Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–15.
ISSN-0037-8712 in press
doi:10.5269/bspm.51825

Common Fixed, Coupled Coincidence and Common Coupled Fixed Point Results in Hyperbolic Valued Metric Spaces

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abstract: In this paper, we obtain existence of unique common fixed point for a contraction mapping on hyperbolic valued metric spaces, and also develop some coupled coincidence point and common coupled fixed point results for two mappings satisfying various contractive conditions in such spaces. We also give some illustrative examples to validate our results.

Key Words: Common fixed point, coupled coincidence point, common coupled fixed point, contractive mapping, hyperbolic valued metric space.

Contents

1. Introduction

As we have known, one of the most powerful tools in modern analysis used for the existence of solutions of many nonlinear problems in many branches of physics and engineering sciences is the Banach fixed point theorem which implies that every contraction mapping on a complete metric space has a unique fixed point. In general, the theorem is known as the Banach contraction principle. Some authors generalized the Banach contraction principle [1] in many different directions. Works noted in [2,3,4,5,6,7, 8,9,10,11,12,13,14] are some relevant examples.

In 2006, Bhashkar and Lakshmikantham [3] introduced the concept of a coupled fixed point of a mapping $F : X \times X \to X$ and established some coupled fixed point theorems in partially ordered complete metric spaces. They also discussed an application of their result by investigating the existence and uniqueness of solution for a periodic boundary value problem. In 2009, Lakshmikantham and Ćirić [7] gave the notion of a coupled coincidence point and proved coupled coincidence and coupled common fixed point results for nonlinear mappings $F : X \times X \to X$ and $g : X \to X$ satisfying certain contractive conditions in partially ordered complete metric spaces. In 2010, Abbas et al. [8] proved coupled coincidence and coupled common fixed point results in cone metric spaces for $w-$compatible mappings.

In 2011, Azam et al. [10] introduced complex valued metric spaces as a generalization of metric spaces. They proved some fixed point theorems for mappings satisfying a rational inequality in complex valued metric spaces. They also applied these results to the existence and uniqueness for a solution of an integral equation. After the publication of this work, some important works on fixed point theorems in this direction have appeared in complex valued metric spaces; for several related examples, see [11,12,15, 16,17,18,19,20].

In 2016, Kumar and Saini [21] defined hyperbolic valued metric space. In [22], we gave some elementary topological concepts and results on hyperbolic valued metric space and then, we introduced two fixed point theorems for hyperbolic valued metric spaces by defining hyperbolic contraction mapping.

2010 *Mathematics Subject Classification:* 47H10, 47H09, 54H25, 30G35.
Submitted January 16, 2020. Published November 02, 2020

Following the same line, as a generalization of fixed point theorems in hyperbolic valued metric spaces [22], we prove existence of unique common fixed point for a contraction mapping on hyperbolic valued metric spaces. Then, we introduce some coupled coincidence point results for such spaces, and also obtain a unique common coupled fixed point of two mappings in such spaces by using the concept of $w-$compatible maps. We also furnish some examples which substantiate our obtained results.

2. Preliminaries

Now, let give basic properties of bicomplex numbers and hyperbolic numbers which will be used in our subsequent discussion. For further details on the following definitions and results, we refer the reader to [23,24]. Let i and j be independent imaginary units such that $i^2 = j^2 = -1$, $ij = ji$ and $\mathbb{C}(i)$ be the set of complex numbers with the imaginary unit i. The set of bicomplex numbers $\mathbb{BC}$ is defined by

$$\mathbb{BC} = \{z = z_1 + jz_2 : z_1, z_2 \in \mathbb{C}(i)\}.$$

The set $\mathbb{BC}$ forms a ring with respect to the addition and multiplication defined as

$$\begin{aligned} z+w &= (z_1 + jz_2) + (w_1 + jw_2) = (z_1 + w_1) + j(z_2 + w_2), \\ z.w &= (z_1 + jz_2).(w_1 + jw_2) = (z_1w_1 - z_2w_2) + j(z_1w_2 + z_2w_1). \end{aligned}$$

The set of hyperbolic numbers $\mathbb{D}$ is defined by

$$\mathbb{D} = \{x + ky : x, y \in \mathbb{R}\},$$

where $k^2 = 1$ and $k = i.j$ and $0, 1 \in \mathbb{D}$.

The set $\mathbb{D}$ is a subring of the set $\mathbb{BC}$, and also $\mathbb{D}$ is a ring and a module over itself.

There are three types of conjugates in $\mathbb{BC}$:

$$\begin{aligned} z^{\dagger_1} &= \overline{z_1} + j\overline{z_2}, \\ z^{\dagger_2} &= z_1 - jz_2, \\ z^{\dagger_3} &= \overline{z_1} - j\overline{z_2}, \end{aligned}$$

where $\overline{z_1}, \overline{z_2}$ are the complex conjugates of $z_1, z_2 \in \mathbb{C}(i)$. Also, we know three types moduli for any $z \in \mathbb{BC}$:

$$\begin{aligned} |z|_i^2 &= z.z^{\dagger_2} = z_1^2 + z_2^2 \in \mathbb{C}(i), \\ |z|_j^2 &= z.z^{\dagger_1} = \left(|z_1|^2 - |z_2|^2\right) + j\left(2\Re\left(z_1.\overline{z_2}\right)\right) \in \mathbb{C}(j), \\ |z|_k^2 &= z.z^{\dagger_3} = \left(|z_1|^2 + |z_2|^2\right) + k\left(-\Im\left(z_1.\overline{z_2}\right)\right) \in \mathbb{D}. \end{aligned}$$

Let $z = z_1 + jz_2$ be any bicomplex number in $\mathbb{BC}$. We say that z is invertible if $|z|_i \neq 0$, that is, $z_1^2 + z_2^2 \neq 0$ and its inverse is given by $z^{-1} = \frac{z^{\dagger_2}}{|z|_i^2}$. If, on the other hand, $z \neq 0$ but $|z|_i = 0$, then z is a zero divisor.

The ring $\mathbb{BC}$ is not a division ring, since one can see that if $e_1 = \frac{1+ij}{2}$ and $e_2 = \frac{1-ij}{2}$, then e_1 and e_2 are zero divisors. The numbers e_1 and e_2 form idempotent basis of bicomplex numbers and hence any bicomplex number $z = z_1 + jz_2$ is uniquely written as

$$z = e_1\beta_1 + e_2\beta_2 \tag{2.1}$$

where $\beta_1 = z_1 - iz_2, \beta_2 = z_1 + iz_2 \in \mathbb{C}(i)$. Formula (2.1) is called the idempotent representation of z.

Let $\alpha = x + ky$ be any hyperbolic number. Then, we have the equality

$$\alpha = e_1\alpha_1 + e_2\alpha_2,$$

where $\alpha_1 = x + y, \alpha_2 = x - y \in \mathbb{R}$. If $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$, then α is called a positive hyperbolic number. Therefore, the set of positive hyperbolic numbers $\mathbb{D}^+$ is denoted by

$$\mathbb{D}^+ = \{\alpha = e_1\alpha_1 + e_2\alpha_2 : \alpha_1 \geq 0, \alpha_2 \geq 0\}.$$

For two hyperbolic numbers α and β; if their difference $\beta - \alpha \in \mathbb{D}^+$ (or $\beta - \alpha \in \mathbb{D}^+ - \{0\}$), then we write $\alpha \precsim \beta$ (or $\alpha \precnsim \beta$). For $\alpha = e_1\alpha_1 + e_2\alpha_2$, $\beta = \beta_1 e_1 + \beta_2 e_2 \in \mathbb{D}$ with real numbers $\alpha_1, \alpha_2, \beta_1$ and β_2, we have that

$$\begin{aligned} \alpha &\precsim \beta \text{ if and only if } \alpha_1 \leq \beta_1 \text{ and } \alpha_2 \leq \beta_2, \\ \alpha &\precnsim \beta \text{ if and only if } \alpha \neq \beta \text{ and } \alpha_1 \leq \beta_1 \text{ and } \alpha_2 \leq \beta_2, \\ \alpha &\prec \beta \text{ if and only if } \alpha_1 < \beta_1 \text{ and } \alpha_2 < \beta_2. \end{aligned}$$

This relation $\precsim$ is reflexive, anti - symmetric, transitive and so defines a partial order on $\mathbb{D}$.

We know that the hyperbolic valued module $|z|_k$ of a bicomplex number $z = e_1\beta_1 + e_2\beta_2$ is also given as

$$|z|_k = e_1 |\beta_1| + e_2 |\beta_2|.$$

One can easily see that

$$\begin{aligned} |z.w|_k &= |z|_k \cdot |w|_k, \\ |z + w|_k &\precsim |z|_k + |w|_k, \\ \left|\frac{z}{w}\right|_k &= \frac{|z|_k}{|w|_k} \end{aligned}$$

for any $z, w \in \mathbb{BC}$.

