Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–7.
ISSN-0037-8712 in press
doi:10.5269/bspm.51864

Existence and Non-existence of Solutions for a (p, q)-Laplacian Steklov System

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Abstract: In this paper, we study the existence and non-existence of weak solutions to the following system:

$$\begin{cases} \Delta_p u = \Delta_q v = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda m|u|^{p-2}u - \varepsilon[(\alpha+1)|u|^{\alpha-1}u|v|^{\beta+1} - f] & \text{on } \partial\Omega, \\ |\nabla v|^{q-2}\frac{\partial v}{\partial \nu} = \lambda n|v|^{q-2}v - \varepsilon[(\beta+1)|v|^{\beta-1}v|u|^{\alpha+1} - g] & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a smooth boundary $\partial\Omega$, $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $\frac{\partial}{\partial \nu}$ is the outer normal derivative, $\varepsilon \in \{0, 1\}$, m, n, f and g are the functions that satisfies some conditions.

Key Words: Steklov system, weights, nonlinear boundary conditions, (p, q)-Lapacian, eigenvalue problem.

Contents

1. Introduction

Consider the system with nonlinear boundary conditions

$$\begin{cases} \Delta_p u = \Delta_q v = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda m|u|^{p-2}u - \varepsilon[(\alpha+1)|u|^{\alpha-1}u|v|^{\beta+1} - f] & \text{on } \partial\Omega, \\ |\nabla v|^{q-2}\frac{\partial v}{\partial \nu} = \lambda n|v|^{q-2}v - \varepsilon[(\beta+1)|v|^{\beta-1}v|u|^{\alpha+1} - g] & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$, with a smooth boundary $\partial\Omega$, $1 < p < +\infty$, $1 < q < +\infty$, $\varepsilon \in \{0, 1\}$ and suppose the following conditions:
$\alpha \geq 0$, $\beta \geq 0$ such that $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1$ and

$$f \in \mathrm{L}^r(\partial\Omega),\ r = \frac{p\bar{p}}{p\bar{p} - \bar{p} + 1},\ \frac{N-1}{p-1} < \bar{p} < \infty \ if\ p < N\ and\ \bar{p} \geq 1\ if\ p \geq N,$$

$$g \in \mathrm{L}^{\bar{r}}(\partial\Omega),\ \bar{r} = \frac{q\bar{q}}{q\bar{q} - \bar{q} + 1},\ \frac{N-1}{q-1} < \bar{q} < \infty \ if\ q < N\ and\ \bar{q} \geq 1\ if\ q \geq N.$$

$$M_{\bar{p}} = \{m \in \mathrm{L}^{\bar{p}}(\partial\Omega), m^+ \not\equiv 0, \int_{\partial\Omega} m d\sigma < 0\},$$

$$M_{\bar{q}} = \{m \in \mathrm{L}^{\bar{q}}(\partial\Omega), m^+ \not\equiv 0, \int_{\partial\Omega} m d\sigma < 0\},$$

$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian. The operator $\triangle_p$ turns up in many mathematical setting: e.g., non-newtonien fluids, reaction-diffusion problems, porous media, astronomy, etc. (see for example [4]).

2010 *Mathematics Subject Classification:* 33E12, 30C45.
Submitted January 20, 2020. Published October 13, 2020

Many publications, such as [6,8], discuss quasilinear elliptic systems involving p-Laplacian operators and show the existence and multiplicity of solutions. The authors in [6] studied the existence of solutions for

$$\begin{cases} -\triangle_p u = F_u(x,u,v) & \text{in } \Omega, \\ -\triangle_q v = F_v(x,u,v) & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

where p, q are real numbers larger than 1.

In [1,5], the authors studied a Dirichlet problem involving critical exponents. The author, in [9], has been interested to the system involving $(p(x), q(x))$-laplacian with Dirichlet conditions, which generalize and improve the result of [1].
Existence results for nonlinear elliptic systems when the nonlinear term appears as a source in the equation complemented with Dirichlet boundary conditions have been studied by various authors; we cite the works [6,10,11].

For the nonlinear boundary condition, the authors in [7] proved the existence of nontrivial solutions of the quasi-linear elliptic system.

$$\begin{cases} \Delta_p u = |u|^{p-2}u, \ \Delta_q v = |v|^{q-2}v & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = F_u(x,u,v), |\nabla v|^{q-2}\frac{\partial v}{\partial \nu} = F_v(x,u,v) & \text{on } \partial\Omega, \end{cases}$$

where (F_u, F_v) is the gradient of some positive potential $F : \partial\Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.
The (p, q) harmonic case for the Steklov system has been studied in [2].

