Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–8.
ISSN-0037-8712 in press
doi:10.5269/bspm.51806

On the Maximum Principle for the Discrete p-Laplacian with Sign-changing Weight

Hamza Chehabi, Omar Chakrone and Mohammed Chehabi

ABSTRACT: This work deals with the maximum principle for the discrete Neumann or Dirichlet problem

$$-\Delta\varphi_p(\Delta u(k-1)) = \lambda m(k)|u(k)|^{p-2}u(k) + h(k) \quad \text{in} \quad [1, n].$$

We study the existence and nonexistence of positive solution and its uniqueness.

Key Words: Difference equations, discrete p-Laplacian, variational methods, maximum principle, discrete Picone's identity.

Contents

1. Introduction

This paper is concerned with the Neumann or Dirichlet problem

$$-\Delta\varphi_p(\Delta u(k-1)) = \lambda m(k)|u(k)|^{p-2}u(k) + h(k) \quad \text{in} \quad [1, n],$$

where n is an integer greater than or equal to 1, $[1, n]$ is the discrete interval $\{1, ..., n\}$, $\Delta u(k) := u(k+1) - u(k)$ is the forward difference operator, $\varphi_p(s) = |s|^{p-2}\, s$, $1 < p < \infty$, h function defined on $[1, n]$ and m changes sign in $[1, n]$. The original form for the maximum principle concerns the continuous problem

$$-\Delta_p u = \lambda m(x)\, |u|^{p-2}\, u + h(x) \text{ in } \Omega,\ Bu = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\Delta_p u := div(|\nabla u|^{p-2}\, \nabla u)$ is the p-Laplacian and $Bu = 0$ represents either the Dirichlet or the Neumann homogeneous boundary conditions (see [7,1]).

The argument here uses a discrete forme of Picone's identity (see [5]). Some of our arguments are inspired by [4,8]. We study the existence and nonexistence of positive solution and its uniqueness depending on the sign of $\sum_{k=1}^{n} m(k)$ and on whether or not λ belongs to $]0, \mu(m)[$ in the Neumann case, and depending whether or not λ belongs to $]\lambda_{-1}(m), \lambda_1(m)[$ in the Dirichlet case, where $\mu(m)$, $\lambda_1(m)$ and $\lambda_{-1}(m)$ are defined in (2.7) and (3.3).

2. Principal eigenvalues in the Neumann case

Consider the Neumann problem

$$\begin{cases} -\Delta\varphi_p(\Delta u(k-1)) = \lambda m(k)|u(k)|^{p-2}u(k) + h(k) \quad \text{in} \quad [1, n], \\ \Delta u(0) = \Delta u(n) = 0. \end{cases} \tag{2.1}$$

Suppose that

$$\exists k_1,\ k_2 \in [1, n]\ ;\ m(k_1)m(k_2) < 0. \tag{2.2}$$

Also, without loss of generality, we can assume that

$$|m(k)| < 1, \quad \forall k \in [1, n]. \tag{2.3}$$

2010 *Mathematics Subject Classification:* 35-XX, 39-XX.
Submitted January 14, 2020. Published October 12, 2020

The class $W=\{u:[0,n+1]\to\mathbb{R}\ ;\ \Delta u(0)=\Delta u(n)=0\}$ is an n-dimensional space under the norm $\|u\|=\left(\sum_{k=1}^{n}|u(k)|^p\right)^{1/p}$.

Solution of (2.1) (or of (2.6)) are exactly the solutions in the sense: $u\in W$ with

$$\sum_{k=1}^{n}\varphi_p(\Delta u(k-1))\Delta v(k-1)=\lambda\sum_{k=1}^{n}m(k)|u(k)|^{p-2}u(k)v(k)+\sum_{k=1}^{n}h(k)v(k),\quad \forall v\in W. \tag{2.4}$$

Our purpose in this preliminary section is to collect some results relative to the principal eigenvalues of

$$\begin{cases} -\Delta\varphi_p(\Delta u(k-1))=\lambda m(k)|u(k)|^{p-2}u(k) \quad \text{in} \quad [1,n],\\ \Delta u(0)=\Delta u(n)=0. \end{cases} \tag{2.5}$$

The fundamental tool is the following form of the maximum principle.

