A New Class of Higher-order Hypergeometric Bernoulli Polynomials Associated with Hermite Polynomials

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Abstract: In this paper, we introduce a new class of higher-order hypergeometric Hermite-Bernoulli numbers and polynomials. We shall provide several properties of higher-order hypergeometric Hermite-Bernoulli polynomials including summation formulae, sums of product identity, recurrence relations.

Key Words: Hermite polynomials, Higher-order hypergeometric Bernoulli polynomials, Higher-order hypergeometric Hermite-Bernoulli polynomials, Recurrence relations.

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1. Introduction

The Bernoulli polynomials $B_n(x)$ are defined by the following generating function

$$\left( \frac{t}{e^t - 1} \right) e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},$$

and $B_n = B_n(0)$ are named Bernoulli numbers. These numbers and polynomials have a long history, which arise from Bernoulli’s calculations of power sums in 1713, that is,

$$\sum_{j=1}^{m} j^n = \frac{B_{n+1}(m+1) - B_{n+1}}{n+1},$$

(see [19], p.5, (2.2)). They have many applications in modern number theory, such as modular forms [11] and Iwasawa theory [9]. A recent book by Arakawa, Ibukiyama and Kaneko [1] give a nice introduction of Bernoulli numbers and polynomials including their connections with zeta functions.

In 1924, Nörlund [14] introduced and studied the generalized higher order Bernoulli polynomials defined by means of the following generating function

$$\left( \frac{t}{e^t - 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!},$$

(1.2)

We also have a similar expression of multiple power sums

$$\sum_{l_1, \ldots, l_n=0}^{m-1} (t + l_1 + \cdots + l_n)^k,$$

in terms of higher order Bernoulli polynomials, (see ([12], Lemma 2.1)).

2010 Mathematics Subject Classification: Primary: 11M35, Secondary: 11B68, 33C45.

Howard ([5], [6]) gave a generalization of Bernoulli polynomials by considering the following generating function

$$\frac{t^2 e^{xt} / 2}{e^t - 1 - t} = \sum_{n=0}^{\infty} A_n^{(a)}(x) \frac{t^n}{n!},$$  \hspace{1cm} (1.3)

and more generally, for all positive integer N

$$\frac{t^N}{e^t - T_{N-1}(t)} e^{xt} = \sum_{n=0}^{\infty} B_{N,n}(x) \frac{t^n}{n!},$$  \hspace{1cm} (1.4)

where $T_{N-1}(t)$ is the Taylor polynomial of order $N - 1$ for the exponential function. For the case $N = 1$ and $N = 2$, (1.4) reduces to (1.1) and (1.3), respectively. We see that the polynomials $B_{N,n}(x)$ have rational coefficients.

The polynomials $B_{N,n}(x)$ are named hypergeometric Bernoulli polynomials, while the numbers $B_{N,n} = B_{N,n}(0)$ are named hypergeometric Bernoulli numbers since the generating function $f(t) = \frac{e^{xt} - T_{N-1}(t)}{e^t}$ can be expressed as $_1F_1(1; N + 1; t)$, where the confluent hypergeometric function $_1F_1(a; b; t)$ is defined by

$$\_1F_1(a; b; t) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{t^n}{n!},$$  \hspace{1cm} (1.5)

and $(a)_n$ is the Pochhammer symbol, (see [20])

$$(a)_0 := 1, \ (a)_n = a(a + 1) \cdots (a + n - 1), (n \in \mathbb{N} := \{1, 2, 3, \cdots \}).$$

For $N, r \in \mathbb{N}$, the higher-order hypergeometric Bernoulli polynomials $B_{N,n}^{(r)}(x)$ are defined by means of the generating function, (see [2], [7], [10])

$$\left(\frac{t^N}{e^t - T_{N-1}(t)}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_{N,n}^{(r)}(x) \frac{t^n}{n!}. \hspace{1cm} (1.6)$$

For $x = 0$ in (1.6), $B_{N,n}^{(r)} = B_{N,n}^{(r)}(0)$ are called the higher order hypergeometric Bernoulli numbers, (see [10], [13]). Again, on taking $r = 1$ in (1.6), $B_{N,n}^{(1)}(x) = B_{N,n}(x)$ are called the hypergeometric Bernoulli polynomials and if we put $x = 0$ in (1.6), $B_{N,n}^{(1)}(0) = B_{N,n}$ are called the hypergeometric Bernoulli numbers.

The 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) $H_n(x, y)$ ([3], [4]) are defined as

$$H_n(x, y) = n! \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{y^r x^{n-2r}}{r!(n-2r)!}. \hspace{1cm} (1.7)$$

It is easily seen that

$$H_n(2x, -1) = H_n(x), H_n(x, -\frac{1}{2}) = He_n(x),$$

where $H_n(x)$ and $He_n(x)$ are called the ordinary Hermite polynomials. Also

$$H_n(x, 0) = x^n.$$  

The generating function for Hermite polynomial $H_n(x, y)$ ([16]-[18]) are given by

$$e^{xt + yt^2} = \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!}. \hspace{1cm} (1.8)$$
The object of this paper is to present a systematic account of these families in a unified and generalized form. We develop some elementary properties and derive the implicit summation formulae for the higher-order hypergeometric Hermite-Bernoulli polynomials by using different analytical means on their respective generating functions. The approach given in recent papers of Pathan and Khan ([16]-[18]) has indeed allowed the derivation of implicit summation formulae in the two-variable higher-order hypergeometric Hermite-Bernoulli polynomials. In addition to this, some relevant connections between Hermite and higher-order hypergeometric Bernoulli polynomials and recurrence relations are given.