The following statements are true for $\alpha, \beta, \gamma \in \mathbb{D}$:

(i) If $\alpha \precsim \beta$ then $\alpha + \gamma \precsim \beta + \gamma$.
(ii) If $\alpha \precsim \beta$ and $\beta \prec \gamma$, then $\alpha \prec \gamma$.
(iii) If $\alpha \prec \beta$ and $0 \prec \gamma$, then $\alpha\gamma \prec \beta\gamma$.
(iv) If $\alpha \prec \beta$ and $\gamma \prec 0$, then $\beta\gamma \prec \alpha\gamma$.
(v) If $\alpha \precsim \beta$ and $0 \prec \gamma$, then $\alpha\gamma \precsim \beta\gamma$.
(vi) If $\alpha \precsim \beta$ and $\gamma \prec 0$, then $\beta\gamma \precsim \alpha\gamma$.
(vii) If $\alpha \precsim \beta$ and $\gamma \precsim \delta$, then $\alpha + \gamma \precsim \beta + \delta$.
(viii) If $\alpha, \beta \in \mathbb{D}^+$, then $\alpha \precsim \beta$ (or $\alpha \prec \beta$) implies that $|\alpha| \leq |\beta|$ (or $|\alpha| < |\beta|$) where $|\cdot|$ shows Euclidean norm in $\mathbb{BC}$ (see [23,24]).
(ix) If $\alpha \in \mathbb{D}^+$, then $|\alpha|_k = \alpha$.

A sequence in $\mathbb{BC}$ is a function defined by $z : \mathbb{N} \to \mathbb{BC}, n \to z_n$. This sequence converges to a point $z^* \in \mathbb{BC}$ if and only if to each $\varepsilon > 0$ there corresponds an $n_0(\varepsilon)$ such that $|z_n - z^*| < \varepsilon$ for all $n \geq n_0(\varepsilon)$. It is denoted by $z_n \to z^*$ as $n \to \infty$. The sequence $z = (z_n)$ is a Cauchy sequence in $\mathbb{BC}$ if and only if to each $\varepsilon > 0$ there corresponds an $n_0(\varepsilon)$ such that $|z_n - z_m| < \varepsilon$ for all $n, m \geq n_0(\varepsilon)$. Also, $z = (z_n)$ converges to a point in $\mathbb{BC}$ if and only if it is a Cauchy sequence in $\mathbb{BC}$. On the other hand, for any sequence (z_n) in $\mathbb{BC}$ such that $z : \mathbb{N} \to \mathbb{BC}$, $z_n = \beta_{1n}e_1 + \beta_{2n}e_2$ and for any $z^* = \beta_1^* e_1 + \beta_2^* e_2 \in \mathbb{BC}$, we have that $z_n \to z^*$ as $n \to \infty$ if and only if $\beta_{1n} \to \beta_1^*$ and $\beta_{2n} \to \beta_2^*$ as $n \to \infty$.

The following popular concept are defined by Kumar and Saini [21].

Definition 2.1. *Let X be a nonempty set and $d_{\mathbb{D}} : X \times X \to \mathbb{D}$ be a function such that for any $x, y, z \in X$, the following properties hold:*

(i) $0 \precsim d_{\mathbb{D}}(x, y)$ and $d_{\mathbb{D}}(x, y) = 0$ if and only if $x = y$,
(ii) $d_{\mathbb{D}}(x, y) = d_{\mathbb{D}}(y, x)$,
(iii) $d_{\mathbb{D}}(x, z) \precsim d_{\mathbb{D}}(x, y) + d_{\mathbb{D}}(y, z)$.

Then $d_{\mathbb{D}}$ is called a hyperbolic valued or $\mathbb{D}$ -valued metric on X and the pair $(X, d_{\mathbb{D}})$ is called a hyperbolic valued or $\mathbb{D}$ -valued metric space (see [21]).

The following definitions and simple properties are recently introduced by Sager and Sağır [22] and they will be needed in the sequel.

Definition 2.2. *Let $(X, d_{\mathbb{D}})$ be a hyperbolic valued metric space, (x_n) be any sequence in X and $x \in X$. If for every $0 \precsim\neq \varepsilon \in \mathbb{D}$ there exists $n_0 \in \mathbb{N}$ depending on ε such that for all $n \geq n_0$*

$$d_{\mathbb{D}}(x_n, x) \precsim\neq \varepsilon,$$

then we say that (x_n) is convergent with respect to the metric $d_{\mathbb{D}}$. We denote this by $\lim_{n\to\infty}{}_{\mathbb{D}} x_n = x$ or $x_n \overset{d_{\mathbb{D}}}{\to} x$ as $n \to \infty$. If for every $0_{\mathbb{D}} \precsim\neq \varepsilon \in \mathbb{D}$ there exists $n_0 \in \mathbb{N}$ depending on ε such that for all $n, m \geq n_0$

$$d_{\mathbb{D}}(x_n, x_m) \precsim\neq \varepsilon,$$

then we say that (x_n) is a Cauchy sequence with respect to the metric $d_{\mathbb{D}}$. If every Cauchy sequence with respect to the metric $d_{\mathbb{D}}$ is convergent with respect to the metric $d_{\mathbb{D}}$ in $(X, d_{\mathbb{D}})$, then we say that $(X, d_{\mathbb{D}})$ is a complete hyperbolic valued metric space.

Corollary 2.3. *Let $(X, d_{\mathbb{D}})$ be a hyperbolic valued metric space, (x_n) be any sequence in X and $x \in X$. Then, the sequence (x_n) converges to x with respect to the metric $d_{\mathbb{D}}$ if and only if $d_{\mathbb{D}}(x_n, x) = |d_{\mathbb{D}}(x_n, x)|_k \to 0$ as $n \to \infty$.*

Corollary 2.4. *Let $(X, d_{\mathbb{D}})$ be a hyperbolic valued metric space and (x_n) be any sequence in X. Then, the sequence (x_n) is a Cauchy sequence with respect to the metric $d_{\mathbb{D}}$ if and only if for all $m \in \mathbb{N}$, $d_{\mathbb{D}}(x_n, x_{n+m}) = |d_{\mathbb{D}}(x_n, x_{n+m})|_k \to 0$ as $n \to \infty$.*

Theorem 2.5. *The following statements are true for $\alpha \in \mathbb{D}$:*
(i) If $\alpha \in \mathbb{D}^+$, $\alpha \neq 1$ and $1 - \alpha$ is invertible, then

$$1 + \alpha + \alpha^2 + ... + \alpha^n = \frac{1 - \alpha^{n+1}}{1 - \alpha}$$

for all $n \in \mathbb{N}$.
(ii) If $\alpha \in \mathbb{D}^+$ and $\alpha \prec 1$, then $0 \precsim \alpha^n \prec 1$ for all $n \in \mathbb{N}$ and $\alpha^n \to 0$.

3. Main Results

3.1. Common Fixed Point Results

In this section, we prove existence of unique common fixed point for a contraction mapping on hyperbolic valued metric spaces. We also give an example which substantiate our main result.

Theorem 3.1. *Let $(X, d_{\mathbb{D}})$ be a complete hyperbolic valued metric space. If S and T are self mappings defined on X satisfying the condition*

$$d_{\mathbb{D}}(Sx, Ty) \precsim \lambda d_{\mathbb{D}}(x, y) + \frac{\mu d_{\mathbb{D}}(x, Sx)\, d_{\mathbb{D}}(y, Ty) + \gamma d_{\mathbb{D}}(y, Sx)\, d_{\mathbb{D}}(x, Ty)}{1 + d_{\mathbb{D}}(x, y)}$$

for all $x, y \in X$, where λ, μ, γ are positive hyperbolic numbers with $\lambda + \mu + \gamma \prec 1$. Then, S and T have a unique common fixed point.

