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The Ruled Surface Obtained by the Natural Mate Curve

Fatma Güler

abstract: The natural mate curve r_1 of r is defined the integral of principal normal vector with any parameter s, of a curve r. We investigate the ruled surface generated by the natural mate curve of any Frenet curve $r = r(s)$ in the Euclidean 3-space. We obtained some necessary and sufficient conditions for this surface to be developable and minimal ruled surface. We research related to be the asymptotic curve and the geodesic curve of the base curve on the ruled surface. Example of our main results are also presented.

Key Words: Curves, ruled surfaces, developable surfaces, minimal surfaces.

Contents

1. Introduction

Ruled surfaces can be defined as the movement of a line along a curve. They are one of the simplest object in geometric modeling. Objects consist of points in the coordinate plane. The position of the object can be known by looking at the surface or its usual equation in terms of x, y and z coordinates. These coordinates can be found by rotating these objectsin a certain direction. Ruled surfaces were common place in civil engineering, ships, designing cars." A one parameter set of lines form a ruled surface, [1,2]. For studying kinematical and positional mechanisms in Euclidean 3-space is essential the curvature of ruled surfaces, [3, 4]." A subsets of ruled surfaces are developable surfaces. These surfaces are characterized by a constant surface normal along each ruling. Gaussian curvature vanishes at all these surfaces points. The developable surfaces are used in CAD, geometric design, surface analysis and manufacturing systems.

The natural mate curve r_1 is orthogonal to r since it is tangent to the principal normal vector field N of any Frenet curve r, [5]. Also, the authors obtained main results be for the natural mate of a Frenet curve to be a helix, a spherical curve, or a curve of constant curvature. In [6], they introduced the definitions of Natural pair and Conjugate pair and give the parametric expression of surface pencil with spatial curves as common lines. Also the authors are give the Frenet frames of Natural mate curve and Conjugate mate curve. In this paper, we give some necessary and sufficient conditions to be developable and minimal ruled surface of the ruled surface generated by the natural mate curve of any Frenet curve. Using the natural mate curve, we give some corollary related to be asymptotic curve and geodesic curve of the base curve on the ruled surface. Also, we give examples conforming our study.

2. Preliminaries

Let $r = r(s) : I \to E^3$ be a curve paratmerized by the arc-lenght parameter s with $\|r'(s)\| = \left\| \frac{dr(s)}{ds} \right\| = 1,\ r''(s) \neq 0.$

Useful property for curves parametrized by the arc-length is the Serret-Frenet formulas. These formulas show the derivatives with respect to arc length of the Frenet frame as a function of the current

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Frenet frame, the curvature and the torsion. Then the Serret-Frenet formulas of $r(s)$ are

$$\begin{aligned} \frac{dT}{ds} &= \kappa(s) N(s) \\ \frac{dN}{ds} &= -\kappa(s) T(s) + \tau(s) B(s), \\ \frac{dB}{ds} &= -\tau(s) N(s) \end{aligned} \tag{2.1}$$

where $T(s) = r'(s)$, $N(s) = r''(s) / \left\| r''(s) \right\|$ and $B(s) = T(s) \times N(s)$ are tangent, principal normal and binormal vector of the curve $r(s)$,the curvature $\kappa(s)$ and the torsion $\tau(s)$ of the curve $r(s)$ at s, respectively.

The trace of an $\overrightarrow{X}$ oriented line along $r(s)$ is generally a ruled surface. A parametric equation of this ruled surface generated by $\overrightarrow{X}$ oriented line is given by

$$\phi(s,v) = r(s) + v\overrightarrow{X}(s) \qquad s,\ v \in I \subset IR \tag{2.2}$$

where $\overrightarrow{X}(s)$ is the director vector and $r(s)$ is the directrix.

Definition 2.1. *The distribution parameter (or drall) of the ruled surface is given as, by [7]*

$$P_{\overrightarrow{X}} = \frac{\det\left(dr/ds, \overrightarrow{X}, d\overrightarrow{X}/ds\right)}{\left\| d\overrightarrow{X}/ds \right\|^2}. \tag{2.3}$$

Lemma 2.2. *The ruled surface Eqn.* (2.2) *is developable if and only if, by [7]*

$$\det\left(dr/ds, \overrightarrow{X}, d\overrightarrow{X}/ds\right) = 0. \tag{2.4}$$

The Gaussian curvature $K(s,v)$ and the mean curvature $H(s,v)$ of the ruled surface is given by the formula:

$$K(s,v) = \frac{LN - M^2}{EG - F^2},\ H(s,v) = \frac{GL + EN - 2FM}{2(EG - F^2)} \tag{2.5}$$

where, the elements of the first and second fundamental forms on the surface $\phi(s,v)$ are defined by

$$E = \|\phi_s\|^2,\ F = \langle \phi_s, \phi_v \rangle,\ G = \|\phi_v\|^2, \tag{2.6}$$

and

$$L = \langle \phi_{ss}, \phi_s \times \phi_v \rangle,\quad N = \langle \phi_{vv}, \phi_s \times \phi_v \rangle,\ M = \langle \phi_{sv}, \phi_s \times \phi_v \rangle, \tag{2.7}$$

,respectively.