In the present paper, we are interested at the existence and non-existence of (p, q)-harmonic solutions, $(u, v) \in \mathrm{W}^{1,\mathrm{p}}(\Omega) \times \mathrm{W}^{1,\mathrm{q}}(\Omega)$, for a Steklov system (1.1).

This paper is organized as follows. In section 2, which has a preliminary character, we collect some results relative to the following Steklov problem (2.1). In section 3, we study the existence and non-existence solutions for our system (1.1). Our proofs are based on variational arguments.

2. Preliminaries

In this section, we collect some results relative to the Steklov eigenvalue problem

$$\begin{cases} \Delta_p u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda m(x)|u|^{p-2}u & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where the weight m is assumed to lie in $M_{\bar{p}} = \{m \in \mathrm{L}^{\bar{\mathrm{p}}}(\partial\Omega), m^+ \not\equiv 0, \int_{\partial\Omega} m d\sigma < 0\}$.
O. Torné in [12] showed, by using infinite dimensional Ljusternik-Schnirelman theory, that the problem (2.1) admits a sequence of eigenvalues:

$$\lambda_k(m,p) = \inf_{C\in\Gamma_k} \sup_{x\in C} \frac{1}{p}\int_\Omega |\nabla u|^p dx,$$

where

$$\Gamma_k = \{C \subset S; C \text{ is symmetric, compact and } \gamma(C) \geq K\},$$

with

$$S = \{u \in \mathrm{W}^{1,\mathrm{p}}(\Omega); \ \frac{1}{p}\int_{\partial\Omega} m|u|^P d\sigma = 1\} \text{ and } \gamma(C) \text{ is the Krasnoselski genus of C.}$$

Let $\lambda_1(m,p) = \inf\{\frac{1}{p}\int_\Omega |\nabla u|^p dx; u \in \mathrm{W}^{1,\mathrm{p}}(\Omega) \text{ and } \frac{1}{p}\int_{\partial\Omega} m|u|^P d\sigma = 1\}$.
This author also showed that if $\int_{\partial\Omega} m d\sigma < 0$, then $\lambda_1(m,p) > 0$.
In [3], A. Anane et al have also proved that there exists an increasing unbounded sequence of positive eigenvalues for the problem (2.1) but by applying an other deformation lemma.
In [3] the authors showed the following result.

Theorem 2.1. *1. If $m, m_0 \in M_{\bar{p}}$, then we have*

$$\frac{1}{\lambda_1} := \frac{1}{\lambda_1(m,p)} = \sup_{u\in A} \frac{1}{p}\int_{\partial\Omega} m|u|^p d\sigma,$$

where

$$A = \{u \in W^{1,p}(\Omega); \frac{1}{p}\int_{\Omega} |\nabla u|^p dx = 1\}.$$

2. *If* $m \leq\not\equiv m_0$*, then* $\lambda_1(m_0, p) < \lambda_1(m, p)$*.*

3. Existence and nonexistence of solutions for a Steklov system

In this section, where $\varepsilon = 1$, we show that the problem (1.1), admits at least a nontrivial solution under some conditions on the positive number λ, we also show the non-existence results for nontrivial solutions of the system (1.1) in the case $\varepsilon = 0$. The following theorem is the main result in this paper.

Theorem 3.1. *Let* $m \in M_{\bar{p}}$*,* $n \in M_{\bar{q}}$ *and* $0 < \lambda < \inf(\lambda_1(m,p), \lambda_1(n,q))$*. Then*

1. *If* $\varepsilon = 1$*, the system* (1.1) *admits at least a solution for any* f, g*.*

2. *If* $\varepsilon = 0$*, the system* (1.1) *has no non-trivial solutions.*

Consider the space $\mathrm{W} = \mathrm{W}^{1,\mathrm{p}}(\Omega) \times \mathrm{W}^{1,\mathrm{q}}(\Omega)$ equipped with the norm

$$\|w\| = \|u\|_{1,p} + \|v\|_{1,q}, for\ w = (u, v) \in \mathrm{W},$$

where

$$\|u\|_{1,p} = \Big(\int_{\Omega} |\nabla u|^p dx + \int_{\Omega} |u|^p dx\Big)^{\frac{1}{p}}$$

and

$$\|v\|_{1,q} = \Big(\int_{\Omega} |\nabla v|^q dx + \int_{\Omega} |v|^q dx\Big)^{\frac{1}{q}}.$$

