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Behavior of the Isothermal Elasticity Operator with Non-linear Friction in a Thin Domain*

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ABSTRACT: This paper deals with the asymptotic behavior of a boundary value problem in a three dimensional thin domain Ω^ε with non-linear friction of Coulomb type. We will establish a variational formulation for the problem and prove the existence and uniqueness of the weak solution. We then study the asymptotic behavior when one dimension of the domain tends to zero. In which case, the uniqueness result of the displacement for the limit problem is also proved.

Key Words: A priori inequalities, asymptotic approach, Coulomb law, elasticity system, free boundary problems, variational inequalities.

Contents

1. Introduction

The theory of variational inequalities represents, in very natural generalization of theory of boundary value problems and allows us to consider new problems arising from many fields of applied Mathematics, such as Mechanics, Physics, the Theory of convex programming, the control and in engineering science. In this paper, we study a problem involving boundary conditions describing real phenomena such as contact and friction. The problem presented in this work is very frequent in applications. For instance the physical domains are defined such that the height is much smaller than the length. These are the assumptions of elasticity and Visco-elasticity of a tire. Other applications are related to the mechanism of ball bearing. Scientific research in mechanics are articulated around two main components: one devoted to the laws of behavior and the other on boundary conditions imposed on the body. For the constitutive law, we consider an isothermal elastic body with Coulomb free boundary friction conditions in the stationary regime occupying a bounded, homogeneous domain $\Omega^\varepsilon \subset \mathbb{R}^3$. As the boundary conditions reflect the binding of the body with the outside world, the boundary Γ^ε of the domain is assumed to be Lipschitz continuous so that the unit outward normal n exists almost everywhere on Γ^ε. The boundary of the domain is assumed to be composed of three portions : ω the bottom of the domain, Γ_1^ε the upper surface, and Γ_L^ε the lateral surface. We suppose that the Dirichlet boundary conditions are satisfied on $\overline{\Gamma}_1^\varepsilon \cup \overline{\Gamma}_L^\varepsilon$, for the displacement. On the bottom surface, the normal displacement is null. However, assuming the friction is sufficiently large, the tangential velocity is unknown and satisfies the Coulomb boundary condition. This law is one of the most spread laws in mathematics and it is more realistic than the law of Tresca. Several works have been done on the mechanical contact with the various laws of behaviour and various friction boundary conditions close to our problem, however these papers were restricted only to the results of existence and uniqueness of the weak solution under several assumptions. Let us mention for example the work by [17] in which the authors worked the mathematical model which describes the quasistatic frictional contact between a pie-zoelectric body and a deformable foundation with the normal compliance condition and a version of Coulomb's law of friction. The dynamic evolution with frictional contact of an elastic body was studied in [16]. They proved the existence of a solution

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in the two-dimensional case and the uniqueness for the one-dimensional shearing problem. An excellent reference on analysis of contact problems involving elastic materials with or without friction is in [1,9]. Existence, uniqueness and regularity results in the study of a new class of variational inequalities were proved in [18]. The authors in [2] studied the asymptotic and numerical analysis for a unilateral contact problem with Coulomb's friction between an elastic body and a thin elastic soft layer. More recently, the asymptotic analysis of a dynamical problem of isothermal and non-isothermal elasticity with non linear friction of Tresca type was studied in [4,17]. Some research papers have been written dealing with both the asymptotic analysis of an incompressible fluid in a three-dimensional thin domain, when one dimension of the fluid domain tends to zero can be found in [5,10,11]. The goal of this paper is to study the asymptotic behavior of a boundary value problem in a three dimensional thin domain Ω^ε with non linear friction of Coulomb type. The use of the small change of variable $z=\dfrac{x_3}{\varepsilon}$, transforms the initial problem posed in the domain Ω^ε into a new problem posed on a fixed domain Ω independent of the parameter ε. We prove some estimates on the displacement independent of the small parameter. The passage to the limit on ε, permits us to obtain the existence and uniqueness of the limit of a weak solution to the problem described in the abstract.

This article is organized as follows. In Section 2 we introduce the notation and we recall the weak formulation of our problem considered. In section 3 we find some estimates and prove convergence theorem by using several inequalities. Finally, we obtain all the properties of our original problem.

2. The model and variational problem

We consider a mathematical problem governed to the stationary equations for elasticity system in a three dimensional bounded domain $\Omega^\varepsilon \subset \mathbb{R}^3$ with boundary Γ^ε. We denote by $\mathbf{S}_3$ the space of second order symmetric tensor on $\mathbb{R}^3$, and $|.|$ the inner product and the Euclidean norm on $\mathbb{R}^3$ and $\mathbf{S}_3$, respectively. Thus, for every $\mathbf{u},\mathbf{v} \in \mathbb{R}^3, \mathbf{u}.\mathbf{v} = u_i.v_i, |\mathbf{v}| = (\mathbf{v}.\mathbf{v})^{\frac{1}{2}}$and for everywhere $\sigma,\tau \in \mathbf{S}_3, \sigma.\tau = \sigma_{ij}\tau_{ij}, |\tau| = (\tau.\tau)^{\frac{1}{2}}$ for $1 \le i,j \le 3$. Let ω be a fixed bounded domain of $\mathbb{R}^3$ of equation $x_3 = 0$. We suppose that ω has a Lipschitz continuous boundary and is the bottom of the domain. The upper surface $\overline{\Gamma}_1^\varepsilon$ is defined by $x_3 = \varepsilon h(x) = \varepsilon h(x_1, x_2)$. We introduce a small parameter ε, that will tend to zero, and a function h on the closure of ω such that $0 < h_* \le h(x) \le h^*$, for all $(x,0)$ in ω. The domain Ω^ε is defined by

$$\Omega^\varepsilon = \left\{(x,\, x_3) \in \mathbb{R}^3 :\, (x,0) \in \omega, \quad 0 < x_3 < \varepsilon h(x)\right\}.$$

Also, we use the following notations

$$H^1(\Omega^\varepsilon)^3 = \left\{ v \in \left(L^2\left(\Omega^\varepsilon\right)\right)^3 : \frac{\partial v_i}{\partial x_j} \in L^2\left(\Omega^\varepsilon\right),\ \forall i,j = 1,2,3 \right\},$$

$$V^\varepsilon = \left\{ v \in H^1(\Omega^\varepsilon)^3 : v = 0 \text{ on } \Gamma_L^\varepsilon \cup\ \Gamma_1^\varepsilon,\ v.n = 0 \text{ on } \omega \right\},$$

$$H(\Omega^\varepsilon) = \left\{ \sigma = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega^\varepsilon) \right\},$$

$$H^1_{\Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon}\left(\Omega^\varepsilon\right) = \left\{ \varphi \in H^1\left(\Omega^\varepsilon\right) : \varphi = 0 \quad \text{on } \Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon \right\}.$$