The first fundamental form I and the second fundamental form II of the ruled surfaces $\phi(s,v)$ are given by in [7]

$$I = E(ds)^2 + 2Fdsdv + G(dv)^2,\ II = L(ds)^2 + 2Mdsdv + N(dv)^2\ . \tag{2.8}$$

Definition 2.3. *Let the curve $r_1(s_1)$ be the integral curve of the principal normal vector $N(s)$ of spatial curve $r(s)$, that is $r_1(s_1) = \int N(s)\,ds$, the curve $r_1(s_1)$ is called the Natural mate curve of spatial curve $r(s)$, and $\{r(s), r_1(s_1)\}$ is called the Natural pair, [5]*

Lemma 2.4. *Let $\{r(s), r_1(s)\}$ be the natural pair and Frenet frames $\{T, N, B\}$ and $\{T_1, N_1, B_1\}$ .The Frenet frame of natural mate curve $r_1(s)$ satisfies the following formula*

$$\begin{bmatrix} T_1(s) \\ N_1(s) \\ B_1(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\cos\varphi & 0 & \sin\varphi \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix} \tag{2.9}$$

where the angle φ is between Darboux vector with binormal vector, the curvature and torsion of $r_1(s)$ satisfies $\kappa_1 = \sqrt{\kappa^2+\tau^2}$, $\tau_1 = \frac{\kappa^2\left(\frac{\tau}{\kappa}\right)'}{\kappa^2+\tau^2}$, respectively, [5].

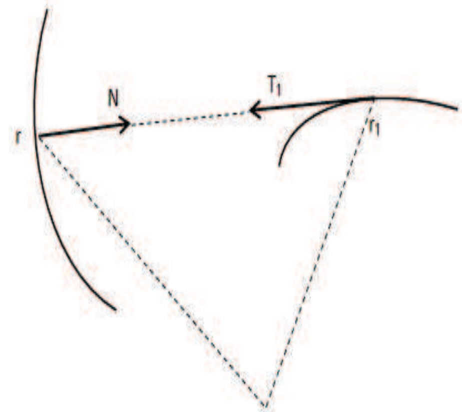

Figure 1: Natural mate pair

3. The Ruled Surface Obtained by the Natural Mate Curve

Let the curve $r_1(s)$ be the natural mate curve of spatial curve $r(s)$, and Frenet frames $\{T_1, N_1, B_1\}$, $\{T, N, B\}$ and the curvatures $\kappa_1(s)$, $\kappa(s) \neq 0$ and the torsions $\tau_1(s), \tau(s) \neq 0$, respectively. Let X be ruled surface which is given by

$$X(s,v) = r_1(s) + vT_1(s). \tag{3.1}$$

Since $r_1(s)$ is the natural mate curve of spatial curve $r(s)$, we can write

$$r_1(s) = r(s) + R(s)N(s), \tag{3.2}$$

where R is distance between corresponding natural pair point.

By using Eqs. (3.1) and (3.2), we have

$$X(s,v) = r(s) + (v + R(s))N(s). \tag{3.3}$$

Differentiating Eq. (3.3) and using Eq. (2.1) , the tangents of the parametric curves of the surface $X(s,v)$ are given by

$$X_s = (1 - \kappa(v+R))T + R'N + \tau(v+R)B, \quad X_v = N, \tag{3.4}$$

Differentiating Eq. (3.4) and using the Eq. (2.1), the second derivatives for the function on the ruled surface $X(s,v)$ are give

$$\begin{gathered} X_{ss} = (-2\kappa R' - \kappa'(v+R))T + \left(1 + R'' - (v+R)\left(\kappa^2+\tau^2\right)\right)N + (2\tau R' + \tau'(v+R))B \\ X_{sv} = -\kappa T + \tau B, \quad X_{vv} = 0. \end{gathered} \tag{3.5}$$

Using Eqs. (2.6),(3.4) , we can write the first fundamental and second fundamental quantities of as follows

$$E=\left(1-\kappa\left(v+R\right)\right)^{2}+R'^{2}+\tau^{2}\left(v+R\right)^{2},\quad F=R',\quad G=1, \tag{3.6}$$

and

$$\begin{gathered}L=\left(\tau\kappa'-\tau'\kappa\right)v^{2}+\left(2\tau R\kappa'-2\tau'\kappa R+\tau'\right)+2\tau R'+\tau' R\left(1-R\kappa\right)+\tau R^{2}\kappa' \\ M=\kappa\tau\left(v+R\right)+\tau\left(1-\kappa\left(v+R\right)\right),\qquad N=0.\end{gathered} \tag{3.7}$$