2. Multiple hypergeometric Hermite-Bernoulli numbers and polynomials

For every positive integer \( N \) and \( r \), the higher-order hypergeometric Hermite-Bernoulli numbers and polynomials \( H^{(r)}_{N,n}(x,y) \) are defined by means of the following generating function defined in a suitable neighborhood of \( t = 0 \):

\[
F_{r,N}(x,y,t) = \frac{1}{1F_1(1;N+1;t)} e^{xt+yt^2} = \left( \frac{t^N}{N!} \right)^r e^{xt} - \sum_{n=0}^{N-1} \frac{t^n}{n!},
\]

(2.1)

For \( x = y = 0 \), \( B^{(r)}_{N,n} = H^{(r)}_{N,n}(0,0) \) are called the higher-order hypergeometric Bernoulli numbers, (see [10, 13]). When \( r = 1 \), we obtain the hypergeometric Hermite-Bernoulli polynomials \( H^{(1)}_{N,n}(x,y) = H^{(1)}_{N,n}(0,0) \) is the hypergeometric Bernoulli numbers, (see [8, 15]). If we put \( N = 1 \), the result reduces to the known result of Pathan and Khan, (see [16]).

Remark 2.1. On setting \( y = 0 \), (2.1) reduces to the known result of Aoki et al. [2] as follows:

\[
F_{r,N}(x,t) = \frac{1}{1F_1(1;N+1;t)} e^{xt} = \left( \frac{t^N}{N!} \right)^r e^{xt} - \sum_{n=0}^{N-1} \frac{t^n}{n!},
\]

(2.2)

In particular in terms of higher-order hypergeometric Bernoulli numbers \( B^{(r)}_{N,n} \) and Hermite polynomials \( H_s(x,y) \), the higher order Hermite-Bernoulli polynomials \( H^{(r)}_{N,n}(x,y) \) are defined as

\[
H^{(r)}_{N,n}(x,y) = \sum_{s=0}^{n} \binom{n}{s} B^{(r)}_{N,n-s} H_s(x,y).
\]

(2.3)

Taking \( r = N = 1 \) and \( x = 0 \) in (2.1) gives the result

\[
\sum_{m=0}^{\infty} \binom{n}{2m} B_{n-2m} y^m = H^{(1)}_{1,n}(0, y).
\]

(2.4)

Using \( e^{it} = \cos t + i \sin t \) and \( N = 1 \), the result reduces to

\[
\sum_{n=0}^{\infty} f(n) = \sum_{n=0}^{\infty} f(2n) + \sum_{n=0}^{\infty} f(2n + 1),
\]

(2.5)
and 
\[ \left( \frac{it}{e^{it} - 1} \right)^r = \left( \frac{it \cos t - 1 - i \sin t}{(\cos t - 1 + i \sin t)(\cos t - 1 - i \sin t)} \right)^r = \left( \frac{it \cos t - 1 - i \sin t}{(\cos t - 1)^2 + (\sin t)^2} \right)^r = \left( \frac{(t \sin t) + it \cos t - 1}{\Omega} \right)^r, \]

where \( \Omega = (\cos t - 1)^2 + (\sin t)^2 \), together with the definition (2.1) and the result (2.5), we get (see Pathan and Khan [16]):

\[ e^{ixt + yt^2} \left( \frac{(t \sin t) + it \cos t - 1}{\Omega} \right)^r = \sum_{n=0}^{\infty} H B_{2n}^{(r)}(x, y) \frac{(-1)^n t^{2n}}{(2n)!} + \sum_{n=0}^{\infty} H B_{2n+1}^{(r)}(x, y) \frac{(-1)^n t^{2n+1}}{(2n + 1)!}, \]

(2.6)

where \( r \geq 1, \Omega = (\cos t - 1)^2 + (\sin t)^2 \).