The spaces $H^1(\Omega^\varepsilon)^3$, V^ε, $H(\Omega^\varepsilon)$ and $H^1_{\Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon}\left(\Omega^\varepsilon\right)$ are a real Hilbert spaces endowed with their natural norms $\|.\|_{1,\Omega^\varepsilon}$ and scalar product $(.,.)_{1,\Omega^\varepsilon}$.
Let Q the real Hilbert space

$$Q = \left\{ \tau = (\tau_{ij}) : \tau_{ij} = \tau_{ji} \in L^2(\Omega^\varepsilon),\, \forall i,j = 1,2,3 \right\},$$

with the canonical inner product

$$(\sigma,\tau)_Q = \int_{\Omega^\varepsilon} \sigma_{ij}\tau_{ij}dx = \int_{\Omega^\varepsilon} \sigma.\tau dx,$$

where

$$(\mathbf{u},\mathbf{v})_{1,\Omega^\varepsilon} = (\epsilon(\mathbf{u}),\epsilon(\mathbf{v}))_Q \quad \forall \mathbf{u},\, \mathbf{v} \in V(\Omega^\varepsilon).$$

Here ϵ denotes the *deformation* operator $\epsilon : H^1(\Omega^\varepsilon)^3 \longrightarrow Q$ defined by

$$\epsilon(\mathbf{u}) = (\epsilon_{ij}(\mathbf{u})), \quad \epsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Let $\mathbf{Q}_\infty$ be the space of forth order fields given by (see [18], page 97)

$$\mathbf{Q}_\infty = \{\mathcal{E} = (e_{ijpq}) : e_{ijpq} = e_{jipq} = e_{pqij} \in L^\infty(\Omega^\varepsilon),\ 1 \leq i,j,p,q \leq 3\},$$

witch is a real Banach space with the norm

$$\|\mathcal{E}\|_{\mathbf{Q}_\infty} = \max_{0 \leq i,j,p,q \leq 3} \|e_{ijpq}\|_{L^\infty(\Omega^\varepsilon)}$$

and moreover

$$\|\mathcal{E}\tau\|_Q \leq 3\|\mathcal{E}\|_{\mathbf{Q}_\infty}\|\tau\|_Q \qquad \forall \mathcal{E} \in \mathbf{Q}_\infty,\ \tau \in Q.$$

Let us introduce a vector function $g = (g_i)_{1 \leq i \leq 3}$, such that

$$\int_{\Gamma_L^\varepsilon} g.n d\sigma = 0\,. \tag{*}$$

We assume that g is in $H^{\frac{1}{2}}(\partial\Omega^\varepsilon)^3$, the space of traces of functions from $H^1(\Omega^\varepsilon)^3$ in $\partial\Omega^\varepsilon$. Due to (*) there exists a function G^ε such that

$$G^\varepsilon \in H^1(\Omega^\varepsilon)^3 \text{ with } G^\varepsilon = g \text{ on } \partial\Omega^\varepsilon.$$

For every element $u \in H^1(\Omega^\varepsilon)^3$ we denote by u_n^ε and u_τ^ε the normal and the tangential components of u on the boundary ω given by

$$u_n^\varepsilon = u^\varepsilon.n, \quad u_{\tau_i}^\varepsilon = u_i^\varepsilon - u_n^\varepsilon.n_i.$$

Also, for a regular function σ^ε, we define its normal and tangential components by

$$\sigma_n^\varepsilon = (\sigma^\varepsilon.n_i)\,.n_j, \qquad \sigma_{\tau_i}^\varepsilon = \sigma_{ij}^\varepsilon.n_j - (\sigma_n^\varepsilon)\,.n_i.$$

We denote by $\mathbf{u}^\varepsilon = (u_i^\varepsilon)_{1 \leq i \leq 3}$, the displacement vectors, by $\sigma^\varepsilon = (\sigma_{ij})_{1 \leq i,j \leq 3}$, the stress tensor and $\epsilon(\mathbf{u}^\varepsilon)$ the linearized strain tensors. We model the materials with elastic constructive law

$$\boldsymbol{\sigma}^\varepsilon = \mathcal{E}\epsilon(\mathbf{u}^\varepsilon),$$

where $\mathcal{E}$ is given linear constitutive functions satisfy

$$\begin{cases} \text{(a) } \mathcal{E} : \Omega^\varepsilon \times \mathbf{S}_3 \to \mathbf{S}_3, \\ \text{(b) There exists } L_{\mathcal{E}} > 0 \text{ such that} \\ \quad |\mathcal{E}(x,\epsilon_1) - \mathcal{E}(x,\epsilon_2)| \leq L_{\mathcal{E}}\,|\epsilon_1 - \epsilon_2| \\ \qquad \forall\,\epsilon_1,\,\epsilon_2 \in \mathbf{S}_3, \text{ a.e. } x \in \Omega^\varepsilon, \\ \text{(c) There exists } m_{\mathcal{E}} > 0 \text{ such that} \\ \qquad (\mathcal{E}(x,\epsilon_1) - \mathcal{E}(x,\epsilon_2)).(\epsilon_1 - \epsilon_2) \geq m_{\mathcal{E}}|\epsilon_1 - \epsilon_2|^2 \\ \qquad \forall\,\epsilon_1,\,\epsilon_2 \in \mathbf{S}_3, \text{ a.e. } x \in \Omega^\varepsilon, \\ \text{(d) The map } x \longmapsto \mathcal{E}(x,0) \text{ is Lebesgue measurable on } \Omega^\varepsilon \\ \qquad \text{for any } \epsilon \in \mathbf{S}_3, \\ \text{(e) The map } x \longmapsto \mathcal{E}(x,0) \in H(\Omega^\varepsilon). \end{cases} \tag{2.0}$$

For given body forces f^ε the problem is described by:
• The stationary elasticity system of equations:

$$-div\,(\sigma^\varepsilon) = f^\varepsilon \quad \text{ in } \Omega^\varepsilon.$$

▶ The upper surface Γ^ε, being assumed to be fixed so :

$$\mathbf{u}^\varepsilon = 0, \quad \text{on } \Gamma_1^\varepsilon.$$

▶ On Γ_L^ε, the displacement is known and parallel to the w-plane:

$$\mathbf{u}^\varepsilon = g \ \text{ with } \ g_3 = 0.$$

• Now, we describe the conditions on the surface ω. We assume that the contact is bilateral, i.e.,

$$u^\varepsilon.n = 0.$$

• The tangential displacement on ω is unknown and satisfies the Coulomb friction law.

$$\left.\begin{array}{l} |\sigma_\tau^\varepsilon| < \mathcal{F}^\varepsilon|\sigma_n^\varepsilon| \implies u_\tau^\varepsilon = s, \\ |\sigma_\tau^\varepsilon| = \mathcal{F}^\varepsilon|\sigma_n^\varepsilon| \implies \exists\beta \geq 0 \text{ such that } u_\tau^\varepsilon = s - \beta\sigma_\tau^\varepsilon. \end{array}\right\}$$

In this law, the tangential stress can reach the border $\mathcal{F}^\varepsilon|\sigma_n^\varepsilon|$ which is called friction threshold, where $\mathcal{F}^\varepsilon \geq 0$ is the coefficient of friction.