Corollary 3.1. *The Gaussian curvature $K\left(s,v\right)$ of the ruled surface $X\left(s,v\right)$ is*

$$K\left(s,v\right)=-\frac{\tau^{2}\left(1+2\kappa R\right)}{\left(1-\kappa\left(v+R\right)\right)^{2}+\tau^{2}\left(v+R\right)^{2}}. \tag{3.8}$$

Corollary 3.2. *The ruled surface $X\left(s,v\right)$ is developable if and only if*

$$R=-\frac{1}{2\kappa}. \tag{3.9}$$

Corollary 3.3. *The mean curvature $H\left(s,v\right)$ of the ruled surface $X\left(s,v\right)$ is*

$$H\left(s,v\right)=\frac{\left(\frac{\kappa}{\tau}\right)'\tau^{2}v^{2}+\left(2R\left(\frac{\kappa}{\tau}\right)'\tau^{2}+\tau'\right)v+\left(\tau' R+R^{2}\left(\frac{\kappa}{\tau}\right)'\tau^{2}\right)}{2\left(\left(1-\kappa\left(v+R\right)\right)^{2}+\tau^{2}\left(v+R\right)^{2}\right)}. \tag{3.10}$$

Corollary 3.4. *The ruled surface $X\left(s,v\right)$ is minimal surface if and only if r is circular helix. So, the natural curve r_1 is planar.*

The unit normal vector to ruled surface $X\left(s,v\right)$ at the point (s,0) is given by

$$\bar{N}\left(s,0\right)=\frac{X_{s}\times X_{v}}{\left\|X_{s}\times X_{v}\right\|}|_{v=0.} \tag{3.11}$$

Thus, from Eq. (3.11) the unit normal vector to ruled surface $X\left(s,v\right)$ at the point (s,0) is

$$\bar{N}\left(s,0\right)=\frac{-\tau RT+\left(1-R\kappa\right)B}{\sqrt{\tau^{2}R^{2}+\left(1-R\kappa\right)^{2}}}. \tag{3.12}$$

The geodesic curvature and the normal curvature of the base curve are

$$k_{g}=\left\langle \bar{N}\times T,T'\right\rangle=\frac{\kappa\left(1-R\kappa\right)}{\sqrt{\tau^{2}R^{2}+\left(1-R\kappa\right)^{2}}},\qquad k_{n}=\left\langle r'',\bar{N}\right\rangle=0. \tag{3.13}$$

Corollary 3.5. *The base curve $r\left(s\right)$ is geodesic curve on the ruled surface $X\left(s,v\right)$ if and only if $R=\frac{1}{\kappa}$. Also, the base curve $r\left(s\right)$ is asymptotic curve on the ruled surface $X\left(s,v\right)$.*

Example 3.6. *Let $r\left(s\right)=\left(\frac{3}{5}\cos s,\frac{3}{5}\sin s,\frac{4}{5}s\right)$ be an arc-length curve. It is easy to show that*

$$\begin{gathered}T\left(s\right)=\left(-\tfrac{3}{5}\sin s,\tfrac{3}{5}\cos s,\tfrac{4}{5}\right)\\ N\left(s\right)=\left(-\cos s,-\sin s,0\right)\\ B(s)=\left(\tfrac{4}{5}\sin s,\tfrac{4}{5}\cos s,\tfrac{3}{5}\right)\end{gathered}$$

and

$$\kappa=\frac{3}{5}.$$

Using definition 2.2 , the natural mate curve $r_1\left(s\right)$ of $r\left(s\right)$ is

$$r_1(s) = (-\sin s, \cos s, a), \qquad a = cons\tan t.$$

Using Eq.(3.3) and ıf we take $R = -\frac{5}{6}$, then Eq.(3.9) is satisfied. Thus, we obtained developable the ruled surface as, (Fig 2.)

$$X(s,v) = \left(\frac{3}{5}\cos s - (\cos s)\left(v - \frac{5}{6}\right), \frac{3}{5}\sin s - (\sin s)\left(v - \frac{5}{6}\right), \frac{4}{5}s\right)$$

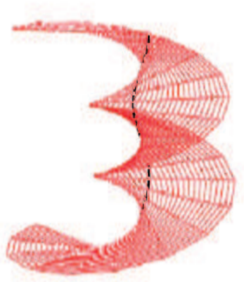

Figure 2: Developable Ruled Surface

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Fatma GÜLER,
Department of Mathematics,
Arts and Science Faculty
Ondokuz Mayis University,
55139 Samsun, Turkey.
E-mail address: `f.guler@omu.edu.tr`