Proof. Let' s first prove the existence. Let x_0 be any point in X. We define $x_{2k+1} = Sx_{2k},\ x_{2k+2} = Tx_{2k+1},\ k = 0, 1, 2, ...$ Then,

$$\begin{aligned} d_{\mathbb{D}}(x_{2k+1}, x_{2k+2}) &= d_{\mathbb{D}}(Sx_{2k}, Tx_{2k+1}) \\ &\precsim \lambda d_{\mathbb{D}}(x_{2k}, x_{2k+1}) + \frac{\mu d_{\mathbb{D}}(x_{2k}, Sx_{2k})\, d_{\mathbb{D}}(x_{2k+1}, Tx_{2k+1})}{1 + d_{\mathbb{D}}(x_{2k}, x_{2k+1})} \\ &\quad + \frac{\gamma d_{\mathbb{D}}(x_{2k+1}, Sx_{2k})\, d_{\mathbb{D}}(x_{2k}, Tx_{2k+1})}{1 + d_{\mathbb{D}}(x_{2k}, x_{2k+1})} \\ &= \lambda d_{\mathbb{D}}(x_{2k}, x_{2k+1}) + \frac{\mu d_{\mathbb{D}}(x_{2k}, x_{2k+1})\, d_{\mathbb{D}}(x_{2k+1}, x_{2k+2})}{1 + d_{\mathbb{D}}(x_{2k}, x_{2k+1})} \\ &\prec \lambda d_{\mathbb{D}}(x_{2k}, x_{2k+1}) + \mu d_{\mathbb{D}}(x_{2k+1}, x_{2k+2}). \end{aligned}$$

This implies that $d_{\mathbb{D}}(x_{2k+1}, x_{2k+2}) \prec \frac{\lambda}{1-\mu} d_{\mathbb{D}}(x_{2k}, x_{2k+1})$. Furthermore,

$$\begin{aligned} d_{\mathbb{D}}(x_{2k+2}, x_{2k+3}) &= d_{\mathbb{D}}(Tx_{2k+1}, Sx_{2k+2}) \\ &= d_{\mathbb{D}}(Sx_{2k+2}, Tx_{2k+1}) \\ &\precsim \lambda d_{\mathbb{D}}(x_{2k+2}, x_{2k+1}) + \frac{\mu d_{\mathbb{D}}(x_{2k+2}, Sx_{2k+2}) d_{\mathbb{D}}(x_{2k+1}, Tx_{2k+1})}{1 + d_{\mathbb{D}}(x_{2k+2}, x_{2k+1})} \\ &\quad + \frac{\gamma d_{\mathbb{D}}(x_{2k+1}, Sx_{2k+2}) d_{\mathbb{D}}(x_{2k+2}, Tx_{2k+1})}{1 + d_{\mathbb{D}}(x_{2k+2}, x_{2k+1})} \\ &= \lambda d_{\mathbb{D}}(x_{2k+2}, x_{2k+1}) + \frac{\mu d_{\mathbb{D}}(x_{2k+2}, Sx_{2k+2}) d_{\mathbb{D}}(x_{2k+1}, x_{2k+2})}{1 + d_{\mathbb{D}}(x_{2k+2}, x_{2k+1})} \\ &\prec \lambda d_{\mathbb{D}}(x_{2k+2}, x_{2k+1}) + \mu d_{\mathbb{D}}(x_{2k+2}, Sx_{2k+2}). \end{aligned}$$

This implies that $d_{\mathbb{D}}(x_{2k+2}, x_{2k+3}) \prec \frac{\lambda}{1-\mu} d_{\mathbb{D}}(x_{2k+2}, x_{2k+1}) = \frac{\lambda}{1-\mu} d_{\mathbb{D}}(x_{2k+1}, x_{2k+2})$. By setting $h = \frac{\lambda}{1-\mu} \prec 1$, we derive

$$d_{\mathbb{D}}(x_n, x_{n+1}) \precsim h d_{\mathbb{D}}(x_{n-1}, x_n) \precsim h^2 d_{\mathbb{D}}(x_{n-2}, x_{n-1}) \precsim ... \precsim h^n d_{\mathbb{D}}(x_0, x_1).$$

Then, using Theorem 2.5, for any $m > n$, we obtain that

$$\begin{aligned} d_{\mathbb{D}}(x_n, x_m) &\precsim d_{\mathbb{D}}(x_n, x_{n+1}) + d_{\mathbb{D}}(x_{n+1}, x_{n+2}) + ... + d_{\mathbb{D}}(x_{m-1}, x_m) \\ &\precsim h^n d_{\mathbb{D}}(x_0, x_1) + h^{n+1} d_{\mathbb{D}}(x_0, x_1) + ... + h^{m-1} d_{\mathbb{D}}(x_0, x_1) \\ &= \left[h^n + h^{n+1} + ... + h^{m-1}\right] d_{\mathbb{D}}(x_0, x_1) \\ &= \frac{h^n - h^m}{1-h} d_{\mathbb{D}}(x_0, x_1) \\ &\precsim \frac{h^n}{1-h} d_{\mathbb{D}}(x_0, x_1). \end{aligned}$$

and taking the limit as $m, n \to \infty$, we conclude that $d_{\mathbb{D}}(x_n, x_m) \to 0$. Therefore, (x_n) is a Cauchy sequence with respect to the metric $d_{\mathbb{D}}$. Since $(X, d_{\mathbb{D}})$ is a complete hyperbolic valued metric space, there exists a point $x \in X$ such that $x_n \overset{d_{\mathbb{D}}}{\to} x$ as $n \to \infty$.

Assume that $x \neq Sx$. Then,

$$\begin{aligned} d_{\mathbb{D}}(x, Sx) &\precsim d_{\mathbb{D}}(x, Tx_{2k+1}) + d_{\mathbb{D}}(Tx_{2k+1}, Sx) \\ &= d_{\mathbb{D}}(x, Tx_{2k+1}) + d_{\mathbb{D}}(Sx, Tx_{2k+1}) \\ &\precsim d_{\mathbb{D}}(x, x_{2k+2}) + \lambda d_{\mathbb{D}}(x_{2k+1}, x) \\ &\quad + \frac{\mu d_{\mathbb{D}}(x, Sx) d_{\mathbb{D}}(x_{2k+1}, Tx_{2k+1}) + \gamma d_{\mathbb{D}}(x_{2k+1}, Sx) d_{\mathbb{D}}(x, Tx_{2k+1})}{1 + d_{\mathbb{D}}(x_{2k+1}, x)}. \end{aligned}$$

Taking $k \to \infty$, one gets $d_{\mathbb{D}}(x, Sx) = 0$ which is a contradiction and hence $x = Sx$. Similarly, we can show that $x = Tx$. Therefore, x is a common fixed point of S and T.

Finally, we show the uniqueness. Assume that x^* is another common fixed point of S and T. Then,

$$\begin{aligned} d_{\mathbb{D}}(x, x^*) &= d_{\mathbb{D}}(Sx, Tx^*) \\ &\precsim \lambda d_{\mathbb{D}}(x, x^*) + \frac{\mu d_{\mathbb{D}}(x, Sx) d_{\mathbb{D}}(x^*, Tx^*) + \gamma d_{\mathbb{D}}(x^*, Sx) d_{\mathbb{D}}(x, Tx^*)}{1 + d_{\mathbb{D}}(x, x^*)} \\ &= \lambda d_{\mathbb{D}}(x, x^*) + \frac{\gamma d_{\mathbb{D}}(x^*, x) d_{\mathbb{D}}(x, x^*)}{1 + d_{\mathbb{D}}(x, x^*)} \\ &\prec (\lambda + \gamma) d_{\mathbb{D}}(x, x^*). \end{aligned}$$

But this is impossible. So, $d_{\mathbb{D}}(x, x^*) = 0,\ x = x^*$ which implies that the fixed point is unique. □

Corollary 3.2. *Let $(X, d_{\mathbb{D}})$ be a complete hyperbolic valued metric space. If T is self mapping defined on X satisfying the condition*

$$d_{\mathbb{D}}(Tx, Ty) \precsim \lambda d_{\mathbb{D}}(x, y) + \frac{\mu d_{\mathbb{D}}(x, Tx)\, d_{\mathbb{D}}(y, Ty) + \gamma d_{\mathbb{D}}(y, Tx)\, d_{\mathbb{D}}(x, Ty)}{1 + d_{\mathbb{D}}(x, y)}$$

for all $x, y \in X$, where λ, μ, γ are positive hyperbolic numbers with $\lambda + \mu + \gamma \prec 1$. Then, T has a unique fixed point.

Proof. By setting $S = T$ in Theorem 3.1, we derive Corollary 3.2. □

Remark 3.3. *If we choose $\lambda = 0$, $\mu = 0$ or $\gamma = 0$ in all possible combinations, we obtain fixed point theorems on hyperbolic valued metric spaces.*

We finish this section with an example which satisfy the requirements of Corollary 3.2 as follows:

Example 3.4. *Let*

$$\begin{aligned} X_1 &= \{\gamma = \gamma_1 e_1 + \gamma_2 e_2 \in \mathbb{D} : \gamma_1 = \gamma_2, \gamma_1 \geq 0\}, \\ X_2 &= \{\gamma = \gamma_1 e_1 + \gamma_2 e_2 \in \mathbb{D} : \gamma_1 = -\gamma_2, \gamma_1 \geq 0\} \end{aligned}$$

and $X = X_1 \cup X_2$. Define a mapping $d_{\mathbb{D}} : X \times X \to \mathbb{D}$ as

$$d_{\mathbb{D}}(\alpha, \beta) = \begin{cases} \frac{7}{6}|\alpha_1 - \beta_1| e_1 + \frac{9}{8}|\alpha_1 - \beta_1| e_2, \ \alpha, \beta \in X_1 \\ \frac{5}{6}|\alpha_1 - \beta_1| e_1 + \frac{7}{8}|\alpha_1 - \beta_1| e_2, \ \alpha, \beta \in X_2 \\ \left(\frac{7}{6}\alpha_1 + \frac{5}{6}\beta_1\right) e_1 + \left(\frac{9}{8}\alpha_1 + \frac{7}{8}\beta_1\right) e_2, \ \alpha \in X_1, \beta \in X_2 \\ \left(\frac{5}{6}\alpha_1 + \frac{7}{6}\beta_1\right) e_1 + \left(\frac{7}{8}\alpha_1 + \frac{9}{8}\beta_1\right) e_2, \ \alpha \in X_2, \beta \in X_1 \end{cases},$$

where $\alpha = \alpha_1 e_1 + \alpha_2 e_2, \beta = \beta_1 e_1 + \beta_2 e_2$, then $(X, d_{\mathbb{D}})$ is a complete hyperbolic valued metric space.