We say that $(u, v) \in \mathrm{W}^{1,\mathrm{p}}(\Omega) \times \mathrm{W}^{1,\mathrm{q}}(\Omega)$ is a weak solution of (1.1) if :

$$\int_{\Omega} |\nabla u|^{p-2}\nabla u \nabla\varphi dx = \int_{\partial\Omega} \lambda m|u|^{p-2}u\varphi d\sigma - \varepsilon[(\alpha+1)\int_{\partial\Omega} |u|^{\alpha-1}u|v|^{\beta+1}\varphi d\sigma - \int_{\partial\Omega} f\varphi d\sigma],$$
$$\int_{\Omega} |\nabla v|^{q-2}\nabla v \nabla\psi dx = \int_{\partial\Omega} \lambda n|v|^{q-2}v\psi d\sigma - \varepsilon[(\beta+1)\int_{\partial\Omega} |v|^{\beta-1}v|u|^{\alpha+1}\psi d\sigma + \int_{\partial\Omega} g\psi d\sigma].$$

for all $(\varphi, \psi) \in \mathrm{W}^{1,\mathrm{p}}(\Omega) \times \mathrm{W}^{1,\mathrm{q}}(\Omega)$, where $d\sigma$ is the $N-1$ dimensional Hausdroff measure.
The energy functional corresponding to the system (1.1) is the functional Φ_ε such that $\Phi_\varepsilon : \mathrm{W} \to \mathbb{R}$ with

$$\Phi_\varepsilon(u,v) = \frac{1}{p}\int_{\Omega} |\nabla u|^p dx - \frac{\lambda}{p}\int_{\partial\Omega} m|u|^p d\sigma + \frac{1}{q}\int_{\Omega} |\nabla v|^q dx - \frac{\lambda}{q}\int_{\partial\Omega} n|v|^q d\sigma$$
$$+\varepsilon\left[\int_{\partial\Omega} |u|^{\alpha+1}|v|^{\beta+1} d\sigma - \int_{\partial\Omega} fu d\sigma - \int_{\partial\Omega} gv d\sigma\right].$$

It is clear that the critical points of the energy functional Φ_ε are the weak solutions of the system (1.1).
To prove the Theorem (3.1), we need the following lemmas.

Lemma 3.2. *If* $\varepsilon = 1$*,* $m \in M_{\bar{p}}$ *and* $n \in M_{\bar{q}}$*, then the functional* $\Phi_{\varepsilon=1}$ *is coercive for,* $0 < \lambda < \inf(\lambda_1(m,p), \lambda_1(n,q))$.

Proof. Suppose by contradiction that $\Phi_{\varepsilon=1}$ is not coercive. Then there exist a sequence $w_n \in \mathrm{W}$ and $c \geq 0$ with $w_n = (u_n, v_n)$ such that $\|w_n\| \to +\infty$ and $|\Phi_{\varepsilon=1}(w_n)| \leq c$.
The condition $|\Phi_{\varepsilon=1}(w_n)| \leq c$ implies that

$$\frac{1}{p}\int_{\Omega} |\nabla u_n|^p dx - \frac{\lambda}{p}\int_{\partial\Omega} m|u_n|^p d\sigma + \frac{1}{q}\int_{\Omega} |\nabla v_n|^q dx - \frac{\lambda}{q}\int_{\partial\Omega} n|v_n|^q d\sigma$$
$$+\int_{\partial\Omega} |u_n|^{\alpha+1}|v_n|^{\beta+1} d\sigma - \int_{\partial\Omega} fu_n d\sigma - \int_{\partial\Omega} gv_n d\sigma \leq c.$$

Since

$$\int_{\partial\Omega} |u_n|^{\alpha+1}|v_n|^{\beta+1} d\sigma \geq 0,$$

then

$$\frac{1}{p}\int_\Omega |\nabla u_n|^p dx - \frac{\lambda}{p}\int_{\partial\Omega} m|u_n|^p d\sigma + \frac{1}{q}\int_\Omega |\nabla v_n|^q dx - \frac{\lambda}{q}\int_{\partial\Omega} n|v_n|^q d\sigma$$
$$-\int_{\partial\Omega} f u_n d\sigma - \int_{\partial\Omega} g v_n d\sigma \leq c.$$