Proposition 2.1. *(see [3]) Let u be a solution of*

$$\begin{cases} -\Delta\varphi_p(\Delta u(k-1))+a_0(k)|u(k)|^{p-2}u(k)=h(k) \quad \textit{in} \quad [1,n],\\ \Delta u(0)=\Delta u(n)=0, \end{cases} \tag{2.6}$$

where $a_0\geq 0$ and $h\gneqq 0$. Then $u>0$ in $[1,n]$.

Proof. Writing $u=u^+-u^-$ with $u^\pm=\max\{\pm u,0\}$ and taking $-u^-$ as testing function in (2.6),

$$-\sum_{k=1}^{n}\varphi_p(\Delta u(k-1))\Delta u^-(k-1)+\sum_{k=1}^{n}a_0(k)|u^-(k)|^p=-\sum_{k=1}^{n}h(k)u^-(k).$$

Distinguishing the cases of sign of $u(k-1)$ and $u(k)$, we prove that

$$\sum_{k=1}^{n}|\Delta u^-(k-1)|^p\leq-\sum_{k=1}^{n}\varphi_p(\Delta u(k-1))\Delta u^-(k-1),$$

then

$$\sum_{k=1}^{n}|\Delta u^-(k-1)|^p+\sum_{k=1}^{n}a_0(k)|u^-(k)|^p\leq-\sum_{k=1}^{n}h(k)u^-(k)\leq 0,$$

therefore $\sum_{k=1}^{n}|\Delta u^-(k-1)|^p=0$ and u^- is constant. If $u^-\not\equiv 0$, since $\sum_{k=1}^{n}h(k)u^-(k)=0$, then $h\equiv 0$ which is absurd. Thus $u\geq 0$.

On the other hand, if $u(k_0)=0$ for some $k_0\in[1,n]$, then $\Delta u(k_0)=u(k_0+1)\geq 0$ and $\Delta u(k_0-1)=-u(k_0-1)\leq 0$, so $\varphi_p(\Delta u(k_0))\geq 0$ and $\varphi_p(\Delta u(k_0-1))\leq 0$. As $-\varphi_p(\Delta u(k_0))+\varphi_p(\Delta u(k_0-1))+a_0(k_0)(u(k_0))^{p-1}=h(k_0)\geq 0$, then $0\leq\varphi_p(\Delta u(k_0))\leq\varphi_p(\Delta u(k_0-1))\leq 0$, from where $u(k_0+1)=u(k_0-1)=0$ and so on, we prove $u\equiv 0$, which contradicts $h\not\equiv 0$. □

Corollary 2.2. *(see [3]) If $u\gneqq 0$ is a solution of (2.1) with $h\geq 0$, then $u>0$.*

The following expression will play a central role in our approach:

$$\mu(m):=\inf\left\{\sum_{k=1}^{n}|\Delta u(k-1)|^p:\ u\in W \ \text{ and } \ \sum_{k=1}^{n}m(k)|u(k)|^p=1\right\}. \tag{2.7}$$

Proposition 2.3. *(see [3]) (i) Suppose that* $\sum_{k=1}^{n} m(k) < 0$. *Then* $\mu(m) > 0$, *every eigenfunction with* $\mu(m)$ *of (2.5) does not change sign in* $[1,n]$ *and does not vanish in* $[1,n]$, *and* $\mu(m)$ *is the unique nonzero principal eigenvalue of (2.5); moreover, the interval* $]0,\mu(m)[$ *does not contain any eigenvalue of (2.5).*

(ii) Suppose that $\sum_{k=1}^{n} m(k) \geq 0$. *Then* $\mu(m) = 0$; *moreover, if* $\sum_{k=1}^{n} m(k) = 0$, *then* 0 *is the unique principal eigenvalue of (2.5).*

Remark 2.4. *If* $\sum_{k=1}^{n} m(k) > 0$, *we apply the Propoition 2.3 to the weight* $(-m)$, *then* $-\mu(-m)$ *is the unique nonzero principal eigenvalue of (2.5).*

Lemma 2.5. *Assume that* $\sum_{k=1}^{n} m(k) < 0$. *Then there exists a constant* $c > 0$ *such that* $\sum_{k=1}^{n} |\Delta u(k-1)|^p \geq c\sum_{k=1}^{n} |u(k)|^p$ *for all* $u \in W$ *with* $\sum_{k=1}^{n} m(k)|u(k)|^p > 0$.

