Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 ON LINE
SPM: www.spm.uem.br/bspm

(3s.) **v. 2022 (40)** : 1–13.
ISSN-0037-8712 IN PRESS
doi:10.5269/bspm.51971

Mathematical Behavior of Solutions of the Kirchhoff Type Equation with Logarithmic Nonlinearity

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ABSTRACT: We consider the existence and decay estimates of solutions for Kirchhoff type equation with damping logarithmic source term. We proved global existence of solutions under suitable conditons by potential well method and the decay estimates result of the solutions for subcritical energy level.

Key Words: Existence, Decay of solution, Logarithmic nonlinearity.

Contents

1. Introduction

In this work, we focus on the following Kirchhoff type equation with nonlinear damping and logarithmic nonlinearity

$$\begin{cases} u_{tt} + M\left(\|\Delta u\|^2\right) \triangle^2 u + a_0 u_t + a_1 |u_t|^{r-1} u_t = |u|^{p-1} \ln |u|^k, & x \in \Omega,\ t > 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = \frac{\partial}{\partial \nu} u(x,t) = 0, & x \in \partial\Omega,\ t \geq 0, \end{cases} \tag{1.1}$$

where $a_0,\ a_1,\ k$ are a positive real number and $M(s) = \beta_1 + \beta_2 s^\gamma$, $\gamma > 0$, $\beta_1 \geq 1$, $\beta_2 > 0$. Also

$$\begin{cases} 2\gamma + 2k \leq p \leq \frac{2(n-2)}{n-4},\ n > 4, \\ 2\gamma + 2k \leq p \leq \infty, \quad n \leq 4, \end{cases}$$

where $\Omega \subset R^n$ $(n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$.

1.1. Wave equation with logarithmic term

Studies of logarithmic nonlinearity have a long history in physics as it occurs naturally in different areas of physics such as inflation cosmology, supersymmetric field theories, quantum mechanics and nuclear physics [4,8].

In [6], Cazenave and Haraux studied the existence of the solution following equation

$$u_{tt} - \triangle u + u = u \ln |u|^k, \tag{1.2}$$

in R^3. Nowadays, there are much more works related to logarithmic nonlinearity in the literature, we refer the interested readers to [7,12,16,17,22,25] and papers cited there in.

2010 *Mathematics Subject Classification:* 35A01, 35B40, 35L25.
Submitted January 31, 2020. Published May 06, 2020

Al-Gharabli and Messaoudi [1,2] studied the following equation

$$u_{tt} + \triangle^2 u + u + h(u_t) = |u| \ln |u|^k ,$$

where $k > 0$ and $h(s) = s$. They proved the local-global existence and the exponential decay rate of the solutions. Then Peyravi [15], studied stability and instability at infinity of solutions to a logarithmic wave equation

$$u_{tt} - \triangle u + u + (g * \triangle u)(t) + h(u_t) u_t + |u|^2 u = |u| \ln |u|^k ,$$

in an bounded domain $\Omega \subset R^3$ with $h(s) = k_0 + k_1 |s|^{m-1}$.

1.2. Kirchhoff type equation

In 1876, Kirchhoff [3] introduced Kirchhoff type equation in order to study the nonlinear vibrations of an elastic string. Kirchhoff model to described the transverse oscillations of stretched string, with local or nonlocal flexural rigidity [24]. So that, Kirchhoff model was very important for many applications in mechanics, elastic theory and other areas of mathematical physics [3]. It is worth noting that there have been mang interesting study of the with initial and boundary value problems for Kirchhoff type equation, for details on Kirchhoff type equation, we refer to see the works [9,13,19,20,23]

The original equation is

$$\rho h u_{tt} + \delta u_t + f = \left(P_0 + \frac{Eh}{2L} \int_0^L |u_x|^2 ds \right) u_{xx},$$

where $t > 0$ and $0 < x < L$. E is the Young modulus, p the mass density, h is the cross-section area, P_0 is the initial axial tension, δ is the resistance modulus, f is the external force.

1.3. Kirchoff type equation with logarithmic term

Then, some authors discussed the following Kirchhoff type equation

$$u_{tt} - M(\|\nabla u\|) \triangle u + g(u_t) = f(u),$$

which including more general function M and dissipative term (see [10,11,14]).

Nowadays, the studies have intensified about analysis of solutions for a class of Kirchhoff equation with logarithmic source term. We refer to work of see [5,21]. In 2019, Yang et. al [21] considered the following equation

$$u_{tt} - M\left(\|\nabla u\|^2\right) \Delta u + |u_t|^{p-1} u_t - \Delta u_t = u^{k-1} \ln |u| , \tag{1.3}$$

where $M(s) = \alpha + \beta s^{\gamma}$, $\gamma > 0, \alpha \geq 1, \beta > 0$. They studied local existence, finite time blow up and asymptotic behavior of solutions in cases subcritical energy and critical energy. And also, they proved finite time blow up solutions in case arbitrary high energy.