To simplify the notation, we do not indicate explicitly the dependence of various functions on the variables $x \in \Omega^\varepsilon \cup \Gamma^\varepsilon$. Then, the classical formulation of the mechanical problem of a frictional bilateral contact with wear may be stated as follows.

Problem $\mathcal{P}^\varepsilon$. Find a displacement field $\mathbf{u}^\varepsilon = ((u_i^\varepsilon))_{1\leq i\leq 3} : \Omega^\varepsilon \to \mathbb{R}^3$ such that

$$\text{Div}\sigma^\varepsilon + \mathbf{f}^\varepsilon = 0, \text{ in } \Omega^\varepsilon, \tag{2.1}$$

$$\sigma^\varepsilon = \mathcal{E}\epsilon(\mathbf{u}^\varepsilon), \text{ in } \Omega^\varepsilon, \tag{2.2}$$

$$u^\varepsilon = 0, \text{ on } \Gamma_1^\varepsilon, \tag{2.3}$$

$$u^\varepsilon = g, \text{ with } g_3 = 0, \text{ on } \Gamma_L^\varepsilon, \tag{2.4}$$

$$u^\varepsilon.n, \text{ on } \omega, \tag{2.5}$$

$$\left.\begin{array}{l} |\sigma_\tau^\varepsilon| < \mathcal{F}^\varepsilon|\sigma_n^\varepsilon| \implies u_\tau^\varepsilon = s \\ |\sigma_\tau^\varepsilon| = \mathcal{F}^\varepsilon|\sigma_n^\varepsilon| \implies \exists\beta \geq 0 \text{ such that } u_\tau^\varepsilon = s - \beta\sigma_\tau^\varepsilon \end{array}\right\}, \text{ on } \omega, \tag{2.6}$$

Remark 2.1. If we have only $\mathbf{u}^\varepsilon \in V^\varepsilon$ and σ_n^ε is defined by duality as an element of $H^{-\frac{1}{2}}(\omega)$, has non sense then the integral $J^\varepsilon(\mathbf{v})$ has no meaning. From the mathematical point of view it is necessary that $R(\sigma_\eta^\varepsilon) = |\sigma_n^\varepsilon|$ with R is a regularization operator (cf. [12,13]) from $H^{\frac{1}{2}}(\omega)$ into $L^2(\omega)$ defined by

$$\forall\tau \in H^{-\frac{1}{2}}(\omega), \ \ R(\tau) \in L^2(\omega), \ \ R(\tau)(x) = \langle\tau, \phi(x-t)\rangle_{H^{-\frac{1}{2}}(\omega), H_{00}^{\frac{1}{2}}(\omega)} \quad \forall x \in \omega,$$

where ϕ is a given positive function of class C^∞ with compact support in ω and $H^{-\frac{1}{2}}(\omega)$ is the dual space to

$$H_{00}^{\frac{1}{2}} = \{v_{|\omega} : v \in H^1(\Omega), v = 0 \ \text{ on } \ \Gamma_1 \cup \Gamma_L\}.$$

Theorem 2.1. *Let* $\mathbf{u}^\varepsilon$ *be a solution of Problem* $\mathcal{P}^\varepsilon$*, with a sufficient regularity, then it satisfies the following variational problem*:

Problem $\mathcal{P}_v^\varepsilon$. *Find* $\mathbf{u}^\varepsilon \in V^\varepsilon$ *such that*

$$a(\mathbf{u}^\varepsilon, \mathbf{v} - \mathbf{u}^\varepsilon) + J^\varepsilon(\mathbf{v}) - J^\varepsilon(\mathbf{u}^\varepsilon) \geq \int_{\Omega^\varepsilon} \mathbf{f}^\varepsilon(\mathbf{v} - \mathbf{u}^\varepsilon)dxdx_3, \quad \forall\mathbf{v} \in V^\varepsilon, \tag{2.7}$$

where

$$a(\mathbf{u}^\varepsilon, \mathbf{v} - \mathbf{u}^\varepsilon) = \int_{\Omega^\varepsilon} \sigma^\varepsilon(\epsilon(\mathbf{v}) - \epsilon(\mathbf{u}^\varepsilon))dxdx_3 = \int_{\Omega^\varepsilon} \mathcal{E}(\epsilon(\mathbf{v}) - \epsilon(\mathbf{u}^\varepsilon))dxdx_3,$$

$$(f^\varepsilon, v) = \int_{\Omega^\varepsilon} f_i v_i dxdx_3, \ \forall v \in H^1(\Omega^\varepsilon)^3,$$

$$J^\varepsilon(\mathbf{v}) = \int_\omega \mathcal{F}^\varepsilon|R(\sigma_\eta^\varepsilon)||v - s|dx.$$

Proof. Multiplying equation (2.1) by $(\varphi - u^{\varepsilon})$, where $\varphi \in V^{\varepsilon}$, and we use Green's formula, we get

$$\int_{\Omega^{\varepsilon}} \sigma^{\varepsilon}(\epsilon(\mathbf{v}) - \epsilon(\mathbf{u}^{\varepsilon}))dxdx_3 - \int_{\Gamma^{\varepsilon}} \sigma^{\varepsilon}_{ij} n_j \left(\varphi_i - u^{\varepsilon}_i\right) d\sigma = \int_{\Omega^{\varepsilon}} f^{\varepsilon}_i \left(\varphi_i - u^{\varepsilon}_i\right) dxdx_3. \tag{2.8}$$

According to the boundary conditions (2.3)-(2.5), we find

$$\int_{\Gamma^{\varepsilon}} \sigma^{\varepsilon}_{ij} n_j \left(\varphi_i - u^{\varepsilon}_i\right) d\sigma = \int_{\omega} \sigma^{\varepsilon}_{\tau_i} \left(\varphi_i - u^{\varepsilon}_i\right) dx.$$

Therefore

$$\int_{\Omega^{\varepsilon}} \sigma^{\varepsilon}(\epsilon(\mathbf{v}) - \epsilon(\mathbf{u}^{\varepsilon}))dxdx_3 - \int_{\omega} \sigma^{\varepsilon}_{\tau_i} \left(\varphi_i - u^{\varepsilon}_i\right) dx = \int_{\Omega^{\varepsilon}} f^{\varepsilon}_i \left(\varphi_i - u^{\varepsilon}_i\right) dxdx_3, \ \ \forall \varphi \in V^{\varepsilon}. \tag{2.9}$$