Proof. Assume by contradiction that for each $j = 1, 2, ...$, there exists $u_j \in W$ with $\sum_{k=1}^{n} m(k)|u_j(k)|^p > 0$ and $\sum_{k=1}^{n} |\Delta u_j(k-1)|^p < \frac{1}{j}\sum_{k=1}^{n} |u_j(k)|^p$, then $u_j \not\equiv 0$. One considers the normalisation $v_j = \frac{u_j}{\|u_j\|}$, for a subsequence $v_j \to v$ in W, $\|v\| = 1$ and $\sum_{k=1}^{n} |\Delta v(k-1)|^p = 0$, then v nontrivial constant and $\sum_{k=1}^{n} m(k)|v(k)|^p \geq 0$, which contradicts $\sum_{k=1}^{n} m(k) < 0$. □

Proposition 2.6. *Suppose that* $\sum_{k=1}^{n} m(k) \leq 0$. *The principal eigenvalues* 0 *and* $\mu(m)$ *are simple.*

Proof. If u is an eigenfunction associated to $\lambda = 0$ of (2.5), then $\sum_{k=1}^{n} |\Delta u(k-1)|^p = 0$ and u is nonzero constant. Now if u and v are two eigenfunctions associated to $\mu(m) > 0$, then, using Proposition 2.3, by replacing if necessary u or v by $-u$ or $-v$, we can assume that $u > 0$ and $v > 0$. Applying Lemma 2.8 below with $\varphi = v$,

$$\mu(m)\sum_{k=1}^{n} m(k)|v(k)|^p \leq \sum_{k=1}^{n} |\Delta v(k-1)|^p. \tag{2.8}$$

In fact, equality holds in (2.8) since v is an eigenfunction associated to $\mu(m)$. Consequently, by Lemma 2.8 below, v is multiple of u. □

Proposition 2.7. *(see [3]) Suppose that* $\sum_{k=1}^{n} m(k) \leq 0$. *If* $\lambda \notin [0,\mu(m)]$, *then problem (2.1) with* $h \geq 0$ *has no solution* $u \gneqq 0$.

Lemma 2.8. *(see [3]) Let* (λ, u) *be a solution of (2.1) with arbitrary* h *and* $u > 0$ *in* $[1,n]$. *Then for any* $\varphi \in W$, *one has*

$$\lambda\sum_{k=1}^{n} m(k)|\varphi(k)|^p + \sum_{k=1}^{n} \frac{h(k)|\varphi(k)|^p}{(u(k))^{p-1}} \leq \sum_{k=1}^{n} |\Delta|\varphi(k-1)||^p. \tag{2.9}$$

Moreover, equality holds in (2.9) if and only if $|\varphi|$ *is a multiple of* u.

Proposition 2.9. *Suppose that* $\sum_{k=1}^{n} m(k) \leq 0$. *Then problem (2.1) with* $h \gneqq 0$ *does not admit any solution if* $\lambda = 0$ *or* $\lambda = \mu(m)$. *It admits a unique solution if* $0 < \lambda < \mu(m)$ *and the latter is strictly positive in* $[1,n]$.

Proof. If $\lambda = 0$, by taking $\varphi = 1$ as testing function in (2.1), we get $\sum_{k=1}^{n} h(k) = 0$, which contradicts $h \gneqq 0$. Reasoning by contradiction, suppose that (2.1) with $\lambda = \mu(m)$ has a solution u, we get $u > 0$ in $[1, n]$. Indeed, if $u^- \not\equiv 0$, then taking $-u^-$ as testing function in (2.1) and as $h \gneqq 0$,

$$\begin{aligned}\sum_{k=1}^{n} |\Delta u^-(k-1)|^p &\leq -\sum_{k=1}^{n} \varphi_p(\Delta u(k-1))\Delta u^-(k-1)\\ &= \mu(m)\sum_{k=1}^{n} m(k)|u^-(k)|^p - \sum_{k=1}^{n} h(k)u^-(k)\\ &\leq \mu(m)\sum_{k=1}^{n} m(k)|u^-(k)|^p,\end{aligned}$$

