx) = 0$, because if $d(x, Fx) > 0$, then

$$\begin{aligned} d(x,Fx) &\leq k\ max\{cd(x,Fx), \frac{d}{2}sd(x,Fx)\} \\ &\leq max\{kc, \frac{kds}{2}\}d(x,Fx) \\ &< d(x,Fx) \quad because \quad max\{kc, \frac{kds}{2}\} < 1. \end{aligned}$$

which is a contradiction, hence $x \in Fx$ and consequently F and G have a common fixed point $x \in X$.
Unicity.
Suppose that $F(x) = \{x\}$ and $y \in X$ is another common fixed point of F and G, then by (ii) we have

$$\begin{aligned} d(x,y) \leq H(Fx,Gy) &\leq kN_{c,d}(x,y) \\ &\leq k\ max\{d(x,y), ds\ d(x,y)\} < d(x,y), \quad because\ (kds < 1). \end{aligned}$$

which is a contradiction. Hence $d(x, y) = 0$ then $x = y$.

So x is the unique common fixed point of F and G.

Example 3.7. .
Let $(X = [0,1], d)$ be a complete $b-$metric space with constant $s = 2$, $d(x,y) = |x-y|^2$. We define $F, G : X \longrightarrow B(X)$, by $Fx = \left[0, \frac{x}{4}\right]$, $Gx = \left[0, \frac{x}{8}\right]$
and
$d(x, Fx) = |x - \frac{x}{4}|^2$ $\quad d(y, Gy) = |y - \frac{y}{8}|^2$ $\quad H(Fx, Gy) = |\frac{x}{4} - \frac{y}{8}|^2$.
(i) It is easy to see that F and G are closed.
(ii) We prove that F and G check

$$\begin{aligned} H(Fx,Gy) &\leq \frac{1}{8}\max\left\{d(x,y), d(x,Fx), d(y,Gy), \frac{1}{2}(d(x,Gy)+d(y,Fx))\right\}. \\ &\leq \frac{1}{8}N_{1,1}(x,y). \end{aligned}$$

Indeed, we have the following situations:

1) If $x \leq \frac{y}{2}$, then

$$\begin{aligned}|\frac{x}{4}-\frac{y}{8}| &= \frac{y}{8}-\frac{x}{4}=\frac{1}{4}(\frac{y}{2}-x)\\ &\leq \frac{1}{4}|y-x|,\end{aligned}$$

from where

$$|\frac{x}{4}-\frac{y}{8}|^2 \leq \frac{1}{16}d(x,y) \leq \frac{1}{8}\max\left\{d(x,y), d(x,Fx), d(y,Gy), \frac{1}{2}(d(x,Gy)+d(y,Fx))\right\}.$$

2) If $x \geq \frac{y}{2}$, we have $d(x,Gy)=|x-\frac{y}{8}|^2$. Then

$$\begin{aligned}|\frac{x}{4}-\frac{y}{8}| &= \frac{x}{4}-\frac{y}{8}=\frac{1}{4}(x-\frac{y}{2})\\ &\leq \frac{1}{4}|x-\frac{y}{8}|,\end{aligned}$$

from where

$$\begin{aligned}|\frac{x}{4}-\frac{y}{8}|^2 &\leq \frac{1}{16}d(x,Gy) \leq \frac{1}{8}(\frac{1}{2}(d(x,Gy)+d(y,Fx)))\\ &\leq \frac{1}{8}\max\left\{d(x,y), d(x,Fx), d(y,Gy), \frac{1}{2}(d(x,Gy)+d(y,Fx))\right\}.\end{aligned}$$

This implies

$$\begin{aligned}H(Fx,Gy) &\leq \frac{1}{8}\max\left\{d(x,y), d(x,Fx), d(y,Gy), \frac{1}{2}(d(x,Gy)+d(y,Fx))\right\},\\ &\leq \frac{1}{8}N_{1,1}(x,y) \qquad \text{for all } x,y\in X.\end{aligned}$$

(iii) We have $ksd=\frac{2}{8}<1$.

So all the conditions of Theorem 3.3 are satisfied, then 0 is the unique common absolutely fixed point of F and G.

4. Consequences of the main result

As a consequence of Theorem 3.1, if $F=G=T$, then we obtain the following corollary

Corollary 4.1 (Theorem 2.1 [19]). *Let (X,d) be a b-metric space of constant s and $T: X \longrightarrow B(X)$ having the property that there exist $c,d\in[0,1]$ with $k\in[0,1)$ such that:*

(i) $ksd<1$,

(ii) $H(Tx,Ty)\leq kN_{c,d}(x,y)$ for all $x,y\in X$.

Then, for every $x_0\in X$, there exist $\gamma\in[0,1)$ and a sequence (x_n) of elements from X such that:
(a) $x_{n+1}\in Tx_n$ for every $n\in\mathbb{N}$,
(b) $d(x_{n+1},x_n)\leq\gamma d(x_n,x_{n-1})$ for every $n\in\mathbb{N}^*$,
(c) (x_n) is Cauchy.

From Theorem 3.3 and $F=G=T$ we obtain the following corollary

Corollary 4.2 (Theorem 3.1 [19]). *Let (X,d) be a complete $b-$metric space with constant $s\geq 1$ and $T: X\longrightarrow B(X)$, satisfying the following conditions:*

(i) T is closed,

(ii) *there exist* $c, d \in [0,1]$*and* $k \in [0,1)$ *such that* $H(Tx, Ty) \leq kN_{c,d}(x, y)$ *for all* $x, y \in X$,

(iii) $ksd < 1$.

Then T *has a fixed point* $x \in X$. *Moreover, if* x *is absolutely fixed, then it is unique.*

From Theorem 3.6 and $F = G = T$ a we obtain corollary

Corollary 4.3 (Theorem 3.2 [19]). *Let* (X, d) *be a complete* $b-$*metric space with constant* $s \geq 1$ *and* $T : X \longrightarrow B(X)$, *satisfying the following conditions:*

(i) d *is* $*$*-continuous,*

(ii) *there exist* $c, d \in [0,1]$*and* $k \in [0,1)$ *such that* $H(Tx, Ty) \leq kN_{c,d}(x, y)$ *for all* $x, y \in X$,

(iii) $ksd < 1$.

Then T *has a fixed point* $x \in X$. *Moreover, if* x *is absolutely fixed, then it is unique.*

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Noreddine Makran,
Department of Mathematical Sciences,
Mohammed Premier University, Oujda,
Morocco.
E-mail address: `makranmakran83@gmail.com`

and

Abdelhak El Haddouchi,
Department of Mathematical Sciences,
University Moulay Ismail,
Faculty of Sciences and Technics,
Errachidia, Morocco.
E-mail address: `b.marzouki@ump.ac.ma`

and

Brahim Marzoukim,
Department of Mathematical Sciences,
Mohammed Premier University, Oujda,
Morocco.
E-mail address: `marzoukib@yahoo.fr`