On setting \( r = 1, x = y = 0 \) in the above results, we get the following well known classical results involving Bernoulli numbers, (see [16]):

\[ \frac{t}{2} \cot \left( \frac{t}{2} \right) = \sum_{n=0}^{\infty} B_{2n} \frac{(-1)^n t^{2n}}{(2n)!}, \quad \frac{t}{2} \coth \left( \frac{t}{2} \right) = \sum_{n=0}^{\infty} B_{2n} t^{2n} (2n)! \cdot \]

**Theorem 2.2.** For \( n \geq 1 \), we have

\[ H_n(x, y) = n!(N)! \sum_{m=0}^{n} \sum_{i_1 + \cdots + i_r = n-m} \frac{H B_{N,m}^{(r)}(x, y)}{m!(N + i_1)! \cdots (N + i_r)!}. \]

(2.7)

**Proof.** From definition (2.1), we have

\[ \left( \frac{t}{N} \right)^r e^{xt + yt^2} = \left( \frac{t + N}{(i + N)!} \right)^r \sum_{n=0}^{\infty} H B_{N,n}^{(r)}(x, y) \frac{t^n}{n!} \]

\[ = r^N \sum_{i=0}^{\infty} \sum_{i_1 + \cdots + i_r = 1} \frac{t!}{(N + i_1)! \cdots (N + i_r)!} \left( \sum_{m=0}^{\infty} H B_{N,m}^{(r)}(x, y) \frac{t^m}{m!} \right) \]

\[ = r^N \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{i_1 + \cdots + i_r = n-m} \frac{H B_{N,m}^{(r)}(x, y)}{m!(N + i_1)! \cdots (N + i_r)!}. \]

Comparing the coefficients of \( t^n \) on both sides, we get (2.7).

**Corollary 2.3.** For \( r = 1 \) in (2.7), we get

\[ H_n(x, y) = n!(N)! \sum_{m=0}^{n} \left( \frac{n + N}{m} \right) H B_{N,m}(x, y). \]

(2.8)

**Corollary 2.4.** For \( x = y = 0 \) in (2.7), the result reduces to the known result of Aoki et al. [2] as follows

\[ \sum_{m=0}^{n} \sum_{i_1 + \cdots + i_r = n-m} \frac{B_{N,m}^{(r)}}{m!(N + i_1)! \cdots (N + i_r)!} = 0. \]

(2.9)

and \( r = 1 \) in (2.8), the result reduces to (see [7]):

\[ \sum_{m=0}^{n} \left( \frac{n + N}{m} \right) B_{N,m}(x, y) = 0. \]

(2.10)
Theorem 2.5. The following relationship holds true:

\[ H_n(x, y) = \sum_{m=0}^{n} \binom{n}{m} m! \Gamma(N+1) \frac{H_{B_{N,n-m}}(x, y)}{\Gamma(N+1+m)} \]  \hspace{1cm} (2.11)

Proof. Using equations (2.1), (1.5) and (1.8), we have

\[ \frac{1}{1 F_1(1; N + 1; t)} e^{xt+yt^2} = \sum_{n=0}^{\infty} H_{B_{N,n}}(x, y) \frac{t^n}{n!} \]

\[ e^{xt+yt^2} = \sum_{n=0}^{\infty} \frac{(1)_m}{(N+1)_m} \frac{t^m}{m!} \sum_{n=0}^{\infty} H_{B_{N,n}}(x, y) \frac{t^n}{n!} \]

\[ \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!} = \sum_{m=0}^{\infty} \frac{(1)_m}{(N+1)_m} \frac{t^m}{m!} \sum_{n=0}^{\infty} H_{B_{N,n}}(x, y) \frac{t^n}{n!} \]

\[ = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} m! \Gamma(N+1) \frac{H_{B_{N,n-m}}(x, y)}{\Gamma(N+1+m)} \frac{t^n}{n!} \]

Comparing the coefficients of \( t^n \) on both sides, we arrive at the obtained result (2.11).

Theorem 2.6. The following relationship holds true:

\[ \int_0^1 (1-x)^{N-1} H_{B^{(r)}_{N,n}}(x, y) dx = (N-1)! \sum_{k=0}^{n} \binom{n}{k} \frac{(n-k)!}{(N+n-k)!} H_{B^{(r)}_{N,k}}(0, y). \]  \hspace{1cm} (2.12)

Proof. From (2.1), we have

\[ \frac{1}{1 F_1(1; N + 1; t)} e^{xt+yt^2} = \sum_{n=0}^{\infty} H_{B^{(r)}_{N,n}}(x, y) \frac{t^n}{n!} \]

\[ e^{xt} \sum_{n=0}^{\infty} H_{B^{(r)}_{N,n}}(0, y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} H_{B^{(r)}_{N,n}}(x, y) \frac{t^n}{n!} \]

\[ \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} H_{B^{(r)}_{N,k}}(0, y) x^{n-k} \frac{t^n}{n!} = \sum_{n=0}^{\infty} H_{B^{(r)}_{N,n}}(x, y) \frac{t^n}{n!} \]

Thus, we have

\[ H_{B^{(r)}_{N,n}}(x, y) = \sum_{k=0}^{n} \binom{n}{k} H_{B^{(r)}_{N,k}}(0, y) x^{n-k}. \]  \hspace{1cm} (2.13)