Consider a mapping T on X with $\gamma = \gamma_1 e_1 + \gamma_2 e_2$ as

$$T\gamma = \begin{cases} \gamma_1 e_1 - \gamma_1 e_2, \ \gamma \in X_1 \\ \frac{\gamma_1}{2} e_1 + \frac{\gamma_1}{2} e_2, \ \gamma \in X_2 \end{cases}$$

and let $S = T$. So, we have

$$
\begin{aligned}
d_{\mathbb{D}}(S\alpha, T\beta) &= d_{\mathbb{D}}(T\alpha, T\beta) \\
&= \begin{cases}
\frac{5}{6}\left|\alpha_1 - \beta_1\right| e_1 + \frac{7}{8}\left|\alpha_1 - \beta_1\right| e_2, & \alpha, \beta \in X_1 \\
\frac{7}{12}\left|\alpha_1 - \beta_1\right| e_1 + \frac{9}{16}\left|\alpha_1 - \beta_1\right| e_2, & \alpha, \beta \in X_2 \\
\left(\frac{5}{6}\alpha_1 + \frac{7}{12}\beta_1\right) e_1 + \left(\frac{7}{8}\alpha_1 + \frac{9}{16}\beta_1\right) e_2, & \alpha \in X_1, \beta \in X_2 \\
\left(\frac{7}{12}\alpha_1 + \frac{5}{6}\beta_1\right) e_1 + \left(\frac{9}{16}\alpha_1 + \frac{7}{8}\beta_1\right) e_2, & \alpha \in X_2, \beta \in X_1
\end{cases} \\
&\precsim \begin{cases}
\left|\alpha_1 - \beta_1\right| e_1 + \left|\alpha_1 - \beta_1\right| e_2, & \alpha, \beta \in X_1 \\
\frac{5}{7}\left|\alpha_1 - \beta_1\right| e_1 + \frac{7}{9}\left|\alpha_1 - \beta_1\right| e_2, & \alpha, \beta \in X_2 \\
\left(\alpha_1 + \frac{5}{7}\beta_1\right) e_1 + \left(\alpha_1 + \frac{7}{9}\beta_1\right) e_2, & \alpha \in X_1, \beta \in X_2 \\
\left(\frac{5}{7}\alpha_1 + \beta_1\right) e_1 + \left(\frac{7}{9}\alpha_1 + \beta_1\right) e_2, & \alpha \in X_2, \beta \in X_1
\end{cases} \\
&= \left(\frac{6}{7}e_1 + \frac{8}{9}e_2\right) \begin{cases}
\frac{7}{6}\left|\alpha_1 - \beta_1\right| e_1 + \frac{9}{8}\left|\alpha_1 - \beta_1\right| e_2, & \alpha, \beta \in X_1 \\
\frac{5}{6}\left|\alpha_1 - \beta_1\right| e_1 + \frac{7}{8}\left|\alpha_1 - \beta_1\right| e_2, & \alpha, \beta \in X_2 \\
\left(\frac{7}{6}\alpha_1 + \frac{5}{6}\beta_1\right) e_1 + \left(\frac{9}{8}\alpha_1 + \frac{7}{8}\beta_1\right) e_2, & \alpha \in X_1, \beta \in X_2 \\
\left(\frac{5}{6}\alpha_1 + \frac{7}{6}\beta_1\right) e_1 + \left(\frac{7}{8}\alpha_1 + \frac{9}{8}\beta_1\right) e_2, & \alpha \in X_2, \beta \in X_1
\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{6}{7}e_1 + \frac{8}{9}e_2\right) d_{\mathbb{D}}(\alpha, \beta) \\
&\precsim \left(\frac{6}{7}e_1 + \frac{8}{9}e_2\right) d_{\mathbb{D}}(\alpha, \beta) \\
&\quad + \frac{\left(\frac{3}{56}e_1 + \frac{1}{20}e_2\right) d_{\mathbb{D}}(x, Sx)\, d_{\mathbb{D}}(y, Ty) + \left(\frac{3}{56}e_1 + \frac{1}{20}e_2\right) d_{\mathbb{D}}(y, Sx)\, d_{\mathbb{D}}(x, Ty)}{1 + d_{\mathbb{D}}(x, y)}
\end{aligned}
$$

where $\lambda = \frac{6}{7}e_1 + \frac{8}{9}e_2$, $\mu = \gamma = \frac{3}{56}e_1 + \frac{1}{20}e_2$. Note that $\frac{6}{7}e_1 + \frac{8}{9}e_2 + 2.\left(\frac{3}{56}e_1 + \frac{1}{20}e_2\right) = \frac{27}{28}e_1 + \frac{89}{90}e_2 \prec 1$. Thus, the conditions of Corollary 3.2 are satisfied. Then, T has a unique fixed point in X. This fixed point is $(0,0)$.

3.2. Coupled Coincidence Point Results

In this section, we introduce some coupled coincidence point theorems for hyperbolic valued metric spaces which give us a sufficient condition for the existence of coupled coincidence points for two mappings. We also present an example in order to deduce results about coupled coincidence points.

Definition 3.5. *An element $(x,y) \in X \times X$ is called a coupled fixed point of mapping $F : X \times X \to X$ if $x = F(x,y)$ and $y = F(y,x)$ (see [3]).*

Definition 3.6. *An element $(x,y) \in X \times X$ is called*

(i) a coupled coincidence point of mappings $F : X \times X \to X$ and $g : X \to X$ if $g(x) = F(x,y)$ and $g(y) = F(y,x)$ and (gx, gy) is called coupled point of coincidence (see [7]),

(ii) a common coupled fixed point of mappings $F : X \times X \to X$ and $g : X \to X$ if $x = g(x) = F(x,y)$ and $y = g(y) = F(y,x)$ (see [8]).

Theorem 3.7. *Let $(X, d_{\mathbb{D}})$ be a hyperbolic valued metric space, $F : X \times X \to X$ and $g : X \to X$ be two mappings and there exist positive hyperbolic numbers λ_i, $i = 1, 2, 3, 4, 5, 6$ with $\sum_{i=1}^{6} \lambda_i \prec 1$ such that the following contractive condition holds for all $x, y, u, v \in X$:*

$$\begin{aligned} d_{\mathbb{D}}\left(F\left(x,y\right),F\left(u,v\right)\right) &\precsim \lambda_1 d_{\mathbb{D}}\left(gx,gu\right)+\lambda_2 d_{\mathbb{D}}\left(gy,gv\right) \\ &\quad+\frac{\lambda_3 d_{\mathbb{D}}\left(F\left(x,y\right),gx\right)+\lambda_4 d_{\mathbb{D}}\left(F\left(x,y\right),gu\right)}{1+d_{\mathbb{D}}\left(u,v\right)} \\ &\quad+\frac{\lambda_5 d_{\mathbb{D}}\left(F\left(u,v\right),gu\right)+\lambda_6 d_{\mathbb{D}}\left(F\left(u,v\right),gx\right)}{1+d_{\mathbb{D}}\left(x,y\right)}. \end{aligned} \tag{3.1}$$

If $F(X \times X) \subset g(X)$ and $g(X)$ is a complete subspace of X, then F and g have a coupled coincidence point in X.