In (2.8) adding and subtract the term $\int_{\omega} \mathcal{F}^{\varepsilon} |R(\sigma^{\varepsilon}_n)| \left(|\varphi - s| - |u^{\varepsilon}_{\tau} - s|\right) dx$, we obtain

$$\int_{\Omega^{\varepsilon}} \sigma^{\varepsilon}(\epsilon(\mathbf{v}) - \epsilon(\mathbf{u}^{\varepsilon}))dxdx_3 - \int_{\omega} \sigma^{\varepsilon}_{\tau_i} \left(\varphi_i - u^{\varepsilon}_i\right) dx + \int_{\omega} \mathcal{F} |R(\sigma^{\varepsilon}_n)| \left(|\varphi - s| - |u^{\varepsilon}_{\tau} - s|\right) dx$$
$$- \int_{\omega} \mathcal{F} |R(\sigma^{\varepsilon}_n)| \left(|\varphi - s| - |u^{\varepsilon}_{\tau} - s|\right) dx = \int_{\Omega^{\epsilon}} f^{\varepsilon}_i \left(\varphi_i - u^{\varepsilon}_i\right) dxdx_3.$$

As the Coulomb friction (2.6) is equivalent to ([12]):

$$\left(u^{\varepsilon}_{\tau} - s\right) \sigma^{\varepsilon}_{\tau} + \mathcal{F}^{\varepsilon} |R(\sigma^{\varepsilon}_n)| |u^{\varepsilon}_{\tau} - s| = 0 \quad \text{on } \omega,$$

we deduce directly the variational inequality (2.7). □

Theorem 2.2. *Assume $f^{\varepsilon} \in L^2(\Omega^{\varepsilon})^3$ and the friction coefficient $\mathcal{F}^{\varepsilon}$ is a non negative function in $\in L^{\infty}(\omega)$, then there exists $u^{\varepsilon} \in V^{\varepsilon}$ solution to the problem P^{ε}. Moreover, for small $\mathcal{F}^{\varepsilon}$ this solution is unique.*

Proof. The proof is similar to that given in [2]. Indeed, for the existence of solution u^{ε} we apply Tichonov's fixed point theorem. Then to proved the uniqueness of u^{ε} we used the same technicals as in [2] and ([18], Theorem 2.1).

3. The problem in a fixed domain

This section is devoted to the study of a priori estimates on the displacement u^{ε} solution of our variational problem. For the asymptotic analysis of problem $(2.1) - (2.6)$, we use the approach which consist in transposing the problem initially posed in the domain Ω^{ε} which depend on a small parameter ε in an equivalent problem posed in the fixed domain Ω which is independent of ε. For that, we introduce a small change of the variable $z = \dfrac{x_3}{\varepsilon}$, so for (x, x_3) in Ω^{ε} we have (x, z) in

$$\Omega = \left\{(x, z) \in \mathbb{R}^3, (x, 0) \in \omega \text{ and } 0 < z < h\left(x\right)\right\},$$

and we denote by $\Gamma = \bar{\omega} \cup \Gamma_L \cup \Gamma_1$ its boundary, then we define the following functions in Ω

$$\hat{u}^{\varepsilon}_3\left(x, z\right) = \varepsilon^{-1} u^{\varepsilon}_3\left(x, x_3\right) \text{ and } \hat{u}^{\varepsilon}_i\left(x, z\right) = u^{\varepsilon}_i\left(x, x_3\right), \, i = 1, 2. \tag{3.1}$$

For the data of problem $(2.1) - (2.6)$, we suppose that they depend of ε in the following manner

$$\begin{cases} \hat{\mathbf{f}}\left(x, z\right) = \varepsilon^2 \mathbf{f}^{\varepsilon}(x, x_3), \\ \widehat{\mathcal{F}} \ = \varepsilon^{-1} \ \mathcal{F}^{\varepsilon}, \\ \hat{g}(x, z) = g^{\varepsilon}(x, x_3), \end{cases} \tag{3.2}$$

with $\hat{\mathbf{f}}$, $\widehat{F}$ and $\hat{g}$ independent of ε.

The vector G^{ε} introduced in section 2 will be defined as follows

$$\begin{cases} \hat{G}_i(x, z) = G^{\varepsilon}_i(x, x_3), & i = 1, 2, \\ \hat{G}_3\left(x, z\right) = \varepsilon^{-1} G^{\varepsilon}_3(x, x_3). \end{cases} \tag{3.3}$$

Now we introduce the functional framework on Ω. For this, we note:

$$V=\left\{\widehat{v}\in H^1(\Omega)^3:\widehat{v}=0 \text{ on } \Gamma_L\cup\Gamma_1 \text{ and } \widehat{v}.n=0 \text{ on } \omega\right\},$$
$$\Pi(V)=\left\{\widehat{v}\in H^1(\Omega)^2:\widehat{v}=(\widehat{v}_1,\widehat{v}_2),\widehat{v}_i=0 \text{ on } \Gamma_1\cup\Gamma_L\ ,\ i=1,2\ \right\},$$
$$V_z=\left\{\widehat{v}=(\widehat{v}_1,\widehat{v}_2)\in L^2(\Omega)^2:\frac{\partial\widehat{v}_i}{\partial z}\in L^2(\Omega),i=1,2;\text{ and } \widehat{v}=0 \text{ on } \Gamma_1\right\}.$$

Assuming $(3.1)-(3.3)$, then problem (2.7) leads to the following form:
Problem $\hat{\mathcal{P}}^\varepsilon$. Find $\hat{\mathbf{u}}^\varepsilon\in V$ such that

$$\hat{\mathbf{a}}\left(\hat{\mathbf{u}},\hat{\mathbf{v}}-\hat{\mathbf{u}}\right)+\hat{J}(\hat{\mathbf{v}})-\hat{J}(\hat{\mathbf{u}})\geq\sum_{i=1}^{2}\left(\hat{f}_i,\hat{v}_i-\hat{u}_i\right)+\varepsilon\left(\hat{f}_3,\hat{v}_3-\hat{u}_3\right),\forall\hat{\mathbf{v}}\in V, \tag{3.4}$$

where

$$\hat{J}(\hat{\mathbf{v}})=\int_\omega\widehat{\mathcal{F}}\left|R(\sigma_\eta^\varepsilon)\right||\hat{\mathbf{v}}_\tau-s|dx,$$
$$\hat{\mathbf{a}}\left(\hat{\mathbf{u}},\hat{\mathbf{v}}-\hat{\mathbf{u}}\right)=\varepsilon^2\sum_{i=1}^{2}\int_{\Omega^\varepsilon}\sigma^\varepsilon(\epsilon(\hat{v}_i)-\epsilon(\hat{u}_i^\varepsilon))dxdx_3+\varepsilon\int_{\Omega^\varepsilon}\sigma^\varepsilon(\epsilon(\hat{v}_3)-\epsilon(\hat{u}_3^\varepsilon))dxdx_3.$$