Therefore, by integrating (2.13) with weight \( (1-x)^{N-1} \) and using the result ([20], p.26(48)), we obtain

\[ \int_0^1 (1-x)^{N-1} H_{B^{(r)}_{N,n}}(x, y) dx = \sum_{k=0}^{n} \binom{n}{k} H_{B^{(r)}_{N,k}}(0, y) \int_0^1 (1-x)^{N-1} x^{n-k} dx \]

\[ = (N-1)! \sum_{k=0}^{n} \binom{n}{k} \frac{(n-k)!}{(N+n-k)!} H_{B^{(r)}_{N,k}}(0, y), \]

which follows from (2.12). This completes the proof.
Theorem 2.7. The following representation for higher-order hypergeometric Hermite-Bernoulli polynomials $H B^{(r)}_{N,n}(x, y)$ involving Hermite-Euler polynomials $H E_n(x, y)$ holds true:

$$H B^{(r)}_{N,n}(x, y) = \frac{1}{2} \left[ \sum_{m=0}^{n} \sum_{k=0}^{m} \left( \begin{array}{c} m \\ n \end{array} \right) \left( \begin{array}{c} m \\ k \end{array} \right) H E_{n-m}(x, y) B^{(r)}_{N,m-k} \\
+ \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) H E_{n-m}(x, y) B^{(r)}_{N,m} \right].$$

(2.14)

Proof. Using generating function for Hermite-Euler polynomials as follows

$$e^{xt+yt^2} = e^t + \frac{1}{2} \sum_{n=0}^{\infty} H E_n(x, y) \frac{t^n}{n!}, \text{ (see [18])}.$$

Substituting this value of $e^{xt+yt^2}$ in (2.1) gives

$$\sum_{n=0}^{\infty} H B^{(r)}_{N,n}(x, y) \frac{t^n}{n!} = \frac{1}{1F_1(1; N+1; t)^r} e^t + \frac{1}{2} \sum_{n=0}^{\infty} H E_n(x, y) \frac{t^n}{n!}$$

$$= \frac{1}{2} \left[ \sum_{n=0}^{\infty} H E_n(x, y) \frac{t^n}{n!} \sum_{m=0}^{n} \sum_{k=0}^{m} B^{(r)}_{N,m-k} \frac{t^m}{(m-k)!k!} \right.\right.$$

$$+ \left[ \sum_{n=0}^{\infty} \sum_{m=0}^{n} H E_{n-m}(x, y) B^{(r)}_{N,m} \frac{t^n}{(n-m)!m!} \right] \frac{1}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we required at the result (2.14).

Theorem 2.8. For $n \geq 0$, $p, q \in \mathbb{R}$, the following formula for higher-order hypergeometric Hermite-Bernoulli polynomials $H B^{(r)}_{N,n}(px, qy)$ holds true:

$$H B^{(r)}_{N,n}(px, qy)$$

$$= n! \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{j=0}^{k} H B^{(r)}_{N,n-k}(x, y)((p-1)x)^{k-j}((q-1)y)^j \frac{t^n}{(n-k-2j)!j!k!}.$$

(2.15)

Proof. Rewrite the generating function (2.1), we have

$$\sum_{n=0}^{\infty} H B^{(r)}_{N,n}(px, qy) \frac{t^n}{n!}$$

$$= \frac{1}{1F_1(1; N+1; t)^r} e^{xt+yt^2} e^{(p-1)x} e^{(q-1)y} t^2$$

(2.16)
For $n \geq 0$, $p, q \in \mathbb{R}$ and $x, y \in \mathbb{C}$, we have

$$H B_{N,n}^{(r)}(px, qy) = \sum_{k=0}^{n} \binom{n}{k} H B_{N,n-k}^{(r)}(x, y) H_k((p-1)x, (q-1)y). \quad (2.17)$$

**Proof.** By using (2.16) and (1.8), we can easily derive (2.17). We omit the proof. \qed

### 3. Summation formulae for higher-order hypergeometric Hermite-Bernoulli polynomials

In this section, we derive the summation formula, the sum of the product of identity and recurrence relations. First, we prove the following results involving higher-order hypergeometric Hermite-Bernoulli polynomials $H B_{N,n}^{(r)}(x, y)$.

**Theorem 3.1.** The following implicit summation formulae for higher-order hypergeometric Hermite-Bernoulli polynomials $H B_{N,n}^{(r)}(x, y)$ holds true:

$$H B_{N,k+l}^{(r)}(z, y) = \sum_{n,p=0}^{k,l} \frac{k! l! ((z-x)^{n+p} H B_{N,k+l-p-n}^{(r)}(x, y))}{(k-n)!(l-p)!n!p!}. \quad (3.1)$$

**Proof.** We replace $t$ by $t + u$ and rewrite the generating function (2.1) as

$$\frac{1}{1+q} \sum_{k,l=0}^{\infty} H B_{N,k+l}^{(r)}(x, y) \frac{t^k u^l}{k! l!} = e^{-(t+u)x} \sum_{k,l=0}^{\infty} H B_{N,k+l}^{(r)}(x, y) \frac{t^k u^l}{k! l!}. \quad (3.2)$$