Proof. Choose $x_0, y_0 \in X$. Then, since $F(X \times X) \subset g(X)$, there exist $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Continuing in this way, we construct two sequences (x_n) and (y_n) in X, such that $g(x_{n+1}) = F(x_n, y_n)$ and $g(y_{n+1}) = F(y_n, x_n)$. Then, we have

$$\begin{aligned} d_{\mathbb{D}}\left(gx_n,gx_{n+1}\right) &= d_{\mathbb{D}}\left(F\left(x_{n-1},y_{n-1}\right),F\left(x_n,y_n\right)\right) \\ &\precsim \lambda_1 d_{\mathbb{D}}\left(gx_{n-1},gx_n\right)+\lambda_2 d_{\mathbb{D}}\left(gy_{n-1},gy_n\right) \\ &\quad+\frac{\lambda_3 d_{\mathbb{D}}\left(F\left(x_{n-1},y_{n-1}\right),gx_{n-1}\right)+\lambda_4 d_{\mathbb{D}}\left(F\left(x_{n-1},y_{n-1}\right),gx_n\right)}{1+d_{\mathbb{D}}\left(x_n,y_n\right)} \\ &\quad+\frac{\lambda_5 d_{\mathbb{D}}\left(F\left(x_n,y_n\right),gx_n\right)+\lambda_6 d_{\mathbb{D}}\left(F\left(x_n,y_n\right),gx_{n-1}\right)}{1+d_{\mathbb{D}}\left(x_{n-1},y_{n-1}\right)} \\ &= \lambda_1 d_{\mathbb{D}}\left(gx_{n-1},gx_n\right)+\lambda_2 d_{\mathbb{D}}\left(gy_{n-1},gy_n\right) \\ &\quad+\frac{\lambda_3 d_{\mathbb{D}}\left(gx_n,gx_{n-1}\right)+\lambda_4 d_{\mathbb{D}}\left(gx_n,gx_n\right)}{1+d_{\mathbb{D}}\left(x_n,y_n\right)} \\ &\quad+\frac{\lambda_5 d_{\mathbb{D}}\left(gx_{n+1},gx_n\right)+\lambda_6 d_{\mathbb{D}}\left(gx_{n+1},gx_{n-1}\right)}{1+d_{\mathbb{D}}\left(x_{n-1},y_{n-1}\right)} \\ &= \lambda_1 d_{\mathbb{D}}\left(gx_{n-1},gx_n\right)+\lambda_2 d_{\mathbb{D}}\left(gy_{n-1},gy_n\right)+\frac{\lambda_3 d_{\mathbb{D}}\left(gx_n,gx_{n-1}\right)}{1+d_{\mathbb{D}}\left(x_n,y_n\right)} \\ &\quad+\frac{\lambda_5 d_{\mathbb{D}}\left(gx_{n+1},gx_n\right)+\lambda_6 d_{\mathbb{D}}\left(gx_{n+1},gx_{n-1}\right)}{1+d_{\mathbb{D}}\left(x_{n-1},y_{n-1}\right)} \\ &\precsim \lambda_1 d_{\mathbb{D}}\left(gx_{n-1},gx_n\right)+\lambda_2 d_{\mathbb{D}}\left(gy_{n-1},gy_n\right)+\lambda_3 d_{\mathbb{D}}\left(gx_n,gx_{n-1}\right) \\ &\quad+\lambda_5 d_{\mathbb{D}}\left(gx_{n+1},gx_n\right)+\lambda_6 d_{\mathbb{D}}\left(gx_{n+1},gx_{n-1}\right) \\ &\precsim \lambda_1 d_{\mathbb{D}}\left(gx_{n-1},gx_n\right)+\lambda_2 d_{\mathbb{D}}\left(gy_{n-1},gy_n\right)+\lambda_3 d_{\mathbb{D}}\left(gx_n,gx_{n-1}\right) \\ &\quad+\lambda_5 d_{\mathbb{D}}\left(gx_{n+1},gx_n\right)+\lambda_6 d_{\mathbb{D}}\left(gx_{n+1},gx_n\right)+\lambda_6 d_{\mathbb{D}}\left(gx_n,gx_{n-1}\right) \\ &= \left(\lambda_1+\lambda_3+\lambda_6\right) d_{\mathbb{D}}\left(gx_{n-1},gx_n\right)+\left(\lambda_5+\lambda_6\right) d_{\mathbb{D}}\left(gx_{n+1},gx_n\right) \\ &\quad+\lambda_2 d_{\mathbb{D}}\left(gy_{n-1},gy_n\right). \end{aligned}$$

Hence,

$$\left(1-\lambda_5-\lambda_6\right) d_{\mathbb{D}}\left(gx_{n+1},gx_n\right) \precsim \left(\lambda_1+\lambda_3+\lambda_6\right) d_{\mathbb{D}}\left(gx_{n-1},gx_n\right)+\lambda_2 d_{\mathbb{D}}\left(gy_{n-1},gy_n\right). \tag{3.2}$$

Similarly, one can prove that

$$\left(1-\lambda_5-\lambda_6\right) d_{\mathbb{D}}\left(gy_{n+1},gy_n\right) \precsim \left(\lambda_1+\lambda_3+\lambda_6\right) d_{\mathbb{D}}\left(gy_{n-1},gy_n\right)+\lambda_2 d_{\mathbb{D}}\left(gx_{n-1},gx_n\right). \tag{3.3}$$

On the other hand, we have

$$\begin{aligned}
d_{\mathbb{D}}(gx_{n+1},gx_n) &= d_{\mathbb{D}}(F(x_n,y_n),F(x_{n-1},y_{n-1}))\\
&\precsim \lambda_1 d_{\mathbb{D}}(gx_n,gx_{n-1})+\lambda_2 d_{\mathbb{D}}(gy_n,gy_{n-1})\\
&\quad+\frac{\lambda_3 d_{\mathbb{D}}(F(x_n,y_n),gx_n)+\lambda_4 d_{\mathbb{D}}(F(x_n,y_n),gx_{n-1})}{1+d_{\mathbb{D}}(x_{n-1},y_{n-1})}\\
&\quad+\frac{\lambda_5 d_{\mathbb{D}}(F(x_{n-1},y_{n-1}),gx_{n-1})+\lambda_6 d_{\mathbb{D}}(F(x_{n-1},y_{n-1}),gx_n)}{1+d_{\mathbb{D}}(x_n,y_n)}\\
&= \lambda_1 d_{\mathbb{D}}(gx_n,gx_{n-1})+\lambda_2 d_{\mathbb{D}}(gy_n,gy_{n-1})\\
&\quad+\frac{\lambda_3 d_{\mathbb{D}}(gx_{n+1},gx_n)+\lambda_4 d_{\mathbb{D}}(gx_{n+1},gx_{n-1})}{1+d_{\mathbb{D}}(x_{n-1},y_{n-1})}\\
&\quad+\frac{\lambda_5 d_{\mathbb{D}}(gx_n,gx_{n-1})+\lambda_6 d_{\mathbb{D}}(gx_n,gx_n)}{1+d_{\mathbb{D}}(x_n,y_n)}\\
&\precsim \lambda_1 d_{\mathbb{D}}(gx_n,gx_{n-1})+\lambda_2 d_{\mathbb{D}}(gy_n,gy_{n-1})+\lambda_3 d_{\mathbb{D}}(gx_{n+1},gx_n)\\
&\quad+\lambda_4 d_{\mathbb{D}}(gx_{n+1},gx_n)+\lambda_4 d_{\mathbb{D}}(gx_n,gx_{n-1})+\lambda_5 d_{\mathbb{D}}(gx_n,gx_{n-1})\\
&= (\lambda_1+\lambda_4+\lambda_5)\, d_{\mathbb{D}}(gx_n,gx_{n-1})+(\lambda_3+\lambda_4)\, d_{\mathbb{D}}(gx_{n+1},gx_n)\\
&\quad+\lambda_2 d_{\mathbb{D}}(gy_n,gy_{n-1}).
\end{aligned}$$

Hence,

$$(1-\lambda_3-\lambda_4)\, d_{\mathbb{D}}(gx_{n+1},gx_n)\precsim(\lambda_1+\lambda_4+\lambda_5)\, d_{\mathbb{D}}(gx_{n-1},gx_n)+\lambda_2 d_{\mathbb{D}}(gy_n,gy_{n-1}). \tag{3.4}$$

and similarly,

$$(1-\lambda_3-\lambda_4)\, d_{\mathbb{D}}(gy_{n+1},gy_n)\precsim(\lambda_1+\lambda_4+\lambda_5)\, d_{\mathbb{D}}(gy_{n-1},gy_n)+\lambda_2 d_{\mathbb{D}}(gx_{n-1},gx_n). \tag{3.5}$$

Put $\delta_n=d_{\mathbb{D}}(gx_{n+1},gx_n)+d_{\mathbb{D}}(gy_{n+1},gy_n)$. Adding inequalities (3.2) and (3.3), we get

$$(1-\lambda_5-\lambda_6)\,\delta_n\precsim(\lambda_1+\lambda_2+\lambda_3+\lambda_6)\,\delta_{n-1} \tag{3.6}$$

and adding inequalities (3.4) and (3.5), we get

$$(1-\lambda_3-\lambda_4)\,\delta_n\precsim(\lambda_1+\lambda_2+\lambda_4+\lambda_5)\,\delta_{n-1}. \tag{3.7}$$

From (3.6) and (3.7), we have

$$(2-\lambda_3-\lambda_4-\lambda_5-\lambda_6)\,\delta_n\precsim(2\lambda_1+2\lambda_2+\lambda_3+\lambda_4+\lambda_5+\lambda_6)\,\delta_{n-1}$$

and so, $\delta_n\precsim\eta\delta_{n-1}$ where $\eta=\frac{2\lambda_1+2\lambda_2+\lambda_3+\lambda_4+\lambda_5+\lambda_6}{2-\lambda_3-\lambda_4-\lambda_5-\lambda_6}\prec 1$. Consequently, we get

$$0\precsim\delta_n\precsim\eta\delta_{n-1}\precsim\ldots\precsim\eta^n\delta_0. \tag{3.8}$$