In the next, we will obtain estimates on $\hat{\mathbf{u}}^\varepsilon$. These estimates will be useful in order to prove the convergence of $\hat{\mathbf{u}}^\varepsilon$ toward the expected function.
Theorem 3.1. *Assuming that $f\in\left(L^2(\Omega)\right)^3$ and the friction coefficient $\widehat{\mathcal{F}}>0$ in $\mathbf{L}^\infty(\omega)$ then there exists positive constant C independent of ε, such that the following estimate holds*

$$\varepsilon^2\sum_{1\leq i,j\leq 2}\left\|\frac{\partial\hat{u}_i^\varepsilon}{\partial x_j}\right\|_{L^2(\Omega)}^2+\varepsilon^2\left\|\frac{\partial\hat{u}_3^\varepsilon}{\partial z}\right\|_{L^2(\Omega)}^2+\sum_{1\leq i\leq 2}\left\|\frac{\partial\hat{u}_i^\varepsilon}{\partial z}\right\|_{L^2(\Omega)}^2+\varepsilon^4\left\|\frac{\partial\hat{u}_3^\varepsilon}{\partial x_i}\right\|_{L^2(\Omega)}^2\leq C. \tag{3.5}$$

Proof. Let $\mathbf{u}^\varepsilon$ be a solution to problem $\mathcal{P}^\varepsilon$, we have

$$a\left(\mathbf{u}^\varepsilon,\mathbf{v}-\mathbf{u}^\varepsilon\right)+J^\varepsilon(\mathbf{v})-J^\varepsilon(\mathbf{u}^\varepsilon)\geq\int_{\Omega^\varepsilon}\mathbf{f}^\varepsilon(\mathbf{v}-\mathbf{u}^\varepsilon)dxdx_3,\forall\mathbf{v}\in V^\varepsilon. \tag{3.6}$$

As $J^\varepsilon(\mathbf{u}^\varepsilon)$ is positive, then

$$a\left(\mathbf{u}^\varepsilon,\mathbf{u}^\varepsilon\right)\leq a\left(\mathbf{u}^\varepsilon,\mathbf{v}\right)+J^\varepsilon(\mathbf{v})+\int_{\Omega^\varepsilon}\mathbf{f}^\varepsilon\mathbf{u}^\varepsilon dxdx_3-\int_{\Omega^\varepsilon}\mathbf{f}^\varepsilon\mathbf{v}dxdx_3,\forall\mathbf{v}\in V^\varepsilon. \tag{3.7}$$

By Korn's inequality, there exists a constant $C_k>0$ independent of ε, such that

$$a\left(\mathbf{u}^\varepsilon,\mathbf{u}^\varepsilon\right)\geq C_K\|\nabla\mathbf{u}^\varepsilon\|_{0,\Omega^\varepsilon}^2. \tag{3.8}$$

On the other hand we have

$$a\left(\mathbf{u}^\varepsilon,\mathbf{v}^\varepsilon\right)\leq\int_{\Omega^\varepsilon}\|\mathcal{E}(\epsilon(\mathbf{v})-\epsilon(\mathbf{u}^\varepsilon)\|_{\mathbb{S}_3}^2dxdx_3\leq\int_{\Omega^\varepsilon}L_\mathcal{E}\|\epsilon(\mathbf{v})-\epsilon(\mathbf{u}^\varepsilon)\|_{L(\Omega^\varepsilon)}^2dxdx_3.$$

We use the fact that $\sum_{i,j=1}^2|\epsilon(\mathbf{u}^\varepsilon)|^2\leq|\nabla\mathbf{u}^\varepsilon|^2$, then

$$a\left(\mathbf{u}^\varepsilon,\mathbf{v}^\varepsilon\right)\leq L_\mathcal{E}\left(\|\nabla\mathbf{v}\|_{0,\Omega^\varepsilon}^2+\|\nabla\mathbf{u}^\varepsilon\|_{0,\Omega^\varepsilon}^2\right).$$

Using Poincaré's inequality to obtain

$$\|\mathbf{u}^\varepsilon\|_{0,\Omega^\varepsilon}\leq\varepsilon h^\star\|\nabla\mathbf{u}^\varepsilon\|_{0,\Omega^\varepsilon},$$

then, similar to (3.8), we have

$$\left|\int_{\Omega^\varepsilon} \mathbf{f}^\varepsilon \mathbf{u}^\varepsilon dx dx_3\right| \le \frac{C_K}{2}\|\nabla \mathbf{u}^\varepsilon\|^2_{0,\Omega^\varepsilon} + \frac{(\varepsilon h^\star)^2}{2C_K}\|\nabla \mathbf{f}^\varepsilon\|^2_{0,\Omega^\varepsilon}, \tag{3.9}$$

$$\left|\int_{\Omega^\varepsilon} \mathbf{f}^\varepsilon \mathbf{v}^\varepsilon dx dx_3\right| \le \frac{C_K}{2}\|\nabla \mathbf{v}^\varepsilon\|^2_{0,\Omega^\varepsilon} + \frac{(\varepsilon h^\star)^2}{2C_K}\|\nabla \mathbf{f}^\varepsilon\|^2_{0,\Omega^\varepsilon}. \tag{3.10}$$

Using (3.8) – (3.10) and choosing $\mathbf{v} = G^\varepsilon$, the variational inequality (3.7) becomes

$$\begin{aligned} C_K\|\nabla \mathbf{u}^\varepsilon\|_{0,\Omega^\varepsilon} \le a(\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon) \le L_{\mathcal{E}}\left(\|\nabla G^\varepsilon\|^2_{0,\Omega^\varepsilon} + \|\nabla \mathbf{u}\|^2_{0,\Omega^\varepsilon}\right) \\ + \frac{C_K}{2}\|\nabla \mathbf{u}^\varepsilon\|^2_{0,\Omega^\varepsilon} + \frac{(\varepsilon h^\star)^2}{2C_K}\|\nabla \mathbf{f}^\varepsilon\|^2_{0,\Omega^\varepsilon} \\ + \frac{C_K}{2}\|\nabla \mathbf{G}^\varepsilon\|^2_{0,\Omega^\varepsilon} + \frac{(\varepsilon h^\star)^2}{2C_K}\|\nabla \mathbf{f}^\varepsilon\|^2_{0,\Omega^\varepsilon}. \end{aligned}$$

