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Some Fixed Point Theorems in Generalized M-Fuzzy Metric Space

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ABSTRACT: In this paper, we define the expansive mapping in G-metric space and we prove some fixed point theorems in generalized M-fuzzy (GM-fuzzy) metric space.

Key Words: Fuzzy metric space, G-metric space, GM-fuzzy metric space, expansive mapping.

Contents

1. Introduction

The theory of fuzzy sets was introduced by Zadeh [18]. Thereafter the introduced notion has evolved in many directions of science and technology, where mathematics has a role. It has been studied by Tripathy and Borgohain [11], Tripathy and Duta [12] for studying the properties of sequences of fuzzy numbers, Tripathy and Ray [16] for studying fuzzy topological spaces, Deb and Saha [1], Dhange [2], Mustafa et. al [5], Sedghi et.al. [9], Sun and Yang [10], Tripathy et. al ([13], [14], [15]), Wang [17] and others for studying fixed point theory in fuzzy settings. Different researcher have interpreted and introduced the concept of fuzzy metric space in different ways. George and Veeramani [3] modified the concept of a fuzzy metric space introduced by Kramosil and Michalek [4] and defined a Hausdorff topology on this fuzzy metric space.

The study of fixed points of a function satisfying certain contractive conditions has been at the center of rigorous research activity. Mustafa and Sims [7] generalized the concept of a metric space. Based on the notion of generalized metric spaces, Mustafa et. al [8] obtained some fixed point theorems for mappings satisfying different contractive conditions.

2. Preliminaries and Definitions

Definition 2.1 A fuzzy set M on an arbitrary set X is a function with domain X and range in [0,1].

Definition 2.2 A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if $([0,1], *)$ is an abelian topological monoid with unit 1 such that $a_1 * b_1 \le a_2 * b_2$ whenever $a_1 \le a_2, b_1 \le b_2$ for all $a_1, a_2, b_1, b_2 \in [0,1]$.

Examples of t-norm
(1) Minimum t-norm $(*M) : *M(x,y) = min\{x,y\}$.
(2) Product t-norm $(*P) : *P(x,y) = x.y$.
(3) Lukasiewicz t-norm $(*L) : *L(x,y) = max\{x+y-1,0\}$.

Definition 2.3 Let X be a non-empty set and let $G : X \times X \times X \to R^+$, be a function satisfying the following properties:
$(G_1) G(x,x,y) > 0$, for all $x, y \in X$, with $x \neq y$;
$(G_2) G(x,y,z) = 0$, if $x = y = z$;

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$(G_3) G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z = y$;
$(G_4) G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$;
(Symmetry in all three variables)
$(G_5) : G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$;
(rectangle inequality).

Then the function G is called Generalized metric or more specifically G-metric on X, and the pair (X,G) is called a G-metric space.

Definition 2.4 The 3-tuple $(X,M,*)$ is called a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in $X^2 \times (0,\infty)$ satisfying the following conditions, for all $x,y,z \in X$ and $t_1,t_2,t > 0$,
(1) $M(x,y,0) = 0$;
(2) $M(x,y,t) = 1$ if and only if $x = y$;
(3) $M(x,y,t) = M(y,x,t)$;
(4) $M(x,y,t_1+t_2) \geq M(x,z,t_1) * M(z,y,t_2)$;
(5) $M(x,y,.) : (0,\infty) \to [0,1]$ is continuous.

Then M is called fuzzy metric on X and $(X,M,*)$ is called fuzzy metric space and $M(x,y,t)$ denotes the degree of nearness between x and y.

Definition 2.5 A 3-tuple $(X,M,*)$ is said to be a Generalized $M(GM)$-fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^3 \times (0,\infty)$ satisfying the following conditions for each $t,s > 0$:
$(M1) M(x,x,y,t) > 0$ for all $x,y \in X$ with $x \neq y$;
$(M2) M(x,x,y,t) \geq M(x,y,z,t)$ for all $x,y,z \in X$ with $y \neq z$;
$(M3) M(x,y,z,t) = 1$ if and only if $x = y = z$;
$(M5) M(x,a,a,t) * M(a,y,z,s) \leq M(x,y,z,t+s)$; (the triangle inequality)
$(M6) M(x,y,z,.) : (0,\infty) \to [0,1]$ is continuous.