so u^- is a minimizer in the definition of $\mu(m)$ and $\sum_{k=1}^{n} h(k)u^-(k) = 0$. Then by Lagrange multiplies, u^- solves (2.1), and consequently by Corollary 2.2, $u^- > 0$ in $[1, n]$, which contradicts $\sum_{k=1}^{n} h(k)u^-(k) = 0$. Thus, $u \gneqq 0$. Applying once more Corollary 2.2, one gets $u > 0$ in $[1, n]$. By Lemma 2.8, we have for a positive eigenfunction φ associated to $\mu(m)$ of (2.5),

$$\mu(m)\sum_{k=1}^{n} m(k)\,(\varphi(k))^p + \sum_{k=1}^{n} \frac{h(k)\,(\varphi(k))^p}{(u(k))^{p-1}} \leq \sum_{k=1}^{n} |\Delta\varphi(k-1)|^p,$$

we deduce $\sum_{k=1}^{n} \frac{h(k)|\varphi(k)|^p}{(u(k))^{p-1}} \leq 0$, which is impossible since $\varphi > 0$ in $[1, n]$ and $h \gneqq 0$.

Suppose that $\lambda \in]0, \mu(m)[$, then by Proposition 2.3, $\sum_{k=1}^{n} m(k) < 0$. To prove the existence of a solution of (2.1), we consider the functional

$$\phi(u) = \frac{1}{p}\sum_{k=1}^{n} |\Delta u(k-1)|^p - \frac{\lambda}{p}\sum_{k=1}^{n} m(k)|u(k)|^p - \sum_{k=1}^{n} h(k)u(k).$$

We distinguish two cases. If $u \in W$ and $\sum_{k=1}^{n} m(k)|u(k)|^p > 0$, by definition of $\mu(m)$ and Lemma 2.5,

$$\begin{aligned}\phi(u) &\geq \frac{1}{p}\left(1 - \frac{\lambda}{\mu(m)}\right)\sum_{k=1}^{n} |\Delta u(k-1)|^p - \sum_{k=1}^{n} h(k)u(k)\\ &\geq c_1\sum_{k=1}^{n} |u(k)|^p - \sum_{k=1}^{n} h(k)u(k),\end{aligned}$$

for some constant $c_1 > 0$. If $u \in W$ and $\sum_{k=1}^{n} m(k)|u(k)|^p \leq 0$, one has, using $\lambda > 0$ and Lemma 2.10 below,

$$\phi(u) \geq c_2\sum_{k=1}^{n} |u(k)|^p - \sum_{k=1}^{n} h(k)u(k),$$

for some constant $c_2 > 0$. So ϕ is coercive on W and has a mininum, thus there exists a solution u of (2.1). Taking $-u^-$ as testing function,

$$\begin{aligned}\sum_{k=1}^{n} |\Delta u^-(k-1)|^p &\leq -\sum_{k=1}^{n} \varphi_p(\Delta u(k-1))\Delta u^-(k-1)\\ &= \lambda\sum_{k=1}^{n} m(k)|u^-(k)|^p - \sum_{k=1}^{n} h(k)u^-(k),\end{aligned}$$

so $\sum_{k=1}^{n} m(k)|u^-(k)|^p \geq 0$, and

$$\begin{aligned}\sum_{k=1}^{n}|\Delta u^{-}(k-1)|^{p} &\leq \mu(m)\sum_{k=1}^{n}m(k)|u^{-}(k)|^{p}-\sum_{k=1}^{n}h(k)u^{-}(k)\\ &\leq \mu(m)\sum_{k=1}^{n}m(k)|u^{-}(k)|^{p}.\end{aligned}$$