Assume that $L_{\mathcal{E}} < \dfrac{C_K}{2}$, then

$$\left(\frac{C_K}{2} - L_{\mathcal{E}}\right)\|\nabla \mathbf{u}^\varepsilon\|_{0,\Omega^\varepsilon} \le \frac{(\varepsilon h^\star)^2}{C_K}\|\nabla \mathbf{f}^\varepsilon\|^2_{0,\Omega^\varepsilon} + \left(L_{\mathcal{E}} + \frac{C_K}{2}\right)\|\nabla \mathbf{G}^\varepsilon\|^2_{0,\Omega^\varepsilon}. \tag{3.11}$$

Multiplying (3.11) by ε $(0 < \varepsilon < 1)$, we have

$$\begin{aligned} \varepsilon\left(\frac{C_K}{2} - L_{\mathcal{E}}\right)\|\nabla \mathbf{u}^\varepsilon\|_{0,\Omega^\varepsilon} \\ \le \frac{(\varepsilon h^\star)^2}{C_K}\|\nabla \hat{\mathbf{f}}\|^2_{0,\Omega} + \left(L_{\mathcal{E}} + \frac{C_K}{2}\right)\|(\nabla \hat{\mathbf{G}}\|^2_{0,\Omega}. \end{aligned}$$

Thus $\varepsilon\|\nabla \mathbf{u}^\varepsilon\|^2_{0,\Omega^\varepsilon} \le C$, where

$$C = \left(\frac{h^{\star 2}}{C_K}\left\|\nabla \hat{\mathbf{f}}\right\|^2_{L^2(\Omega^\varepsilon)} + \left(\frac{C_K}{2} + L_{\mathcal{E}}\right)\left\|\nabla \hat{G}\right\|^2_{L^2(\Omega^\varepsilon)}\right)\left(\frac{C_K}{2} - L_{\mathcal{E}}\right)^{-1}$$

$$\varepsilon\|\nabla \mathbf{u}^\varepsilon\|^2_{0,\Omega^\varepsilon} = \varepsilon^2\sum_{1\le i,j\le 2}\left\|\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j}\right\|^2_{L^2(\Omega)} + \varepsilon^2\left\|\frac{\partial \hat{u}_3^\varepsilon}{\partial z}\right\|^2_{L^2(\Omega)} + \sum_{1\le i\le 2}\left\|\frac{\partial \hat{u}_i^\varepsilon}{\partial z}\right\|^2_{L^2(\Omega)} + \varepsilon^4\left\|\frac{\partial \hat{u}_3^\varepsilon}{\partial x_i}\right\|^2_{L^2(\Omega)}.$$

Which completes the proof. □

Theorem 3.2. *Under the same assumptions of Theorem 3.1 and the inequality (3.5) hold, then there exists* $\hat{\mathbf{u}}^* = (\hat{u}_i^*), i = 1,2$ *in* V_z, *such that*

$$\hat{u}_i^\varepsilon \rightharpoonup \hat{u}_i^* \quad i = 1,2 \text{ weakly in } V_z, \tag{3.12}$$

$$\varepsilon\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} \rightharpoonup 0 \quad i,j = 1,2 \text{ weakly in } L^2(\Omega), \tag{3.13}$$

$$\varepsilon^2\frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \rightharpoonup 0 \quad i = 1,2 \text{ weakly in } L^2(\Omega), \tag{3.14}$$

$$\varepsilon\frac{\partial \hat{u}_3^\varepsilon}{\partial z} \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega), \tag{3.15}$$

$$\varepsilon\hat{u}_3^\varepsilon \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega). \tag{3.16}$$

Proof. The convergences of (3.12) – (3.16) are a direct result of inequalities (3.5).

To be able to pass to the limit in the Problem $\hat{\mathcal{P}}^\varepsilon$, we must prove the convergence of the integral term defined on ω. The following lemma is adapted for this case.

Lemma 3.1 ([2]). *There exists a subsequence of* $R(\sigma_\eta^\varepsilon(\hat{u}^\varepsilon))$ *converging strongly towards* $R(\sigma_\eta^\varepsilon(\hat{u}^*))$ *in* $L^2(\omega)$.

4. Main results and limit problem

Theorem 4.1. *With the same assumptions of Theorem 3.2, the solution* $\hat{u}^*$ *satisfy*

$$\hat{u}_i^\varepsilon \rightharpoonup \hat{u}_i^* \quad \textit{for } i=1,2 \textit{ strongly in } \; V_z, \tag{4.1}$$

$$-\lambda \frac{\partial^2 \hat{u}_i^\star}{\partial z^2} = \hat{f}_i \quad \textit{for } i=1,2, \text{ in } L^2(\Omega)^2, \; \lambda \in \mathbb{R} \tag{4.2}$$

$$\frac{1}{2}\sum_{i=1}^{2}\int_\Omega \mathcal{E}\left(\frac{\partial \hat{u}_i^\star}{\partial z}\right)\frac{\partial}{\partial z}(\hat{v}_i - \hat{u}_i^\star)dxdz + \int_\omega \widehat{\mathcal{F}}\left|R(\sigma_\eta^*(\hat{u}^*))\right|(|\hat{v}-s| - |\hat{u}^* - s|)dx$$
$$\geq \sum_{i=1}^{2}\int_\Omega \hat{f}_i(\hat{v}_i - \hat{u}_i^\star)dxdz\,, \quad \forall \hat{v}\in \Pi\,(V)\,. \tag{4.3}$$

Proof. The proof of the strong convergence of $(\hat{u}_1^\varepsilon, \hat{u}_2^\varepsilon)$ to $(\hat{u}_1^*, \hat{u}_2^*)$ in V_z using the same techniques from ([7], Proof of Theorem 4.2). By the convergence of Theorem 3.2 and as J is convex and lower semi-continus, the inequality (3.4) became

$$\frac{1}{2}\sum_{i=1}^{2}\int_\Omega \mathcal{E}\left(\frac{\partial \hat{u}_i^\star}{\partial z}\right)\frac{\partial}{\partial z}(\hat{v}_i - \hat{u}_i^\star)dxdz + \int_\omega \widehat{\mathcal{F}}\left|R(\sigma_\eta^*(\hat{u}^*))\right|(|\hat{v}-s| - |\hat{u}^* - s|)dx$$
$$\geq \sum_{i=1}^{2}\left(\hat{f}_i, \hat{v}_i - \hat{u}_i\right). \tag{4.4}$$

From ([6]; Lemma 5.3), we can choose $\hat{v}_i$ in (4.4) such that

$$\hat{v}_i = \hat{u}_i^\star \pm \varphi_i,\; \varphi_i \in H_0^1(\Omega)\; i=1,2$$

we get

$$\frac{1}{2}\sum_{i=1}^{2}\int_\Omega \mathcal{E}\left(\frac{\partial \hat{u}_i^\star}{\partial z}\right)\frac{\partial \varphi_i}{\partial z}dxdz = \sum_{i=1}^{2}\int_\Omega \hat{f}_i\varphi_i dxdz.$$