A GM-fuzzy metric space is said to be symmetric if $M(x,y,y,t) = M(x,x,y,t)$ for all $x,y \in X$ and $t > 0$.

Example 2.1 Let X be a non-empty set and G be the G-metric on X.
Denote $a * b = a.b$ for all $a,b \in [0,1]$, For each $t > 0$:

$$M(x,y,z,t) = \frac{t}{t + G(x,y,z)}.$$

Then $(X,M,*)$ is a GM-fuzzy metric space.

Definition 2.6 Let $(X,M,*)$ be a GM-fuzzy metric space. Then
(a) A sequence $\{x_n\}$ in X is said to coverage to x if and only if
$M(x_m,x_n,x,t) \to 1$, as $n \to \infty$, $m \to \infty$, for each $t > 0$.
(b) A sequence $\{x_n\}$ in X is said to be a G-Cauchy sequence if $M(x_m,x_n,x_l,t) \to 1$ as $m \to \infty, n \to \infty, l \to \infty$ for each $t > 0$.
(c) A GM-fuzzy metric space in which every Cauchy sequence is convergent is said to be G-complete.

Lemma 2.7 If $(X,M,*)$ be a GM-fuzzy metric space, then $M(x,y,z,t)$ is non-decreasing with respect to t for all $x,y,z \in X$.

Through out this article we assume that $\underset{n\to\infty}{lim} M(x_n,y,z,t) = 1$ and that N is the set of all natural numbers and that R^+ is the set of all positive real numbers.

Lemma 2.8. Let $(X,M,*)$ be a GM-fuzzy metric space. Then the following properties are equivalent:

1) $\{x_n\}$ is convergent to x.
2) $M(x_n, x_n, x, t) \to 1$, as $n \to \infty$.
3) $M(x_n, x, x, t) \to 1$, as $n \to \infty$.
4) $M(x_m, x_n, x, t) \to 1$, as $m, n \to \infty$.

Lemma 2.9. Let $(X, M, *)$ be a GM-fuzzy metric space, then the following are equivalent:

1) The sequence $\{x_n\}$ is G-Cauchy.
2) For every $\varepsilon \in (0, 1)$ and $t > 0$, there exists $k \in N$ such that
$M(x_n, x_m, x_m, t) > 1 - \varepsilon$ for $n, m \geq k$.

Definition 2.10. Let $(X, M, *)$ be a GM-fuzzy metric space. The following conditions are satisfied:
$\lim_{n\to\infty} M(x_n, y_n, z_n, t_n) = M(x, y, z, t)$,
whenever $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, $\lim_{n\to\infty} z_n = z$
and $\lim_{n\to\infty} M(x, y, z, t_n) = M(x, y, z, t)$,
then M is called a continuous function on $X^3 \times (0, \infty)$.

Lemma 2.11 Let $(X, M, *)$ be a GM-fuzzy metric space. Then M is a continuous function on $X^3 \times (0, \infty)$.

Lemma 2.12 Let $(X, M, *)$ be a complete GM- fuzzy metric space and $T : X \to X$ be a mapping satisfies the following conditions for all $x, y, z \in X$ and $t > 0$,

$$kM(Tx, Ty, Tz, t) \geq M(x, y, z, t), \text{ where } k \in [0, 1) \tag{2.1}$$

Lemma 2.13 Let $(X, M, *)$ be a complete GM-fuzzy metric space and $T : X \to X$ be a mapping satisfies the following conditions for all $x, y \in X$ and $t > 0$

$kM(Tx, Ty, Ty, t) \geq M(x, y, y, t)$,
where $k \in [0, 1)$. Then T has a unique fixed point.

Definition 2.14 Let $(X, M, *)$ be a GM-fuzzy metric space and T be a self mapping on X. Then T is called expansive mapping if there exists a constant $a \geq 1$, such that for all $x, y, z \in X$ and $t > 0$, we have

$M(Tx, Ty, Tz, t) \geq aM(x, y, z, t)$.

3. Main Results

Theorem 3.1 Let $(X, M, *)$ be a complete GM-fuzzy metric space. If there exists a constant $a \leq 1$ and a onto self mapping T on X, such that for all $x, y, z \in X$ and $t > 0$,

$$M(Tx, Ty, Tz, t) \leq aM(x, y, z, t). \tag{3.1}$$

Then T has a unique fixed point.