If $u^{-}\not\equiv 0$, then u^{-} is an eigenfunction associated to $\mu(m)$, consequently $u^{-}>0$ and $\sum_{k=1}^{n}h(k)u^{-}(k)=0$, which contradicts $h\gneqq 0$, then $u\geq 0$ and applying Corollary 2.2, one gets $u>0$ in $[1,n]$. We will now prove unicity, suppose that v is a solution of (2.1). Applying Lemma 2.8 with $\varphi=v>0$,

$$\begin{aligned}\lambda\sum_{k=1}^{n}m(k)v(k)^{p}+\sum_{k=1}^{n}\frac{h(k)v(k)^{p}}{(u(k))^{p-1}} &\leq \sum_{k=1}^{n}|\Delta v(k-1)|^{p}\\ &= \lambda\sum_{k=1}^{n}m(k)v(k)^{p}+\sum_{k=1}^{n}h(k)v(k),\end{aligned}\tag{2.10}$$

one gets, $\sum_{k=1}^{n}h(k)v(k)\left(1-\left(\frac{v(k)}{u(k)}\right)^{p-1}\right)\geq 0.$

Interchanging u and v, we get $\sum_{k=1}^{n}h(k)u(k)\left(1-\left(\frac{u(k)}{v(k)}\right)^{p-1}\right)\geq 0$, and adding, we obtain

$$\sum_{k=1}^{n}h(k)\left[v(k)\left(1-\left(\frac{v(k)}{u(k)}\right)^{p-1}\right)+u(k)\left(1-\left(\frac{u(k)}{v(k)}\right)^{p-1}\right)\right]\geq 0.\tag{2.11}$$

Let $A(k)=v(k)\left(1-\left(\frac{v(k)}{u(k)}\right)^{p-1}\right)+u(k)\left(1-\left(\frac{u(k)}{v(k)}\right)^{p-1}\right)$ for $k\in[1,n]$, we get

$$A(k) = \frac{(u(k))^{p}}{(v(k))^{p-1}}\left[\left(\left(\frac{v(k)}{u(k)}\right)^{p}-1\right)\left(1-\left(\frac{v(k)}{u(k)}\right)^{p-1}\right)\right] \leq 0,$$

which implies that equality holds in (2.11). It follows that equality also holds in (2.10). Lemma 2.8 gives that $v=cu$, for some constant c. Replacing in (2.1) and using the fact that $h\not\equiv 0$, we get $c=1$ and $v=u$. ☐

Lemma 2.10. *Assume that* $\sum_{k=1}^{n}m(k)\neq 0$ *and let* $\lambda>0$. *Then there exists a constant* $c>0$ *such that*

$$\sum_{k=1}^{n}|\Delta u(k-1)|^{p}-\lambda\sum_{k=1}^{n}m(k)|u(k)|^{p}\geq c\sum_{k=1}^{n}|u(k)|^{p},$$

for all $u\in W$ *and* $\sum_{k=1}^{n}m(k)|u(k)|^{p}\leq 0$.

Proof. Assume by contradiction that for each $j=1,2,...$, there exists $u_j\in W$ such that $\sum_{k=1}^{n}m(k)|u_j(k)|^{p}\leq 0$ and $\sum_{k=1}^{n}|\Delta u_j(k-1)|^{p}-\lambda\sum_{k=1}^{n}m(k)|u_j(k)|^{p}<\frac{1}{j}\sum_{k=1}^{n}|u_j(k)|^{p}$, then $u_j\not\equiv 0$. Considering $v_j=\frac{u_j}{\|u_j\|}$, one has

$$\sum_{k=1}^{n}|\Delta v_j(k-1)|^{p}\leq\sum_{k=1}^{n}|\Delta v_j(k-1)|^{p}-\lambda\sum_{k=1}^{n}m(k)|v_j(k)|^{p}\to 0.$$

It follows that for a subsequence, v_k converges in W to a nonzero function v such that $\sum_{k=1}^{n}|\Delta v(k-1)|^{p}=0$, then v is a nonzero constant and $-\lambda\sum_{k=1}^{n}m(k)|v(k)|^{p}=0$. This contradicts $\sum_{k=1}^{n}m(k)\neq 0$. ☐