Thus

$$\frac{1}{p}\int_\Omega |\nabla u_n|^p dx - \frac{\lambda}{p}\int_{\partial\Omega} m|u_n|^p d\sigma - \int_{\partial\Omega} f u_n d\sigma - \int_{\partial\Omega} g v_n d\sigma \leq c, \tag{3.1}$$

and

$$\frac{1}{q}\int_\Omega |\nabla v_n|^q dx - \frac{\lambda}{q}\int_{\partial\Omega} n|v_n|^q d\sigma - \int_{\partial\Omega} f u_n d\sigma - \int_{\partial\Omega} g v_n d\sigma \leq c. \tag{3.2}$$

As $0 < \lambda < \inf(\lambda_1(m,p), \lambda_1(n,q))$, then we have

$$\Big(1 - \frac{\lambda}{\lambda_1(m,p)}\Big)\frac{1}{p}\int_\Omega |\nabla u_n|^p dx - \int_{\partial\Omega} f u_n d\sigma - \int_{\partial\Omega} g v_n d\sigma \leq c,$$

and

$$\Big(1 - \frac{\lambda}{\lambda_1(n,q)}\Big)\frac{1}{q}\int_\Omega |\nabla v_n|^q dx - \int_{\partial\Omega} f u_n d\sigma - \int_{\partial\Omega} g v_n d\sigma \leq c.$$

Put $\tilde{u}_n = \frac{u_n}{\|w_n\|}$ and $\tilde{v}_n = \frac{v_n}{\|w_n\|}$, dividing by $\|w_n\|^p$ and $\|w_n\|^q$, we obtain

$$\Big(1 - \frac{\lambda}{\lambda_1(m,p)}\Big)\frac{1}{p}\int_\Omega |\nabla \tilde{u}_n|^p dx - \frac{1}{\|w_n\|^p}\Big(\int_{\partial\Omega} f u_n d\sigma + \int_{\partial\Omega} g v_n d\sigma\Big) \leq \frac{c}{\|w_n\|^p},$$

$$\Big(1 - \frac{\lambda}{\lambda_1(n,q)}\Big)\frac{1}{q}\int_\Omega |\nabla \tilde{v}_n|^q dx - \frac{1}{\|w_n\|^q}\Big(\int_{\partial\Omega} f u_n d\sigma - \int_{\partial\Omega} g v_n d\sigma\Big) \leq \frac{c}{\|w_n\|^q}.$$

Since $\tilde{u}_n$ is bounded, for a further subsequence still denoted $\tilde{u}_n$, $\tilde{u}_n \rightharpoonup \tilde{u}$ weakly in $\mathrm{W}^{1,\mathrm{p}}(\Omega)$ and $\tilde{u}_n \to \tilde{u}$ strongly in $\mathrm{L}^{\mathrm{p}}(\Omega)$. On the other hand, we have

$$\int_\Omega |\nabla \tilde{u}|^p dx + \int_\Omega |\tilde{u}|^p dx \leq \liminf_{n \longrightarrow +\infty} \Big(\int_\Omega |\nabla \tilde{u}_n|^p dx + \int_\Omega |\tilde{u}_n|^p dx\Big).$$

Passing to the limit, we obtain $\frac{1}{p}\int_\Omega |\nabla \tilde{u}|^p dx = 0$. Thus $\tilde{u} = cst = c_1$ and $\|\tilde{u}_n\|_{1,p} \longrightarrow \|\tilde{u}\|_{1,p}$. Since $\mathrm{W}^{1,\mathrm{p}}(\Omega)$ is uniformly convex and reflexive, $\tilde{u}_n \longrightarrow cst = c_1$ strongly in $\mathrm{W}^{1,\mathrm{p}}(\Omega)$. By a similar argument, we show that $\tilde{v}_n \longrightarrow cst = c_2$ strongly in $\mathrm{W}^{1,\mathrm{q}}(\Omega)$.
Dividing (3.1) and (3.2) respectively by $\|w_n\|^p$ and $\|w_n\|^q$ and passing to the limit, we obtain

$$-\frac{\lambda|c_1|^p}{p}\int_{\partial\Omega} m d\sigma \leq 0$$

and

$$-\frac{\lambda|c_2|^q}{q}\int_{\partial\Omega} n d\sigma \leq 0.$$

Since $\int_{\partial\Omega} m d\sigma < 0$ and $\int_{\partial\Omega} n d\sigma < 0$, then $c_1 = c_2 = 0$. Consequently $\|\tilde{w_n}\| \longrightarrow 0$, where $\tilde{w_n} = (\tilde{u}_n, \tilde{v}_n)$. This contradicts $\|\tilde{w_n}\| = 1$. Finally, $\Phi_{\varepsilon=1}$ is coercive. □