If $m>n$, we have

$$d_{\mathbb{D}}(gx_n,gx_m)\precsim d_{\mathbb{D}}(gx_n,gx_{n+1})+d_{\mathbb{D}}(gx_{n+1},gx_{n+2})+\ldots+d_{\mathbb{D}}(gx_{m-1},gx_m) \tag{3.9}$$

and

$$d_{\mathbb{D}}(gy_n,gy_m)\precsim d_{\mathbb{D}}(gy_n,gy_{n+1})+d_{\mathbb{D}}(gy_{n+1},gy_{n+2})+\ldots+d_{\mathbb{D}}(gy_{m-1},gy_m). \tag{3.10}$$

Now, (3.8), (3.9), (3.10) and Theorem 2.5 (i) imply that

$$\begin{aligned}
d_{\mathbb{D}}(gx_n,gx_m)+d_{\mathbb{D}}(gy_n,gy_m) &\precsim d_{\mathbb{D}}(gx_n,gx_{n+1})+d_{\mathbb{D}}(gy_n,gy_{n+1})+d_{\mathbb{D}}(gx_{n+1},gx_{n+2})\\
&\quad+d_{\mathbb{D}}(gy_{n+1},gy_{n+2})+\ldots+d_{\mathbb{D}}(gx_{m-1},gx_m)+d_{\mathbb{D}}(gy_{m-1},gy_m)\\
&= \delta_n+\delta_{n+1}+\delta_{n+2}+\ldots+\delta_{m-1}\\
&\precsim \left(\eta^n+\eta^{n+1}+\eta^{n+2}+\ldots+\eta^{m-1}\right)\delta_0\\
&= \eta^n\left(1+\eta+\ldots+\eta^{m-n-1}\right)\delta_0\\
&= \eta^n\frac{1-\eta^{m-n}}{1-\eta}\delta_0\\
&\precsim \frac{\eta^n}{1-\eta}\delta_0.
\end{aligned}$$

Since $\frac{\eta^n}{1-\eta}\delta_0 \to 0$ as $n \to \infty$ by Theorem 2.5 (ii), we obtain that $d_{\mathbb{D}}(gx_n, gx_m) \to 0$ and $d_{\mathbb{D}}(gy_n, gy_m) \to 0$ as $m, n \to \infty$. In this case, (gx_n) and (gy_n) are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, there exist x^* and y^* such that $gx_n \to gx^*$ and $gy_n \to gy^*$ as $n \to \infty$.

Now, we shall show that $gx^* = F(x^*, y^*)$ and $gy^* = F(y^*, x^*)$. For that, we have

$$\begin{aligned} d_{\mathbb{D}}(F(x^*,y^*),gx^*) &\precsim d_{\mathbb{D}}(F(x^*,y^*),gx_{n+1}) + d_{\mathbb{D}}(gx_{n+1},gx^*) \\ &= d_{\mathbb{D}}(F(x^*,y^*),F(x_n,y_n)) + d_{\mathbb{D}}(gx_{n+1},gx^*) \\ &\precsim \lambda_1 d_{\mathbb{D}}(gx^*,gx_n) + \lambda_2 d_{\mathbb{D}}(gy^*,gy_n) \\ &\quad + \frac{\lambda_3 d_{\mathbb{D}}(F(x^*,y^*),gx^*) + \lambda_4 d_{\mathbb{D}}(F(x^*,y^*),gx_n)}{1+d_{\mathbb{D}}(x_n,y_n)} \\ &\quad + \frac{\lambda_5 d_{\mathbb{D}}(F(x_n,y_n),gx_n) + \lambda_6 d_{\mathbb{D}}(F(x_n,y_n),gx^*)}{1+d_{\mathbb{D}}(x^*,y^*)} + d_{\mathbb{D}}(gx_{n+1},gx^*) \\ &\precsim \lambda_1 d_{\mathbb{D}}(gx^*,gx_n) + \lambda_2 d_{\mathbb{D}}(gy^*,gy_n) \\ &\quad + \frac{\lambda_3 d_{\mathbb{D}}(F(x^*,y^*),gx^*) + \lambda_4 \left[d_{\mathbb{D}}(F(x^*,y^*),gx^*) + d_{\mathbb{D}}(gx^*,gx_n)\right]}{1+d_{\mathbb{D}}(x_n,y_n)} \\ &\quad + \frac{\lambda_5 \left[d_{\mathbb{D}}(gx_{n+1},gx^*) + d_{\mathbb{D}}(gx^*,gx_n)\right] + \lambda_6 d_{\mathbb{D}}(gx_{n+1},gx^*)}{1+d_{\mathbb{D}}(x^*,y^*)} \\ &\quad + d_{\mathbb{D}}(gx_{n+1},gx^*) \\ &\precsim \lambda_1 d_{\mathbb{D}}(gx^*,gx_n) + \lambda_2 d_{\mathbb{D}}(gy^*,gy_n) + \lambda_3 d_{\mathbb{D}}(F(x^*,y^*),gx^*) \\ &\quad + \lambda_4 d_{\mathbb{D}}(F(x^*,y^*),gx^*) + \lambda_4 d_{\mathbb{D}}(gx^*,gx_n) + \lambda_5 d_{\mathbb{D}}(gx_{n+1},gx^*) \\ &\quad + \lambda_5 d_{\mathbb{D}}(gx^*,gx_n) + \lambda_6 d_{\mathbb{D}}(gx_{n+1},gx^*) + d_{\mathbb{D}}(gx_{n+1},gx^*). \end{aligned}$$

This implies that

$$\begin{aligned} (1-\lambda_3-\lambda_4) d_{\mathbb{D}}(F(x^*,y^*),gx^*) &\precsim (\lambda_1+\lambda_4+\lambda_5) d_{\mathbb{D}}(gx_n,gx^*) + (\lambda_5+\lambda_6) d_{\mathbb{D}}(gx_{n+1},gx^*) \\ &\quad + \lambda_2 d_{\mathbb{D}}(gy_n,gy^*) \end{aligned}$$

and so,

$$\begin{aligned} d_{\mathbb{D}}(F(x^*,y^*),gx^*) &\precsim \frac{\lambda_1+\lambda_4+\lambda_5}{1-\lambda_3-\lambda_4} d_{\mathbb{D}}(gx_n,gx^*) + \frac{\lambda_5+\lambda_6}{1-\lambda_3-\lambda_4} d_{\mathbb{D}}(gx_{n+1},gx^*) \\ &\quad + \frac{\lambda_2}{1-\lambda_3-\lambda_4} d_{\mathbb{D}}(gy_n,gy^*). \end{aligned}$$

Letting $n \to \infty$ we get $d_{\mathbb{D}}(F(x^*,y^*),gx^*) = 0$ and hence $F(x^*,y^*) = gx^*$. By similar way, we obtain $F(y^*,x^*) = gy^*$. Therefore, (x^*,y^*) is a coupled coincidence of F and g. □

By setting $g = I_X$ where I_X is the identity mapping on X in Theorem 3.7, we deduce the following coupled fixed point theorem.

Corollary 3.8. *Let $(X, d_{\mathbb{D}})$ be a hyperbolic valued metric space and $F : X \times X \to X$. Suppose that there exist positive hyperbolic numbers λ_i, $i = 1,2,3,4,5,6$ with $\sum_{i=1}^{6} \lambda_i \prec 1$ such that the following contractive condition holds for all $x, y, u, v \in X$:*

$$\begin{aligned} d_{\mathbb{D}}(F(x,y),F(u,v)) &\precsim \lambda_1 d_{\mathbb{D}}(x,u) + \lambda_2 d_{\mathbb{D}}(y,v) \\ &\quad + \frac{\lambda_3 d_{\mathbb{D}}(F(x,y),x) + \lambda_4 d_{\mathbb{D}}(F(x,y),u)}{1+d_{\mathbb{D}}(u,v)} \\ &\quad + \frac{\lambda_5 d_{\mathbb{D}}(F(u,v),u) + \lambda_6 d_{\mathbb{D}}(F(u,v),x)}{1+d_{\mathbb{D}}(x,y)}. \end{aligned}$$

Then F has a coupled fixed point in X.

By setting $\lambda = \lambda_1 = \lambda_2,\ \mu = \lambda_3 = \lambda_4$ and $\gamma = \lambda_5 = \lambda_6$ in Theorem 3.7, we get the following corollary.