3. Principal eigenvalues in the Dirichlet case

Consider the Dirichlet problem

$$\begin{cases} -\Delta\varphi_p(\Delta u(k-1)) = \lambda m(k)|u(k)|^{p-2}u(k) + h(k) & \text{in} \quad [1,n], \\ u(0) = u(n+1) = 0, \end{cases} \tag{3.1}$$

m and h are as before with (2.2) and (2.3). There are two principal eigenvalues : $\lambda_1(m) > 0$ and $\lambda_{-1}(m) = -\lambda_1(-m)$ of the problem

$$\begin{cases} -\Delta\varphi_p(\Delta u(k-1)) = \lambda m(k)|u(k)|^{p-2}u(k) & \text{in} \quad [1,n], \\ u(0) = u(n+1) = 0, \end{cases} \tag{3.2}$$

where

$$\lambda_1(m) = \inf\left\{\sum_{k=1}^{n+1}|\Delta u(k-1)|^p : \ u \in W_0, \ \sum_{k=1}^{n} m(k)|u(k)|^p = 1\right\}, \tag{3.3}$$

and $W_0 = \{u : [0, n+1] \to \mathbb{R} \ ; \ u(0) = u(n+1) = 0\}$ is an n-dimensional Banach space under the norm

$$\|u\| = \left(\sum_{k=1}^{n+1}|\Delta u(k-1)|^p\right)^{\frac{1}{p}}.$$

These eigenvalues are simple and the corresponding eigenfunctions can be taken strictly positive in [1,n] (see [2]).

Remark 3.1. *The norms* $\left(\sum\limits_{k=1}^{n+1}|\Delta u(k-1)|^p\right)^{\frac{1}{p}}$ *and* $\left(\sum\limits_{k=1}^{n}|u(k)|^p\right)^{1/p}$ *are equivalent in* W_0*, so there exists a constant* $c > 0$ *such that* $\sum\limits_{k=1}^{n+1}|\Delta u(k-1)|^p \geq c\sum\limits_{k=1}^{n}|u(k)|^p$ *for all* $u \in W_0$*.*

Proposition 3.2. *Let u be a solution of*

$$\begin{cases} -\Delta\varphi_p(\Delta u(k-1)) + a_0(k)|u(k)|^{p-2}u(k) = h(k) & \text{in} \quad [1,n], \\ u(0) = u(n+1) = 0, \end{cases} \tag{3.4}$$

where $a_0 \geq 0$ and $h \gneqq 0$. Then $u > 0$ in $[1,n]$.

Proof. As in the proposition 2.1, writing $u = u^+ - u^-$ and taking $-u^-$ as testing function in (3.4), we obtain

$$\sum_{k=1}^{n+1}|\Delta u^-(k-1)|^p + \sum_{k=1}^{n+1} a_0(k)|u^-(k)|^p \leq -\sum_{k=1}^{n+1} h(k)u^-(k) \leq 0,$$

therefore $\sum\limits_{k=1}^{n+1}|\Delta u^-(k-1)|^p = 0$ and $u^- = 0$, thus $u \geq 0$.

On the other hand, if $u(k_0) = 0$ for some $k_0 \in [1,n]$, then as in Proposition 2.1, we obtain $u(k_0+1) = u(k_0-1) = 0$ and so on, we prove $u \equiv 0$, which contradicts $h \not\equiv 0$. □

Remark 3.3. *The corollary 2.2 and Lemma 2.8 remain true in the Dirichlet case.*

Proposition 3.4. *If $\lambda \notin [\lambda_{-1}(m), \lambda_1(m)]$, then problem (3.1) with $h \geq 0$ has no solution $u \gneqq 0$.*

Proof. As in Proposition 2.7, assume that there exists a solution $u \gneqq 0$ of (3.1) for some $\lambda \in \mathbb{R}$ and some $h \geq 0$. We get

$$\lambda\sum_{k=1}^{n+1} m(k)|v(k)|^p \leq \sum_{k=1}^{n+1}|\Delta v(k-1)|^p,$$

for all $v \in W_0$ with $v \geq 0$. This implies $\lambda \leq \lambda_1(m)$, as well as $-\lambda \leq \lambda_1(-m) = -\lambda_{-1}(m)$, thus $\lambda \in [\lambda_{-1}(m), \lambda_1(m)]$. □

Proposition 3.5. *Problem (3.1) with $h \gneqq 0$ does not have any solution if $\lambda = \lambda_{-1}(m)$ or $\lambda = \lambda_1(m)$. It admits a unique solution if, $\lambda_{-1}(m) < \lambda < \lambda_1(m)$ and the latter is strictly positive in $[1, n]$.*