Lemma 3.3. *If $m \in M_{\bar{p}}$ and $n \in M_{\bar{q}}$, then the energy functional $\Phi_{\varepsilon=1}$ is a weakly lower semicontinuous.*

Proof. It suffices to see that the trace mapping $W \longrightarrow L^{\frac{p\bar{p}}{p-1}}(\partial\Omega) \times L^{\frac{q\bar{q}}{q-1}}(\partial\Omega)$ is compact. Indeed, if we have $W^{1,p}(\Omega) \times W^{1,q}(\Omega) \subset L^{\frac{p\bar{p}}{p-1}}(\partial\Omega) \times L^{\frac{q\bar{q}}{q-1}}(\partial\Omega)$ with compact injection, then for any bounded part in W, it is relatively compact in $L^{\frac{p\bar{p}}{p-1}}(\partial\Omega) \times L^{\frac{q\bar{q}}{q-1}}(\partial\Omega)$.
Let (u_n, v_n) be a bounded sequence in W, it means that u_n is bounded in $W^{1,p}(\Omega)$ and v_n is bounded in $W^{1,q}(\Omega)$. For the subsequences , there exists $(u, v) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and $L^{\frac{p\bar{p}}{p-1}}(\partial\Omega)$ and $v_n \rightharpoonup v$ weakly in $W^{1,q}(\Omega)$, strongly in $L^q(\Omega)$ and $L^{\frac{q\bar{q}}{p-1}}(\partial\Omega)$. Thus

$$\int_\Omega |\nabla u|^p dx \le \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^p dx$$

and

$$\frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\int_{\partial\Omega} m|u|^p d\sigma + \int_{\partial\Omega} |u|^{\alpha+1}|v|^{\beta+1} d\sigma - \int_{\partial\Omega} fu d\sigma \le$$

$$\liminf_{n\to\infty}\left[\frac{1}{p}\int_\Omega |\nabla u_n|^p dx - \frac{\lambda}{p}\int_{\partial\Omega} m|u_n|^p d\sigma + \int_{\partial\Omega} |u_n|^{\alpha+1}|v_n|^{\beta+1} d\sigma - \int_{\partial\Omega} fu_n d\sigma\right].$$

We have the same result, if we replace u_n by v_n. This implies that

$$\Phi_{\varepsilon=1}(u, v) \le \liminf_{n\to\infty} \Phi_{\varepsilon=1}(u_n, v_n),$$

consequently $\Phi_{\varepsilon=1}$ is weakly lower semi-continuous. □

Proof of Theorem 3.1. 1. By Lemma 3.2, $\Phi_{\varepsilon=1}$ is coercive and by Lemma 3.3 $\Phi_{\varepsilon=1}$ is weakly lower semicontinuous. Furthermore $\Phi_{\varepsilon=1}$ is continuously differentiable.Thus the proof is complete by using the minimum principle.

2. For $\varepsilon = 0$, the system (1.1) becomes

$$\begin{cases} \Delta_p u = \Delta_q v = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial\nu} = \lambda m|u|^{p-2}u & \text{on } \partial\Omega, \\ |\nabla v|^{q-2}\frac{\partial v}{\partial\nu} = \lambda n|v|^{q-2}v & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Affirm that if $0 < \lambda < \inf(\lambda_1(m,p), \lambda_1(n,q))$, then the system (3.3) has no non-trivial solution. Indeed, suppose, by contradiction, that the system (3.3) have a non-trivial solution (u, v) such that $u \neq 0$ or $v \neq 0$. Then, we obtain

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi dx = \int_{\partial\Omega} \lambda m|u|^{p-2}u\varphi d\sigma,$$

$$\int_\Omega |\nabla v|^{q-2}\nabla v\nabla\phi dx = \int_{\partial\Omega} \lambda n|v|^{q-2}v\phi d\sigma.$$

For all $(\varphi, \psi) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$. Thus for $\varphi = u$ and $\psi = v$, we obtain

$$\int_\Omega |\nabla u|^p dx = \lambda\int_{\partial\Omega} m|u|^p d\sigma \text{ and } \int_\Omega |\nabla v|^q dx = \lambda\int_{\partial\Omega} n|v|^q d\sigma.$$

So, we distinguish two cases:

1. If $\int_\Omega |\nabla u|^p dx = \int_\Omega |\nabla v|^q dx = 0$, then $u = cst$ and $v = cst$. So, we have $0 = \lambda|cst|^p \int_{\partial\Omega} md\sigma$ and $0 = \lambda|cst|^q \int_{\partial\Omega} nd\sigma$. Since $\int_{\partial\Omega} md\sigma < 0$ and $\int_{\partial\Omega} nd\sigma < 0$, $u = v = 0$. This contradicts the fact that $u \neq 0$ or $v \neq 0$.