Replacing $x$ by $z$ in the above equation and equating the resulting equation to the above equation, we get

$$e^{(z-x)(t+u)} \sum_{k,l=0}^{\infty} H B_{N,k+l}^{(r)}(x, y) \frac{t^k u^l}{k! l!} = \sum_{k,l=0}^{\infty} H B_{N,k+l}^{(r)}(z, y) \frac{t^k u^l}{k! l!}. \quad (3.3)$$
On expanding exponential function (3.3) gives

\[
\sum_{M=0}^{\infty} \frac{[z - x](t + u)^M}{M!} \sum_{k,l=0}^{\infty} H B_{N,k+l}^{(r)}(x,y) \frac{t^k u^l}{k! l!} = \sum_{k,l=0}^{\infty} H B_{N,k+l}^{(r)}(z,y) \frac{t^k u^l}{k! l!},
\]  

(3.4)

which on using formula ([20], p.52(2))

\[
\sum_{M=0}^{\infty} f(M) \frac{(x + y)^M}{M!} = \sum_{n,m=0}^{\infty} f(n + m) \frac{x^n y^m}{n! m!},
\]  

(3.5)

in the left hand side becomes

\[
\sum_{n,p=0}^{\infty} \frac{(z - x)^{n+p}}{n!p!} \sum_{k,l=0}^{\infty} H B_{N,k+l}^{(r)}(x,y) \frac{t^k u^l}{k! l!} = \sum_{k,l=0}^{\infty} H B_{N,k+l}^{(r)}(z,y) \frac{t^k u^l}{k! l!}.
\]  

(3.6)

Now replacing \( k \) by \( k - n \), \( l \) by \( l - p \) and using the lemma ([20], p.100(1)) in the left hand side of (3.6), we get

\[
\sum_{n,p=0}^{\infty} \sum_{k,l=0}^{\infty} \frac{(z - x)^{n+p}}{n!p!} H B_{N,k+l-n-p}^{(r)}(x,y) \frac{t^k u^l}{(k-n)!(l-p)!} = \sum_{k,l=0}^{\infty} H B_{N,k+l}^{(r)}(z,y) \frac{t^k u^l}{k! l!}.
\]  

(3.7)

Finally on equating the coefficients of the like powers of \( t \) and \( u \) in the above equation, we get the required result.

**Corollary 3.2.** On taking \( l = 0 \) in Theorem 3.1, the result reduces to

\[
H B_{N,k}^{(r)}(z,y) = \sum_{n=0}^{k} \binom{k}{n} (z - x)^n H B_{N,k-n}^{(r)}(x,y).
\]  

(3.8)

**Corollary 3.3.** On replacing \( z \) by \( z + x \) and setting \( y = 0 \) in Theorem (3.1), we get the following result involving higher-order hypergeometric Hermite-Bernoulli polynomials of one variable:

\[
H B_{N,k+l}^{(r)}(z + x) = \sum_{n,m=0}^{k,l} \frac{k!l! z^{n+m} H B_{N,k+l-m-n}^{(r)}(x)}{(k-n)!(l-m)!n!m!},
\]  

(3.9)

whereas by setting \( z = 0 \) in Theorem 3.1, we get another result involving hypergeometric Hermite-Bernoulli polynomials of one and two variables:

\[
H B_{N,k+l}^{(r)}(y) = \sum_{n,m=0}^{k,l} \frac{k!l! (-x)^{n+m} H B_{N,k+l-m-n}^{(r)}(x,y)}{(k-n)!(l-m)!n!m!}.
\]  

(3.10)

**Theorem 3.4.** The following implicit summation formulae for higher-order hypergeometric Hermite-Bernoulli polynomials \( H B_{N,n}^{(r)}(x,y) \) holds true:

\[
H B_{N,n}^{(r)}(x,y) = \sum_{m=0}^{n} \binom{n}{m} B_{N,n-m}^{(r)}(x-z) H_m(z,y).
\]  

(3.11)

**Proof.** By exploiting the generating function (2.1) and using (1.8), we can write equation (2.1) as

\[
\frac{1}{\Gamma(1; N + 1; t)} e^{(x-z)t} e^{zt+y^2} = \sum_{n=0}^{\infty} B_{N,n}^{(r)}(x-z) \frac{t^n}{n!} \sum_{m=0}^{\infty} H_m(z,y) \frac{t^m}{m!}.
\]
Replacing \( n \) by \( n - m \) in above equation and using lemma ([20], p.101(1)), we get

\[
\sum_{n=0}^{\infty} H B_{N,n}^{(r)}(x,y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} B_{N,n-m}^{(r)}(x-z)H_m(z,y) \frac{t^n}{(n-m)!m!}.
\]

On equating the coefficients of the like powers of \( t \), we get (3.11).