Now, by the Green formula, we deduce

$$-\frac{1}{2}\sum_{i=1}^{2}\int_\Omega \mathcal{E}\left(\frac{\partial \hat{u}_i^\star}{\partial z}\right)\varphi_i dxdz = \sum_{i=1}^{2}\int_\Omega \hat{f}_i\varphi_i dxdz.$$

Using the fact that $\mathcal{E}$ is linear, then there exists a real λ such that

$$-\lambda\frac{\partial^2 \hat{u}_i^\star}{\partial z^2} = \hat{f}_i \;\text{ in } H^{-1}(\Omega)\,.$$

We know that $\hat{f}_i \in L^2(\Omega)$, then (4.2) is true in $L^2(\Omega)$. □

Theorem 4.2. *Under the assumptions of the previous Theorem 4.1, we have*

$$\int_\omega \widehat{\mathcal{F}}\left|R(\sigma_\eta^*(s^*))\right||(|\varphi + s^* - s| - |s^* - s|)dx - \lambda\int_\omega \tau^*\varphi dx \geq 0, \forall \varphi \in L^2(\omega)^2 \tag{4.5}$$

$$\begin{cases} \lambda|\tau^*| < \widehat{\mathcal{F}}\left|R(\sigma_\eta^*(s^*))\right| \Rightarrow s^* = s, \\ \lambda|\tau^*| = \widehat{\mathcal{F}}\left|R(\sigma_\eta^*(s^*))\right| \Rightarrow \exists\beta \geq 0, \;\textit{such that}\, s^* = s + \beta\lambda\tau^*, \end{cases} \quad \textit{on } \;\omega, \tag{4.6}$$

where

$$s^\star = \hat{\mathbf{u}}^\star(x,0), \qquad \tau^* = \frac{\partial \hat{\mathbf{u}}^\star}{\partial z}(x,0).$$

Also the limit function $\mathbf{\hat{u}}^\star$ *satisfies the weak generalized equation*

$$\int_\omega \left(\tilde{F} + \lambda \int_0^h \mathbf{\hat{u}}^\star(x,z)dz \right) \nabla\varphi(x)dx = 0, \ \forall \varphi \in H^1(\omega) \tag{4.7}$$

where

$$\tilde{F}(x) = \int_0^h F(x,z)dz - hF(x,h) \ \textit{and} \ F(x,z) = \int_0^z \int_0^s \hat{f}_i(x,\theta)d\theta ds.$$

Proof. The inequality (3.4) can be written as

$$\begin{aligned}
&\sum_{i,j=1}^{2} \int_\Omega \mathcal{E} \left(\frac{1}{2} \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i} \right) \frac{\partial}{\partial x_j} \left(v_i - \hat{u}_i^\varepsilon \right) dxdz \right) \\
&+ \sum_{i=1}^{2} \int_\Omega \mathcal{E} \left(\frac{1}{2} \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \right) \left[\frac{\partial}{\partial z} \left(v_i - \hat{u}_i^\varepsilon \right) + \varepsilon^2 \frac{\partial}{\partial x_i} \left(v_3 - \hat{u}_3^\varepsilon \right) \right] dxdz \right) \\
&+ \varepsilon^2 \int_\Omega \mathcal{E} \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \frac{\partial}{\partial z} \left(v_3 - \hat{u}_3^\varepsilon \right) dxdz + \int_\omega \widehat{\mathfrak{F}} \left| R(\sigma_\eta^*(s^*)) \right| \left(|\varphi + s^* - s| - |s^* - s| \right) dx - \lambda \int_\omega \tau^* \varphi dx \\
&\geq \sum_{i=1}^{2} \left(\hat{f}_i, \hat{v}_i - \hat{u}_i \right) + \varepsilon \left(\hat{f}_3, \hat{v}_3 - \hat{u}_3 \right).
\end{aligned} \tag{4.8}$$

Passing to the limit, then using the Green formula and we choose in the variational formulation (4.8), $\hat{v}_i = \hat{u}_i^\star + \varphi_i$, where $\varphi_i \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega), i = 1,2$, with

$$H^1_{\Gamma_1 \cup \Gamma_L}(\Omega) = \{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1 \cup \Gamma_L \},$$

we obtain

$$\begin{aligned}
&-\lambda \sum_{i=1}^{2} \int_\Omega \frac{\partial^2 \hat{u}_i^\star}{\partial z^2} \varphi_i dxdz + \int_\omega \widehat{\mathfrak{F}} \left| R(\sigma_\eta^*(s^*)) \right| \left(|\varphi + s^* - s| - |s^* - s| \right) dx - \lambda \int_\omega \tau^* \varphi dx \\
&\geq \sum_{i=1}^{2} \int_\Omega \hat{f}_i \varphi_i dxdz.
\end{aligned}$$

On the other hand, from (4.2) we deduce that for $\varphi = (\varphi_1, \varphi_2) \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega)^2$

$$\int_\omega \widehat{\mathfrak{F}} \left| R(\sigma_\eta^*(s^*)) \right| \left(|\varphi + s^* - s| - |s^* - s| \right) dx - \lambda \int_\omega \tau^* \varphi dx \geq 0. \tag{4.9}$$

Inequality (4.9) holds also for all $\varphi \in D(\omega)^2$ and by the fact that $D(\omega)$ is dens in $L(\omega)$ we find (4.5). The proof of (4.6) is similar to those given in case of the problem of fluid-solid with Coulomb law ([7]. For the rest of proof, we integrate twice (4.2) between 0 and z, we obtain

$$\begin{aligned}
\int_0^z \int_0^\xi \hat{f}_i(x,\alpha)\, d\alpha d\xi &= -\lambda \int_0^z \int_0^\xi \frac{\partial^2 \hat{u}_i^\star}{\partial z^2}(x,\alpha)\, d\alpha d\xi \\
&= -\lambda \int_0^z \left[\frac{\partial \hat{u}_i^\star}{\partial \xi}(x,\xi) - \frac{\partial \hat{u}_i^\star}{\partial \xi}(x,0) \right] d\xi \\
&= -\lambda \left(\hat{u}_i^\star(x,z) - \hat{u}_i^\star(x,0) - z\tau^\star \right),
\end{aligned}$$

then

$$\hat{u}_i^\star(x,z) = s^\star(x,0) + z\tau^\star - \frac{1}{\lambda} \int_0^z \int_0^\xi \hat{f}_i(x,\alpha)\, d\alpha d\xi. \tag{4.10}$$