Proof. Under the assumption, if $Tx = Ty$, then
$1 = M(Tx, Ty, Ty, t) \leq aM(x, y, y, t)$.

Which implies $M(x, y, y, t) = 1 \Rightarrow x = y$.
Hence, T is injective and invertible.

Let h be the inverse mapping of T,
then $M(x, y, z, t) = M(T(hx), T(hy), T(hz), t) \leq aM(hx, hy, hz, t)$.
Thus, for all $x, y, z \in X$ and $t > 0$.

we have, $aM(hx, hy, hz, t) \geq M(x, y, z, t)$.

Applying, Lemma 2.12, we conclude that inverse mapping h has a unique fixed point $u \in X$; $h(u) = u$.
But, $u = T(h(u)) = T(u)$.
This gives that u is also a fixed point of T.

Suppose there exists another fixed point $v \neq u$ such that $Tv = v$.
Then, $Tv = v = T(h(v)) = h(T(v))$.

So, Tv is another fixed point of h.

By uniqueness, we conclude that $u = Tv = v$, which implies that u is a unique fixed point of T.

Theorem 3.2 Let$(X, M, *)$ be a complete GM-fuzzy metric space. If there exists a constant $c \leq 1$ and a surjective self mapping T on X, such that for all $x, y \in X$ and $t > 0$,
$M(Tx, Ty, Ty, t) \leq cM(x, y, y, t)$,
Then T has a unique fixed point.

Proof. Under the assumption, if $Tx = Ty$,
then $1 = M(Tx, Tx, Ty, t) \leq cM(x, x, y, t)$
Which implies $M(x, x, y, t) = 1$.
$\Rightarrow x = y$
and hence T is invertible.

Let h be the inverse mapping of T,
So, $M(x, y, y, t) = M(T(hx), T(hy), T(hy), t) \leq cM(hx, hy, hy, t)$.
Then, for all $x, y \in X$, we have
$cM(hx, hy, hy, t) \geq M(x, y, y, t)$.
Applying Lemma 2.12 on the inverse mapping h, and use argument similar to that in Proof Theorem 3.1, we conclude that T has unique fixed point.

Corollary 3.3. Let $(X, M, *)$ be a complete GM-fuzzy metric space. If there exists a constant $k \leq 1$ and surjective self mapping on X, such that for all $x, y, z \in X$ and $t > 0$.

$$M(Tx, Ty, Tz, t) \leq k\{M(x, z, z, t) * M(y, z, z, t)\}. \tag{3.2}$$

Then T has a unique fixed point.

Proof. The proof follows from Theorem 3.2 by taking $z = y$ in condition (3.2).

Theorem 3.4 Let $(X, M, *)$ be a complete GM-fuzzy metric space and let $T : X \to X$be a surjective mapping satisfying the following condition for all $x, y, z \in X$ and $t > 0$,

$$M(T(x), T(y), T(z), t) \leq k\ max\{(M(x, z, z, t/2) * M(y, z, z, t/2)), (M(z, y, y, t/2)$$
$$*M(x, y, y, t/2)), (M(z, x, x, t/2) * M(y, x, x, t/2)), \tag{3.3}$$

where $k \leq 1$. Then T has a unique fixed point.

Proof. Condition (3.3) implies T is injective and therefore invertible.
Let h be the inverse mapping of T.
By condition (4), for all $x, y, z \in X, t > 0$ We have,

$$M(x, y, z, t) = M(T(hx), T(hy), T(hz), t)$$
$$\leq k\ max\{(M(hx, hz, hz, t/2) * M(hy, hz, hz, t/2)), (M(hz, hy, hy, t/2)$$
$$*M(hx, hy, hy, t/2)), (M(hz, hx, hx, t/2) * M(hy, hx, hx, t/2))\} \tag{3.4}$$

By $(M4)$, we have

$$Max\{(M(hx,hz,hz,t/2)*M(hy,hz,hz,t/2)),(M(hz,hy,hy,t/2)*M(hx,hy,hy,t/2)),$$
$$(M(hz,hx,hx,t/2)*M(hy,hx,hx,t/2))\} \leq M(hx,hy,hz,t). \tag{3.5}$$

Thus equation (3.4) implies

$$kM(hx,hy,hz,t) \geq M(x,y,z,t). \tag{3.6}$$

Applying, Theorem 3.1 with the help of (3.6).
We conclude that the inverse mapping h has a unique fixed point $u \in X$ Such that $h(u) = u$.
But $u = T(h(u)) = T(u)$,
Which shows that u is also a fixed point of T.
To show u is unique fixed point, we can use the same argument in Theorem 3.4.