**Corollary 3.5.** Letting \( z = x \) in Theorem 3.2 gives

\[
H B_{N,n}^{(r)}(x,y) = \sum_{m=0}^{n} \binom{n}{m} B_{N,n-m}^{(r)}H_m(x,y).
\]

**Theorem 3.6.** The following implicit summation formulae for higher-order hypergeometric Hermite-Bernoulli polynomials \( H B_{N,n}^{(r)}(x,y) \) holds true:

\[
H B_{N,n}^{(r)}(x+1,y) = \sum_{m=0}^{n} \binom{n}{m} H B_{N,n-m}^{(r)}(x,y).
\]

**Proof.** Using the generating function (2.1), we have

\[
\sum_{n=0}^{\infty} H B_{N,n}^{(r)}(x+1,y) \frac{t^n}{n!} - \sum_{n=0}^{\infty} H B_{N,n}^{(r)}(x,y) \frac{t^n}{n!} = \frac{1}{1F_1(1; N + 1; t^r)(e^t - 1)e^{zt+y2t}}
\]

\[
= \sum_{n=0}^{\infty} H B_{N,n}^{(r)}(x,y) \frac{t^n}{n!} \left( \sum_{m=0}^{\infty} \frac{t^m}{m!} - 1 \right)
\]

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} H B_{N,n-m}^{(r)}(x,y) \frac{t^n}{n!} - \sum_{n=0}^{\infty} H B_{N,n}^{(r)}(x,y) \frac{t^n}{n!}.
\]

Finally equating the coefficients of the like powers of \( t \), we get (3.13). \( \square \)

**Theorem 3.7.** The following implicit summation formula involving higher-order hypergeometric Hermite-Bernoulli polynomials \( H B_{N,n}^{(r)}(x,y) \) holds true:

\[
H B_{N,n}^{(r)}(z+x,u+y) = \sum_{m=0}^{n} \binom{n}{m} H B_{N,n-m}^{(r)}(x,y)H_m(z,u).
\]

**Proof.** We replace \( x \) by \( x+z \) and \( y \) by \( y+u \) in (2.1), use (1.2) and rewrite the generating function as

\[
\frac{1}{1F_1(1; N + 1; t^r)e^{zt+y2t}} = \sum_{n=0}^{\infty} H B_{N,n}^{(r)}(x,y) \frac{t^n}{n!} \sum_{m=0}^{\infty} H_m(x,y) \frac{t^m}{m!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H B_{N,n}^{(r)}(x,y)H_m(x,y) \frac{t^{n+m}}{n!m!}.
\]

Replacing \( n \) by \( n - m \) in above equation, we have

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{n} H B_{N,n-m}^{(r)}(x,y)H_m(x,y) \frac{t^n}{(n-m)!m!}.
\]

Comparing the coefficients of \( t \) on both sides, we get the result (3.14). \( \square \)
Theorem 3.8. The following implicit summation formula involving higher-order hypergeometric Hermite-Bernoulli polynomials $H^{(r)}_{N,n}(x,y)$ holds true:

$$H^{(r)}_{N,n}(y,x) = \sum_{k=0}^\infty B^{(r)}_{N,n-2k}(y) \frac{x^k}{(n-2k)!k!}.$$  \hspace{1cm} (3.15)

Proof. We replace $x$ by $y$ and $y$ by $x$ in equation (2.1) to get

$$\sum_{n=0}^\infty H^{(r)}_{N,n}(y,x) \frac{t^n}{n!} = \sum_{n=0}^\infty B^{(r)}_{N,n-2k}(y) \frac{t^n}{n!} \sum_{k=0}^\infty \frac{x^k t^{2k}}{k!}.$$  \hspace{1cm} (3.16)

Now replacing $n$ by $n-2k$ and comparing the coefficients of $t$, we get the result (3.15). \hfill \Box

Theorem 3.9. The following implicit summation formula involving higher-order hypergeometric Hermite-Bernoulli polynomials $H^{(r)}_{N,n}(x,y)$ holds true:

$$H^{(r)}_{N,n}(z,u) = \sum_{m=0}^n \binom{n}{m} H_m(\alpha - x + z, \beta - y + u) H^{(r)}_{N,n-m}(x - \alpha, y - \beta), \hspace{1cm} (3.17)$$

and

$$H^{(r)}_{N,n}(z - \alpha - x, u - \beta + y) = \sum_{m=0}^n \binom{n}{m} H_m(z,u) H^{(r)}_{N,n-m}(x - \alpha, y - \beta). \hspace{1cm} (3.18)$$

Proof. By exploiting the generating function (2.1), we can write

$$\sum_{n=0}^\infty H^{(r)}_{N,n}(z,u) \frac{t^n}{n!} = \frac{1}{1F_1(1;N+1;t)^r} e^{zt+ut^2}$$

$$= e^{-(x-z)\alpha - (y-u)\beta} t^{(x-z)\alpha + (y-u)\beta} \sum_{n=0}^\infty H^{(r)}_{N,n}(x - \alpha, y - \beta) \frac{t^n}{n!},$$

which yields

$$\sum_{n=0}^\infty H^{(r)}_{N,n}(x,y) \frac{t^n}{n!} = \sum_{m=0}^n H_m(\alpha - x + z, \beta - y + u) \frac{t^m}{m!} \sum_{n=0}^\infty H^{(r)}_{N,n}(x - \alpha, y - \beta) \frac{t^n}{n!}.$$  \hspace{1cm} (3.19)

Replacing $n$ by $n-m$ in above equation and comparing the coefficients of $t$, we obtain (3.17). On replacing $z$ by $z - \alpha - x$ and $u$ by $u - \beta + y$ in (3.16), we get (3.17). \hfill \Box