As $\hat{u}_i^\star(x, h(x)) = 0$, we have

$$s^\star(x,0) + h\tau^\star = \frac{1}{\lambda}\int_0^h \int_0^\xi \hat{f}_i(x,\alpha)\, d\alpha d\xi. \tag{4.11}$$

Integrating (4.10) between 0 and h give

$$\int_0^h \hat{u}_i^\star(x,z)\,dz = \int_0^h s^\star(x,0)\,dz + \int_0^h z\tau^\star dz - \frac{1}{\lambda}\int_0^h \int_0^z \int_0^\xi \hat{f}_i(x,\alpha)\,d\alpha d\xi dz,$$

$$\int_0^h \hat{u}_i^\star(x,z)\,dz = hs^\star(x,0) + \frac{1}{2}h^2\tau^\star - \frac{1}{\lambda}\int_0^h \int_0^z \int_0^\xi \hat{f}_i(x,\alpha)\,d\alpha d\xi dz,$$

we deduce that

$$\int_0^h \hat{u}_i^\star(x,z)\,dz + \widetilde{\mathcal{F}}(x) - \frac{1}{2}h^2\tau^\star = 0,$$

so, we give (4.7). □

Theorem 4.3. *Under the assumptions of Theorem 4.1, there exists a positive constant $\mathcal{F}^*$ (sufficiently small), such that for $\left\|\widehat{\mathcal{F}}\right\|_{L^\infty(\omega)} \le \mathcal{F}^*$ then the solution of the limit problem (4.2) is unique in V_z.*

Proof. Let $\hat{\mathbf{u}}_1^\star$ and $\hat{\mathbf{u}}_2^\star$ be two solutions of the problem (4.2). For any $\hat{v}_i \in \Pi(V)$, we have

$$\frac{1}{2}\sum_{i=1}^2 \int_\Omega \mathcal{E}\left(\frac{\partial \hat{u}_{1i}^\star}{\partial z}\right)\frac{\partial}{\partial z}(\hat{v}_i - \hat{u}_{1i}^\star)dxdz + \int_\omega \widehat{\mathcal{F}}\left|R(\sigma_\eta^*(\hat{\mathbf{u}}_1^\star))\right|(|\hat{v}-s| - |\hat{\mathbf{u}}_1^\star - s|)dx$$
$$\ge \sum_{i=1}^2 \int_\Omega \hat{f}_i(\hat{v}_i - \hat{u}_{1i}^\star)dxdz$$

$$\frac{1}{2}\sum_{i=1}^2 \int_\Omega \mathcal{E}\left(\frac{\partial \hat{u}_{2i}^\star}{\partial z}\right)\frac{\partial}{\partial z}(\hat{v}_i - \hat{u}_{2i}^\star)dxdz + \int_\omega \widehat{\mathcal{F}}\left|R(\sigma_\eta^*(\hat{\mathbf{u}}_2^\star))\right|(|\hat{v}-s| - |\hat{\mathbf{u}}_2^\star - s|)dx$$
$$\ge \sum_{i=1}^2 \int_\Omega \hat{f}_i(\hat{v}_i - \hat{u}_{2i}^\star)dxdz$$

Taking $\hat{v}_i = \hat{u}_{2i}^\star$ in the first and $\hat{v}_i = \hat{u}_{1i}^\star$ in the second equation then summing the two inequalities above taking in mind that $\mathcal{E}$ is linear, we get :

$$\frac{1}{2}\sum_{i=1}^2 \int_\Omega \mathcal{E}\left(\frac{\partial \hat{u}_{1i}^\star}{\partial z}\right)\frac{\partial}{\partial z}(\hat{u}_{2i}^\star - \hat{u}_{1i}^\star)dxdz + \frac{1}{2}\sum_{i=1}^2 \int_\Omega \mathcal{E}\left(\frac{\partial \hat{u}_{2i}^\star}{\partial z}\right)\frac{\partial}{\partial z}(\hat{u}_{1i}^\star - \hat{u}_{2i}^\star)dxdz$$
$$+ \int_\omega \widehat{\mathcal{F}}\left(\left|R(\sigma_\eta^*(\hat{\mathbf{u}}_2^\star))\right| - \left|R(\sigma_\eta^*(\hat{\mathbf{u}}_1^\star))\right|\right)(|\hat{\mathbf{u}}_1^\star - s| - |\hat{\mathbf{u}}_2^\star - s|)\,dx$$
$$\ge 0,$$

then

$$\frac{1}{2}\sum_{i=1}^2 \int_\Omega \mathcal{E}\left|\frac{\partial}{\partial z}(\hat{u}_{1i}^\star - \hat{u}_{2i}^\star)\right|^2 dxdz \le \int_\omega \widehat{\mathcal{F}}\left(\left|R(\sigma_\eta^*(\hat{\mathbf{u}}_2^\star))\right| - \left|R(\sigma_\eta^*(\hat{\mathbf{u}}_1^\star))\right|\right)|\hat{\mathbf{u}}_1^\star - \hat{\mathbf{u}}_2^\star|dx,$$

this implies

$$\|\mathcal{E}\|_{\mathbf{Q}\infty}\left\|\frac{\partial}{\partial z}\left(\hat{\mathbf{u}}_1^{\star}-\hat{\mathbf{u}}_2^{\star}\right)\right\|_{0,\Omega}^2\leq\int_{\omega}\widehat{\mathcal{F}}\left(\left|R(\sigma_\eta^*(\hat{\mathbf{u}}_2^{\star}))\right|-\left|R(\sigma_\eta^*(\hat{\mathbf{u}}_1^{\star}))\right|\right)|\hat{\mathbf{u}}_1^{\star}-\hat{\mathbf{u}}_2^{\star}|dx.$$

By Poincaré's inequality, we find

$$\|\mathcal{E}\|_{\mathbf{Q}\infty}\left\|\frac{\partial}{\partial z}\left(\hat{\mathbf{u}}_1^{\star}-\hat{\mathbf{u}}_2^{\star}\right)\right\|_{0,\Omega}^2\leq(h^*)^2\left\|\widehat{\mathcal{F}}\right\|_{L^\infty(\omega)}\left(\int_{\omega}\left|R(\sigma_\eta^*(\hat{\mathbf{u}}_2^{\star}))-R(\sigma_\eta^*(\hat{\mathbf{u}}_1^{\star}))\right|^2dx\right)^{\frac{1}{2}}\|\hat{\mathbf{u}}_1^{\star}-\hat{\mathbf{u}}_2^{\star}\|_{0,\omega}.$$

As $\left\|\widehat{\mathcal{F}}\right\|_{L^\infty(\omega)}\leq\mathcal{F}^*$, then we have $\|\hat{\mathbf{u}}_1^{\star}-\hat{\mathbf{u}}_2^{\star}\|_{V_z}=0$. □

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