Corollary 3.9. *Let $(X, d_{\mathbb{D}})$ be a hyperbolic valued metric space, $F : X \times X \to X$ and $g : X \to X$. Suppose that there exist positive hyperbolic numbers $\lambda,\ \mu,\ \gamma$ with $\lambda + \mu + \gamma \prec \frac{1}{2}$ such that the following contractive condition holds for all $x, y, u, v \in X$:*

$$\begin{aligned} d_{\mathbb{D}}(F(x,y), F(u,v)) \precsim\ & \lambda\left[d_{\mathbb{D}}(gx,gu) + d_{\mathbb{D}}(gy,gv)\right] \\ & + \mu\frac{d_{\mathbb{D}}(F(x,y),gx) + d_{\mathbb{D}}(F(x,y),gu)}{1 + d_{\mathbb{D}}(u,v)} \\ & + \gamma\frac{d_{\mathbb{D}}(F(u,v),gu) + d_{\mathbb{D}}(F(u,v),gx)}{1 + d_{\mathbb{D}}(x,y)}. \end{aligned}$$

If $F(X \times X) \subset g(X)$ and $g(X)$ is a complete subspace of X, then F and g have a coupled coincidence point in X.

Remark 3.10. *If we take $\lambda_i = 0,\ i = 1,2,3,4,5,6$ in all possible combinations, we obtain coupled coincidence point theorems on hyperbolic valued metric spaces.*

The following example illustrate Theorem 3.7. In this example, we will use definition and some properties of $|\cdot|_k$ which are given in the preliminaries.

Example 3.11. *Let $d_{\mathbb{D}} : \mathbb{BC} \times \mathbb{BC} \to \mathbb{D}$, $d_{\mathbb{D}}(x,y) = |x-y|_k$. Then, $(\mathbb{BC}, d_{\mathbb{D}})$ is a hyperbolic valued metric space. Define mappings $F : \mathbb{BC} \times \mathbb{BC} \to \mathbb{BC}$ and $g : \mathbb{BC} \to \mathbb{BC}$ as*

$$F(x,y) = \frac{x+ij}{9j} + iy,\ \ g(x) = 4(2i+j)x.$$

Then, $F(\mathbb{BC} \times \mathbb{BC}) \subset \mathbb{BC} = g(\mathbb{BC})$ and $g(\mathbb{BC})$ is a complete subspace of $\mathbb{BC}$. Now we get

$$\begin{aligned} d_{\mathbb{D}}(F(x,y), F(u,v)) &= d_{\mathbb{D}}\left(\frac{x+ij}{9j} + iy, \frac{u+ij}{9j} + iv\right) \\ &= \left|\frac{x-u}{9j} + i(y-v)\right|_k \\ &\precsim \left|\frac{x-u}{9j}\right|_k + |i(y-v)|_k \\ &= \frac{|x-u|_k}{|9j|_k} + |y-v|_k \\ &= \frac{|x-u|_k}{9} + |y-v|_k \\ &= \left(\frac{1}{9}e_1 + \frac{1}{9}e_2\right)|x-u|_k + |y-v|_k \\ &\precsim (2e_1 + 4e_2)|x-u|_k + \left(\frac{4}{3}e_1 + 3e_2\right)|y-v|_k \\ &= \left(\frac{1}{2}e_1 + \frac{1}{3}e_2\right)4(e_1+3e_2)|x-u|_k + \left(\frac{1}{3}e_1 + \frac{1}{4}e_2\right)4(e_1+3e_2)|y-v|_k \\ &= \left(\frac{1}{2}e_1 + \frac{1}{3}e_2\right)d_{\mathbb{D}}(gx,gu) + \left(\frac{1}{3}e_1 + \frac{1}{4}e_2\right)d_{\mathbb{D}}(gy,gv) \\ &\precsim \left(\frac{1}{2}e_1 + \frac{1}{3}e_2\right)d_{\mathbb{D}}(gx,gu) + \left(\frac{1}{3}e_1 + \frac{1}{4}e_2\right)d_{\mathbb{D}}(gy,gv) \\ &\quad + \frac{\left(\frac{1}{30}e_1 + \frac{1}{12}e_2\right)d_{\mathbb{D}}(F(x,y),gx) + \left(\frac{1}{30}e_1 + \frac{1}{12}e_2\right)d_{\mathbb{D}}(F(x,y),gu)}{1 + d_{\mathbb{D}}(u,v)} \\ &\quad + \frac{\left(\frac{1}{30}e_1 + \frac{1}{12}e_2\right)d_{\mathbb{D}}(F(u,v),gu) + \left(\frac{1}{30}e_1 + \frac{1}{12}e_2\right)d_{\mathbb{D}}(F(u,v),gx)}{1 + d_{\mathbb{D}}(x,y)} \end{aligned}$$

where $\lambda_1 = \frac{1}{2}e_1 + \frac{1}{3}e_2$, $\lambda_2 = \frac{1}{3}e_1 + \frac{1}{4}e_2$, $\lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \frac{1}{30}e_1 + \frac{1}{12}e_2$. *Note that* $\left(\frac{1}{2}e_1 + \frac{1}{3}e_2\right) + \left(\frac{1}{3}e_1 + \frac{1}{4}e_2\right) + 4.\left(\frac{1}{30}e_1 + \frac{1}{12}e_2\right) = \frac{29}{30}e_1 + \frac{11}{12}e_2 \prec 1$. *Thus, the conditions of Theorem 3.7 are satisfied. Then,* F *and* g *have a coupled coincidence point in* $\mathbb{BC}$. *This coupled coincidence point is* $(0,0)$.

3.3. Common Coupled Fixed Point Results

In this section, we obtain a unique common coupled fixed point of two mappings in hyperbolic valued metric spaces by using the concept of $w-$compatibility. As an application of our main result in this section, we discuss a problem for the existence and uniqueness of coupled coincidence points for two mappings.

Definition 3.12. *The mappings* $F : X \times X \to X$ *and* $g : X \to X$ *are called* $w-$*compatible if* $g(F(x,y)) = F(gx, gy)$ *whenever* $gx = F(x,y)$ *and* $gy = F(y,x)$ *(see [8]).*

Theorem 3.13. *In addition to hypotheses of Theorem 3.7, if the mappings* $F : X \times X \to X$ *and* $g : X \to X$ *are* $w-$*compatible, then* F *and* g *have a unique common coupled fixed point. Also, common fixed point of* F *and* g *is of the form* (u,u) *for some* $u \in X$.

Proof. Theorem 3.7 implies that there exists a coupled coincidence point (x^*, y^*) of F and g. First, we will show that the coupled point of coincidence is unique. Suppose that there exists another coupled point of coincidence such that $gx' = F(x', y')$ and $gy' = F(y', x')$ where $(x', y') \in X \times X$. Using (3.1), we get

$$\begin{aligned} d_{\mathbb{D}}(gx^*, gx') &= d_{\mathbb{D}}(F(x^*, y^*), F(x', y')) \\ &\precsim \lambda_1 d_{\mathbb{D}}(gx^*, gx') + \lambda_2 d_{\mathbb{D}}(gy^*, gy') \\ &\quad + \frac{\lambda_3 d_{\mathbb{D}}(F(x^*, y^*), gx^*) + \lambda_4 d_{\mathbb{D}}(F(x^*, y^*), gx')}{1 + d_{\mathbb{D}}(x', y')} \\ &\quad + \frac{\lambda_5 d_{\mathbb{D}}(F(x', y'), gx') + \lambda_6 d_{\mathbb{D}}(F(x', y'), gx^*)}{1 + d_{\mathbb{D}}(x^*, y^*)} \\ &= \lambda_1 d_{\mathbb{D}}(gx^*, gx') + \lambda_2 d_{\mathbb{D}}(gy^*, gy') + \frac{\lambda_4 d_{\mathbb{D}}(gx^*, gx')}{1 + d_{\mathbb{D}}(x', y')} + \frac{\lambda_6 d_{\mathbb{D}}(gx', gx^*)}{1 + d_{\mathbb{D}}(x^*, y^*)} \\ &\precsim \lambda_1 d_{\mathbb{D}}(gx^*, gx') + \lambda_2 d_{\mathbb{D}}(gy^*, gy') + \lambda_4 d_{\mathbb{D}}(gx^*, gx') + \lambda_6 d_{\mathbb{D}}(gx', gx^*). \end{aligned}$$

Hence

$$d_{\mathbb{D}}(gx^*, gx') \precsim (\lambda_1 + \lambda_4 + \lambda_6) d_{\mathbb{D}}(gx^*, gx') + \lambda_2 d_{\mathbb{D}}(gy^*, gy'). \tag{3.11}$$

By a similar way, we can prove that

$$d_{\mathbb{D}}(gy^*, gy') \precsim (\lambda_1 + \lambda_4 + \lambda_6) d_{\mathbb{D}}(gy^*, gy') + \lambda_2 d_{\mathbb{D}}(gx^*, gx'). \tag{3.12}$$

By adding inequalities (3.11) and (3.12), we get

$$d_{\mathbb{D}}(gx^*, gx') + d_{\mathbb{D}}(gy^*, gy') \precsim (\lambda_1 + \lambda_2 + \lambda_4 + \lambda_6)\left[d_{\mathbb{D}}(gx^*, gx') + d_{\mathbb{D}}(gy^*, gy')\right].$$

Since $\lambda_1 + \lambda_2 + \lambda_4 + \lambda_6 \prec 1$, we have $d_{\mathbb{D}}(gx^*, gx') + d_{\mathbb{D}}(gy^*, gy') = 0$ and so $d_{\mathbb{D}}(gx^*, gx') = d_{\mathbb{D}}(gy^*, gy') = 0$. In this case, we obtain that $gx^* = gx'$ and $gy^* = gy'$. Therefore, the unique coupled point of coincidence of F and g is (gx^*, gy^*). On the other hand, similarly, we can show that $gx^* = gy'$ and $gy^* = gx'$. Thus, we get $gx^* = gy^*$.