Theorem 3.5: Let $(X, M, *)$ be a complete non symmetric GM-fuzzy metric space and let $T : X \to X$ be a surjective mapping satisfying the following condition for all $x, y \in X, t > 0$,

$$M(T(x),T(y),T(y),t) \leq kmax\{M(x,y,y,t),M(y,x,x,t)\}. \tag{3.7}$$

When $k \leq 1$. Then T has a unique fixed point.

Proof: Since $Max\{M(x,y,y,t),M(y,x,x,t)\} \leq M(x,y,y,t)$,
then from (3.7), we deduce

$$M(T(x),T(y),T(y),t) \leq kM(x,y,y,t) \text{ for all } x,y \in X, t > 0. \tag{3.8}$$

From (3.8), it is clear that Theorem 3.2 implies that T has a unique fixed point.

Corollary 3.6: Let $(X, M, *)$ be a complete non-symmetric GM-fuzzy metric space, and let $T : X \to X$ be a surjective mapping satisfying the following condition for all $x, y, z \in X$, $t > 0$,
$M(T(x),T(y),T(z),t) \leq kmax\{(M(x,y,y,t/2)*M(y,x,x,t/2)),$
$(M(x,z,z,t/2)*M(z,x,x,t/2))(M(z,y,y,t/2)*(M(y,z,z,t/2))\}$, when $k \leq 1$. Then T has a unique fixed point.

Proof: Follows from the Theorem 3.5 on taking $z = y$.

Corollary 3.7: Let $(X, M, *)$ be a complete GM-fuzzy metric space and let $T : X \to X$ be a surjective mapping satisfying the following condition for all $x, y, z \in X$, $t > 0$,

$$M(T(x),T(y),T(z),t) \leq k\{M(x,Tx,Tx,t/2)*M(Tx,y,z,t/2)\}, \tag{3.9}$$

where $k \leq 1$. Then T has a unique fixed point.

Proof: From (M_4), we have
$M(x,Tx,Tx,t/2)*M(Tx,y,z,t/2) \leq M(x,y,z,t)$.
Then condition (10) becomes
$M(T(x),T(y),T(z),t) \leq kM(x,y,z,t)$ for all $x, y, z \in X$ and the proof follows from (3.1).

Theorem 3.8: Let $(X, M, *)$ be a complete GM-fuzzy metric space and $T : X \to X$ be an onto and continuous mapping satisfying the followings condition for all $x \in X$ and $t > 0$,

$$M(T(x),T^2(x),T^3(x),t) \leq aM(x,Tx,T^2x,t). \tag{3.10}$$

Where $a \leq 1$. Then T has a fixed point.

Proof: : Let $x_0 \in X$, since T is onto, so there exists an element x_1 satisfying $x_1 \in T^{-1}(x_0)$. By the same argument we can pick up $x_n \in T^{-1}(x_{n-1})$ where $n = 2, 3, 4, 5, \ldots$

Let $x_n \neq x_{n-1}$, then there is a sequence x_n with $x_n \neq x_{n-1}$ and $T(x_n) = x_{n-1}$.

Then (3.10) implies

$$M(x_{n-1}, x_{n-2}, x_{n-3}, t) = M(Tx_n, T^2x_n, T^3x_n, t) \leq aM(x_n, Tx_n, T^2x_n, t)$$

$$= aM(x_n, x_{n-1}, x_{n-2}, t). \tag{3.11}$$

Therefore, we have

$M(x_n, x_{n-1}, x_{n-2}, t) \geq \frac{1}{a} M(x_{n-1}, x_{n-2}, x_{n-3}, t)$.

Let $q = \frac{1}{a}$, then $q \geq 1$.

It can be easily verified that the sequence $\{x_n\}$ is a Cauchy and by completeness of $(X, M, *)$, the sequence $\{x_n\}$ converges to a point $u \in X$.

Since T is continuous, then

$T(x_n) = x_{n-1} \to T(u)$ as $n \to \infty$.

Hence, $T(u) = u$, which shows that u is a fixed point of T.

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Competing Interest.. The authors declare that the article is free from the competing interest.

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