2. If $\int_\Omega |\nabla u|^p dx > 0$ or $\int_\Omega |\nabla v|^q dx > 0$, then

$$0 < \frac{\int_\Omega |\nabla u|^p dx}{\int_{\partial\Omega} m|u|^p d\sigma} = \lambda \text{ or } 0 < \frac{\int_\Omega |\nabla v|^q dx}{\int_{\partial\Omega} n|v|^q d\sigma} = \lambda.$$

Thus

$$\lambda_1(m,p) \le \frac{\int_\Omega |\nabla u|^p dx}{\int_{\partial\Omega} m|u|^p d\sigma} = \lambda \text{ or } \lambda_1(n,q) \le \frac{\int_\Omega |\nabla v|^q dx}{\int_{\partial\Omega} n|v|^p d\sigma} = \lambda.$$

So

$$\lambda_1(m,p) \le \lambda \text{ or } \lambda_1(n,q) \le \lambda.$$

It follows that

$$\inf(\lambda_1(m,p), \lambda_1(n,q)) \le \lambda.$$

This contradicts our assumption.

□

Corollary 3.4. *1. For $\varepsilon = 1$, let $m, m_0 \in M_{\bar{p}}$ and $n, n_0 \in M_{\bar{q}}$, if $m \lneqq m_0$, $n \lneqq n_0$ on $\partial\Omega$ and $\lambda = \lambda_1(m_0,p) = \lambda_1(n_0,q)$, then the system (1.1) admits at least a solution for any f, g.*

2. For $\varepsilon = 0$, let $m \in M_{\bar{p}}$, $n \in M_{\bar{q}}$, if $0 < \lambda \le \inf(\lambda_1(m,p), \lambda_1(n,q)) < \sup(\lambda_1(m,p), \lambda_1(n,q))$, then the system (1.1) has no non-trivial solution $(u,v) \in W$ in the sense that $u \not\equiv 0$ and $v \not\equiv 0$.

3. For $\varepsilon = 0$, let $m \in M_{\bar{p}}$, $n \in M_{\bar{q}}$, if $\lambda = \lambda_1(m,p) = \lambda_1(n,q)$), then the system (1.1) has infinitely many solutions.

Proof. 1. Let $m, m_0 \in M_{\bar{p}}$, $n, n_0 \in M_{\bar{q}}$, by Theorem 2.1, if $m \lneqq m_0$ and $n \lneqq n_0$, then $\lambda = \lambda_1(m_0,p) < \lambda_1(m,p)$, and $\lambda = \lambda_1(n_0,q) < \lambda_1(n,q)$ this implies that $\lambda < \inf(\lambda_1(m,p), \lambda_1(n,q))$. According to Theorem 3.1 the proof is complete.

2.
- if $0 < \lambda < \inf(\lambda_1(m,p), \lambda_1(n,q))$, we use the Theorem 2.1.
- If $0 < \lambda = \inf(\lambda_1(m,p), \lambda_1(n,q)) < \sup(\lambda_1(m,p), \lambda_1(n,q))$, then we have two cases. First case: if $\lambda = \lambda_1(m,p) < \lambda_1(n,q)$), the non-trivial solutions are of the form $(\alpha\varphi_1(m,p), 0)$, where $\varphi_1(m,p)$ is an eigenfunction of system (2.1) associated to $\lambda_1(m,p)$.
  Second case: if $\lambda = \lambda_1(n,q) < \lambda_1(m,p)$), the non-trivial solutions are of the form $(0, \beta\varphi_1(n,q))$, where $\varphi_1(n,q)$ is an eigenfunction of system (2.1) (with q and n instead p and m) of associated to $\lambda_1(n,q)$.

3. We use the simplicity of the first eigenvalue $\lambda_1(k,r)$ of the system (2.1), where $k \equiv m$ and $r = p$ or $k \equiv n$ and $r = q$.

□

Acknowledgment

We would like to thank the anonymous referee for valuable suggestions.

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