Proof. The proof of this proposition follows almost the same lines as that of Proposition 2.9. Reasoning by contradiction, suppose that (3.1) with $\lambda = \lambda_1(m)$ has a solution u, we get $u > 0$ in $[1, n]$. By Lemma 2.8, we have for an eigenfunction φ associated to $\lambda_1(m)$ of (3.2), $\varphi > 0$ (see [2]),

$$\lambda_1(m) \sum_{k=1}^{n+1} m(k)\,(\varphi(k))^p + \sum_{k=1}^{n+1} \frac{h(k)\,(\varphi(k))^p}{(u(k))^{p-1}} \le \sum_{k=1}^{n+1} |\Delta\varphi(k-1)|^p,$$

we deduce $\sum_{k=1}^{n+1} \frac{h(k)|\varphi(k)|^p}{(u(k))^{p-1}} \le 0$, which is impossible since $h \gneqq 0$.

Suppose that $\lambda \in [0, \lambda_1(m)[$, we consider the functional

$$\phi(u) = \frac{1}{p} \sum_{k=1}^{n+1} |\Delta u(k-1)|^p - \frac{\lambda}{p} \sum_{k=1}^{n} m(k)|u(k)|^p - \sum_{k=1}^{n} h(k)u(k).$$

By definition of $\lambda_1(m)$ and Remark 3.1,

$$\begin{aligned} \phi(u) &\ge \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1(m)}\right) \sum_{k=1}^{n+1} |\Delta u(k-1)|^p - \sum_{k=1}^{n} h(k)u(k) \\ &\ge c \sum_{k=1}^{n} |u(k)|^p - \sum_{k=1}^{n} h(k)u(k), \end{aligned}$$

for some constant $c > 0$ and for all $u \in W_0$. Then ϕ is coercive on W_0, so it has a mininum, thus there exists a solution u of (3.1). One gets $u > 0$ in $[1, n]$. The unicity is proved as in Proposition 2.9. The cases $\lambda = \lambda_{-1}(m)$ or $\lambda \in]\lambda_{-1}(m), 0[$ can be treated in the same way with the weight (-m). □

References

1. A.Anane, O.Chakrone, N,Moradi, Maximum and Anti-Maximum principles for the p-Laplacian with a Nonlinear Boundary condition, Electronic Journal of Differential Equations, Conference **14** (2006) 95-107.

2. M.Chehabi, O.Chakrone, Properties of the First Eigenvalue with Sign-changing Weight of the Discrete p-Laplacian and Applications, Bol. Soc. Paran. Mat. (3s.) **v.36** 2 (2018) 151-167.

3. H.Chehabi, O Chakrone, M.Chehabi, On the antimaximum principle for the discrete p-Laplacian with sign-changing weight, Applied Mathematics and Computation **342** (2019) 112-117.

4. Y. Huang, On eigenvalue problems for the p-Laplacian with Neumann boundary conditions, Proc. Amer. Math. Soc. **109** (1990) 177-184.

5. W. Allegretto, Y. Huang, A Picone's identity for the p-Laplacian and applications, Nonlinear Anal. T.M.A. **32** (1998) 819-830.

6. R. L. Frank, R. Seiringer, Non-linear ground state representations and sharp Hardy inequalities, J. Funct. Anal. **255** (2008), 3407-3430.

7. T. Godoy, J.P. Gossez, S. Paczka, On the antimaximum principle for the p-Laplacian with indefinite weight, Nonlinear Analysis. **51** (2002) 449-467.

8. T. Godoy, J.P. Gossez, S. Paczka, Antimaximum principle for elliptic problems with weight, Electr.J. Differential Equations **1999** (1999) 1-15.

H. Chehabi,
Department of Mathematics and Computer, Laboratory Nonlinear Analysis,
Faculty of Science, University Mohammed 1st, Oujda,
Morocco.
E-mail address: chehabi.hamza@gmail.com

and

O. Chakrone,
Department of Mathematics and Computer, Laboratory Nonlinear Analysis,
Faculty of Science, University Mohammed 1st, Oujda,
Morocco.
E-mail address: chakrone@yahoo.fr

and

M. Chehabi,
Department of Mathematics and Computer, Laboratory Nonlinear Analysis,
Faculty of Science, University Mohammed 1st, Oujda,
Morocco.
E-mail address: chehbmed@gmail.com