Corollary 3.10. On setting $z = u = 0$ in (3.16), we have the following result for higher-order hypergeometric Hermite-Bernoulli polynomials $H^{(r)}_{N,n}(x,y)$ holds true:

$$B^{(r)}_{N,n} = \sum_{m=0}^n \binom{n}{m} H_m(\alpha - x, \beta - y) H^{(r)}_{N,n-m}(x - \alpha, y - \beta).$$
Theorem 3.11. The following implicit summation formula involving higher-order hypergeometric Hermite-Bernoulli polynomials $H_{N,n}(x, y)$ holds true:

$$
\sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{B^{(r)}_{N,n-2m}(x)B^{(r)}_{N,m}(y)}{(n-2m)!m!} = \sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{H_{N,n-2m}(x, y)B^{(r)}_{N,m}}{(n-2m)!m!},
$$

(3.18)

and

$$
\sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{B^{(r)}_{N,n-2m}(x)B^{(r)}_{N,m}(y)}{(n-2m)!m!} = \sum_{k=0}^{n} \sum_{m=0}^{\left\lfloor \frac{n-k}{2} \right\rfloor} \frac{B^{(r)}_{N,n-k-2m}B^{(r)}_{N,m}H_{k}(x, y)}{(n-k-2m)!mk!}.
$$

(3.19)

Proof. Consider the definition of (2.1), we have

$$
\sum_{n=0}^{\infty} B^{(r)}_{N,n}(y) \frac{t^{2n}}{n!} = \frac{1}{1F_{1}(1; N + 1; t^{2})} e^{yt^{2}},
$$

(3.20)

where $x$ is replaced by $y$ and $t$ is replaced by $t^{2}$ in (2.1). On multiplying (2.1) and (3.20), we have

$$
\sum_{n=0}^{\infty} B^{(r)}_{N,n}(x) \frac{t^{n}}{n!} \sum_{m=0}^{\infty} B^{(r)}_{N,m}(y) \frac{t^{2m}}{m!} = \frac{1}{1F_{1}(1; N + 1; t)} \frac{1}{1F_{1}(1; N + 1; t^{2})} e^{xt+yt^{2}},
$$

(3.21)

$$
\sum_{n=0}^{\infty} \sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor} B^{(r)}_{N,n-2m}(x)B^{(r)}_{N,m}(y) \frac{t^{n}}{(n-2m)!m!} = \sum_{n=0}^{\infty} \sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor} B^{(r)}_{N,n}(x, y) \frac{t^{n}}{n!} \sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor} B^{(r)}_{N,m} \frac{t^{2m}}{m!}.
$$

Using the Cauchy product and comparing the coefficients of $t$, we obtain (3.18). Another way of defining the right hand side of equation (3.21) is suggested by replacing $e^{xt+yt^{2}}$ by its series representation

$$
\sum_{n=0}^{\infty} B^{(r)}_{N,n}(x) \frac{t^{n}}{n!} \sum_{m=0}^{\infty} B^{(r)}_{N,m}(y) \frac{t^{2m}}{m!} = \frac{1}{1F_{1}(1; N + 1; t)} \frac{1}{1F_{1}(1; N + 1; t^{2})} e^{xt+yt^{2}},
$$

$$
\sum_{n=0}^{\infty} \sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor} B^{(r)}_{N,n-2m}(x)B^{(r)}_{N,m}(y) \frac{t^{n}}{(n-2m)!m!} = \sum_{k=0}^{\infty} H_{k}(x, y) \frac{t^{k}}{k!} \sum_{n=0}^{\infty} \sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor} B^{(r)}_{N,n} \frac{t^{n}}{n!} \sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor} B^{(r)}_{N,m} \frac{t^{2m}}{m!}.
$$

Using the Cauchy product and comparing the coefficients of $t$, we get (3.19). □

Corollary 3.12. For $y = 0$ in Theorem 3.7, we have the following result for higher-order hypergeometric Bernoulli polynomials $H_{N,n}(x, y)$ holds true:

$$
\sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{B^{(r)}_{N,n-2m}(x)B^{(r)}_{N,m}}{(n-2m)!m!} = \sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{H_{N,n-2m}(x, 0)B^{(r)}_{N,m}}{(n-2m)!m!},
$$

and

$$
\sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{B^{(r)}_{N,n-2m}(x)B^{(r)}_{N,m}}{(n-2m)!m!} = \sum_{k=0}^{n} \sum_{m=0}^{\left\lfloor \frac{n-k}{2} \right\rfloor} \frac{B^{(r)}_{N,n-k-2m}B^{(r)}_{N,m}y^{k}}{(n-k-2m)!mk!}.
$$