Let $u = gx^* = F(x^*, y^*)$. Using condition of $w-$compatible of F and g, we get

$$gu = g(gx^*) = g(F(x^*, y^*)) = F(gx^*, gy^*) = F(gx^*, gx^*) = F(u,u).$$

Then, (gu, gu) is a coupled point of coincidence of F and g and so $gu = gx^*$. Thus, $u = gu = F(u,u)$. This statement implies that (u,u) is the unique common coupled fixed point of F and g. □

Corollary 3.14. *In addition to hypotheses of Corollary 3.9, if the mappings* $F : X \times X \to X$ *and* $g : X \to X$ *are* $w-$*compatible, then* F *and* g *have a unique common coupled fixed point.*

Proof. Taking $\lambda = \lambda_1 = \lambda_2,\ \mu = \lambda_3 = \lambda_4$ and $\gamma = \lambda_5 = \lambda_6$ in Theorem 3.13, we get Corollary 3.14. □

Remark 3.15. *If we take $\lambda_i = 0,\ i = 1,2,3,4,5,6$ in all possible combinations, we obtain common coupled fixed point theorems on hyperbolic valued metric spaces.*

We give an example which supports Theorem 3.13.

Example 3.16. *Let X and $d_{\mathbb{D}}$ be as in Example 3.4. Then, $(X, d_{\mathbb{D}})$ is a complete hyperbolic valued metric space. Now, we define mappings $F : X \times X \to X$ and $g : X \to X$ as*

$$F(\alpha,\beta) = \frac{\alpha_1}{3}e_1 + \frac{\alpha_2}{3}e_2,\ \ g(\gamma) = \begin{cases} \gamma_1 e_1 - \gamma_1 e_2,\ \gamma \in X_1 \\ \\ \gamma_1 e_1 + \gamma_1 e_2,\ \gamma \in X_2 \end{cases}$$

where $\alpha = \alpha_1 e_1 + \alpha_2 e_2, \beta = \beta_1 e_1 + \beta_2 e_2, \gamma = \gamma_1 e_1 + \gamma_2 e_2$. Then, $F(X \times X) \subset X = g(X)$ and $g(X)$ is a complete subspace of X. Also, it is easy to show that the mappings $F : X \times X \to X$ and $g : X \to X$ are $w-$compatible. On the other hand, we obtain that

$$\begin{aligned}
d_{\mathbb{D}}(F(\alpha,\beta), F(u,v)) &= d_{\mathbb{D}}\left(\frac{\alpha_1}{3}e_1 + \frac{\alpha_2}{3}e_2, \frac{u_1}{3}e_1 + \frac{u_2}{3}e_2\right) \\
&= \begin{cases} \frac{7}{18}|\alpha_1 - u_1| e_1 + \frac{3}{8}|\alpha_1 - u_1| e_2,\ F(\alpha,\beta), F(u,v) \in X_1 \\ \frac{5}{18}|\alpha_1 - u_1| e_1 + \frac{7}{24}|\alpha_1 - u_1| e_2,\ F(\alpha,\beta), F(u,v) \in X_2 \\ \left(\frac{7}{18}\alpha_1 + \frac{5}{18}\beta_1\right) e_1 + \left(\frac{3}{8}\alpha_1 + \frac{7}{24}\beta_1\right) e_2,\ F(\alpha,\beta) \in X_1, F(u,v) \in X_2 \\ \left(\frac{5}{18}\alpha_1 + \frac{7}{18}\beta_1\right) e_1 + \left(\frac{7}{24}\alpha_1 + \frac{3}{8}\beta_1\right) e_2,\ F(\alpha,\beta) \in X_2, F(u,v) \in X_1 \end{cases} \\
&\precsim \begin{cases} \frac{5}{12}|\alpha_1 - u_1| e_1 + \frac{7}{12}|\alpha_1 - u_1| e_2,\ F(\alpha,\beta), F(u,v) \in X_1 \\ \frac{7}{12}|\alpha_1 - u_1| e_1 + \frac{3}{4}|\alpha_1 - u_1| e_2,\ F(\alpha,\beta), F(u,v) \in X_2 \\ \left(\frac{5}{12}\alpha_1 + \frac{7}{12}\beta_1\right) e_1 + \left(\frac{7}{12}\alpha_1 + \frac{3}{4}\beta_1\right) e_2,\ F(\alpha,\beta) \in X_1, F(u,v) \in X_2 \\ \left(\frac{7}{12}\alpha_1 + \frac{5}{12}\beta_1\right) e_1 + \left(\frac{3}{4}\alpha_1 + \frac{7}{12}\beta_1\right) e_2,\ F(\alpha,\beta) \in X_2, F(u,v) \in X_1 \end{cases} \\
&= \left(\frac{1}{2}e_1 + \frac{2}{3}e_2\right) \begin{cases} \frac{5}{6}|\alpha_1 - u_1| e_1 + \frac{7}{8}|\alpha_1 - u_1| e_2,\ \alpha, u \in X_1 \\ \frac{7}{6}|\alpha_1 - u_1| e_1 + \frac{9}{8}|\alpha_1 - u_1| e_2,\ \alpha, u \in X_2 \\ \left(\frac{5}{6}\alpha_1 + \frac{7}{6}u_1\right) e_1 + \left(\frac{7}{8}\alpha_1 + \frac{9}{8}u_1\right) e_2,\ \alpha \in X_1, u \in X_2 \\ \left(\frac{7}{6}\alpha_1 + \frac{5}{6}u_1\right) e_1 + \left(\frac{9}{8}\alpha_1 + \frac{7}{8}u_1\right) e_2,\ \alpha \in X_2, u \in X_1 \end{cases} \\
&= \left(\frac{1}{2}e_1 + \frac{2}{3}e_2\right) d_{\mathbb{D}}(g\alpha, gu)
\end{aligned}$$

$$\precsim \left(\frac{1}{2}e_1+\frac{2}{3}e_2\right)d_{\mathbb{D}}\left(g\alpha,gu\right)+\left(\frac{2}{21}e_1+\frac{3}{47}e_2\right)d_{\mathbb{D}}\left(g\beta,gv\right)+ \frac{\left(\frac{2}{21}e_1+\frac{3}{47}e_2\right)d_{\mathbb{D}}\left(F\left(\alpha,\beta\right),g\alpha\right)+\left(\frac{2}{21}e_1+\frac{3}{47}e_2\right)d_{\mathbb{D}}\left(F\left(\alpha,\beta\right),gu\right)}{1+d_{\mathbb{D}}\left(u,v\right)}+ \frac{\left(\frac{2}{21}e_1+\frac{3}{47}e_2\right)d_{\mathbb{D}}\left(F\left(u,v\right),gu\right)+\left(\frac{2}{21}e_1+\frac{3}{47}e_2\right)d_{\mathbb{D}}\left(F\left(u,v\right),g\alpha\right)}{1+d_{\mathbb{D}}\left(\alpha,\beta\right)}$$

where $\lambda_1=\frac{1}{2}e_1+\frac{2}{3}e_2,\ \lambda_2=\ \lambda_3=\lambda_4=\lambda_5=\lambda_6=\frac{2}{21}e_1+\frac{3}{47}e_2$. *Note that* $\left(\frac{1}{2}e_1+\frac{2}{3}e_2\right)+5.\left(\frac{2}{21}e_1+\frac{3}{47}e_2\right)=\frac{41}{42}e_1+\frac{139}{141}e_2\prec 1$. *Therefore, all the conditions of Theorem 3.13 hold. Then* F *and* g *have a unique common coupled fixed point and this common fixed point of* F *and* g *is* $(0,0)$.

4. Conclusion

In this work, we have proved the existence of unique common fixed point for contraction mappings and a coupled coincidence and unique common coupled fixed point for two mappings on hyperbolic valued metric spaces. We also have discussed some illustrative examples which substantiate the authenticity of our newly proved results and distinguish them from existing ones. We hope that the results will help the researchers in the literature of fixed point theory.

Acknowledgments

The authors would like to thank the anonymous referees and the handling editor for their careful reading and suggestions that improved the presentation of this paper.

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