Theorem 3.13. The following implicit summation formula involving higher-order hypergeometric Hermite-Bernoulli polynomials $H_{N,n}(x, y)$ holds true:

$$
\sum_{m=0}^{n} \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{B^{(k)}_{N,m,n}(x, y)}{(n-m-2r)!x^{n-m-2r}} = \sum_{m=0}^{n} \frac{B^{(k)}_{N,m}H_{N,n-m}(x, y)}{x^{m}m!(n-m-2r)!}.n = \sum_{m=0}^{n} \frac{B^{(k)}_{N,m}H_{N,n-m}(x, y)}{x^{m}m!(n-m-2r)!}.
$$

(3.22)
Proof. On replacing $t$ by $\frac{1}{x}$ and $r$ by $k$, we can write equation (2.1) as
\[
\sum_{n=0}^{\infty} H_{N,n}^{(k)}(x, y) \frac{t^n}{x^n n!} = \frac{1}{1F_1(1; N + 1; \frac{1}{x})} e^{t+y \frac{x^2}{2}}.
\] (3.23)

Now interchanging $x$ and $y$, we have
\[
\sum_{n=0}^{\infty} H_{N,n}^{(k)}(y, x) \frac{t^n}{y^n n!} = \frac{1}{1F_1(1; N + 1; \frac{1}{y})} e^{t+x \frac{y^2}{2}}.
\] (3.24)

Comparison of (3.23) and (3.24) yields
\[
e^{\frac{x^2}{2y^2} - \frac{y^2}{2x^2}} \frac{1}{1F_1(1; N + 1; \frac{1}{x})} \sum_{n=0}^{\infty} H_{N,n}^{(k)}(x, y) \frac{t^n}{x^n n!}
= \frac{1}{1F_1(1; N + 1; \frac{1}{y})} \sum_{n=0}^{\infty} H_{N,n}^{(k)}(y, x) \frac{t^n}{y^n n!}
= \sum_{m=0}^{\infty} B_{N,m}^{(k)} \frac{t^m}{x^m m!} \sum_{n=0}^{\infty} H_{N,n}^{(k)}(x, y) \frac{t^n}{x^n n!}.
\]

Using the Cauchy product and comparing the coefficients of $t$, we get (3.22). \qed

Theorem 3.14. The following implicit summation formula involving higher-order hypergeometric Hermite-Bernoulli polynomials $H_{N,n}^{(r)}(x, y)$ holds true:
\[
H_{N,n}^{(r)}(w, u)H_{N,m}^{(r)}(W, U) = \sum_{s,k=0}^{m,n} \binom{n}{s} \binom{m}{k} H_s(w - x, u - y)H_{N,n-s}^{(r)}(x, y) \times H_k(W - X, U - Y)H_{N,m-k}^{(r)}(X, Y).
\] (3.25)

Proof. Consider the product of higher-order hypergeometric Hermite-Bernoulli polynomials, equation (2.1) in the following form
\[
\frac{1}{1F_1(1; N + 1; t)} e^{xt} \frac{1}{1F_1(1; N + 1; T)} e^{yT^2}
= \sum_{n=0}^{\infty} H_{N,n}^{(r)}(x, y) \frac{t^n}{n!} \sum_{m=0}^{\infty} H_{N,m}^{(r)}(X, Y) \frac{T^m}{m!}.
\] (3.26)

Replacing $x$ by $w$, $y$ by $u$, $X$ by $W$ and $Y$ by $U$ in (3.26) and equating the resultant equation to itself, we find
\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{N,n}^{(r)}(w, u)H_{N,m}^{(r)}(W, U) \frac{t^n T^m}{n! m!}
= \exp \left((w - x)t + (u - y)t^2\right) \exp \left((W - X)T + (U - Y)T^2\right)
\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{N,n}^{(r)}(x, y) \frac{t^n}{n!} \sum_{m=0}^{\infty} H_{N,m}^{(r)}(X, Y) \frac{T^m}{m!}.
\]
\[ \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} H_s(w - x, u - y) H_{N,n}^{(r)}(x, y) \frac{t^{n+s}}{n!s!} \times H_k(W - X, U - Y) H_{N,m}^{(r)}(X, Y) \frac{T^{m+k}}{m!k!}. \]

Finally, replacing \( n \) by \( n - s \) and \( m \) by \( m - k \) in the r.h.s. of the above equation and then equating the coefficients of like powers of \( t \) and \( T \), we get assertion (3.25) of Theorem (3.8). \( \square \)

**Remark 3.15.** Replacing \( u \) by \( y \) and \( U \) by \( Y \) in assertion (3.25) of Theorem (3.9), we deduce the following consequence of Theorem (3.9).

**Corollary 3.16.** The following implicit summation formula involving higher-order hypergeometric Hermite-Bernoulli polynomials \( H_{N,n}^{(r)}(x, y) \) holds true:

\[ H_{N,n}^{(r)}(w, y) H_{N,m}^{(r)}(W, Y) = \sum_{s,k=0}^{m,n} \binom{n}{s} \binom{m}{k} (w - x)^s H_{N,n-s}^{(r)}(x, y) \times (W - X)^k H_{N,m-k}^{(r)}(X, Y). \]

**Acknowledgments**

The author Waseem A. Khan thanks to Prince Mohammad Bin Fahd University, Saudi Arabia for providing facilities and support.

**References**


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