Motivated by the above studies, we established the global existence and decay estimates of the solution for the problem (1.1). The results can be also viewed as a improved proof to the global existence theorem in [18].

The rest of our work is organized as follows. In section 2, we gave some notations and lemmas which will be used throughout this paper. In section 3, we proved the global existence of the solutions of the problem. The decay estimates result are presented in section 4.

2. Preliminaries

In this work, we give some useful lemmas and assumptions, which have an essantial role in our proof. Let $W^{m,p}(\Omega)$ be the usual Sobolev space. Specially, $W^{m,2}(\Omega)$ and $W^{0,p}(\Omega)$ will be marked by $H^m(\Omega)$ and $L^p(\Omega)$, respectively. We denote $\|.\| = \|.\|_{L^2(\Omega)}$ and $\|.\|_p = \|.\|_{L^p(\Omega)}$. Moreover, C_i $(i = 1, 2, ...)$ are arbitrary constants.

We define the energy functional $E(t)$ of the problem (1.1) by

$$E(t) = \frac{1}{2} \|u_t\|^2 + \frac{\beta_1}{2} \|\Delta u\|^2 + \frac{\beta_2}{2\gamma + 2} \|\Delta u\|^{2\gamma+2} - \frac{k}{p} \int_{\Omega} u^p \ln |u| \, dx + \frac{k}{p^2} \|u\|_p^p . \tag{2.1}$$

Lemma 2.1. *$E(t)$ is a nonincreasing function for $t \geq 0$ and*

$$E'(t) = -a_0 \|u_t\|^2 - a_1 \|u_t\|_{r+1}^{r+1} \leq 0. \tag{2.2}$$

Proof. By multiplying the equation in (1.1) by u_t and integrating on Ω, we have

$$\begin{aligned} & \int_\Omega u_{tt} u_t dx + \int_\Omega M\left(\|\Delta u\|^2\right) \Delta u \Delta u_t dx + a_0 \int_\Omega u_t u_t dx + a_1 \int_\Omega |u_t|^{r-1} u_t u_t dx \\ = & \int_\Omega |u|^{p-1} \ln |u|^k u_t dx, \end{aligned}$$

$$\begin{aligned} & \frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2 + \frac{\beta_1}{2}\|\Delta u\|^2 + \frac{\beta_2}{2\gamma+2}\|\Delta u\|^{2\gamma+2} - \frac{k}{p}\int_\Omega u^p \ln |u| \, dx + \frac{k}{p^2}\|u\|_p^p\right) \\ = & -a_0 \|u_t\|^2 - a_1 \|u_t\|_{r+1}^{r+1} \end{aligned}$$

and

$$E'(t) = -a_0 \|u_t\|^2 - a_1 \|u_t\|_{r+1}^{r+1} \leq 0, \tag{2.3}$$

$$\begin{aligned} E(t) + a_0 \int_0^t \int_\Omega |u_t|^2 dxdt + a_1 \int_0^t \int_\Omega |u_t|^{r+1} dxdt &= E(0) \\ E(t) &\leq E(0), \end{aligned} \tag{2.4}$$

where

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{\beta_1}{2}\|\Delta u\|^2 + \frac{\beta_2}{2\gamma+2}\|\Delta u\|^{2\gamma+2} - \frac{k}{p}\int_\Omega u^p \ln |u| \, dx + \frac{k}{p^2}\|u\|_p^p. \tag{2.5}$$

□

Lemma 2.2. *[6] (Logarithmic Gronwall Inequality).*

Let $c > 0$, $\gamma \in L^1(0, T; R^+)$ and assume that the function $w : [0, T] \to [1, \infty)$ satisfies

$$w(t) \leq c\left(1 + \int_0^t \gamma(s) w(s) \ln w(s) \, ds\right), \ 0 \leq t \leq T,$$

then

$$w(t) \leq c e^{\int_0^t c\gamma(s) ds}, \ 0 \leq t \leq T.$$

3. Global existence

In this section, we prove the global existence of solution of the problem (1.1) .
Now, we define the following functionals

$$J(u) = \frac{\beta_1}{2}\|\Delta u\|^2 + \frac{\beta_2}{2\gamma+2}\|\Delta u\|^{2\gamma+2} - \frac{k}{p}\int_\Omega u^p \ln |u| \, dx + \frac{k}{p^2}\|u\|_p^p \tag{3.1}$$

and

$$I(t) = \beta_1 \|\Delta u\|^2 + \beta_2 \|\Delta u\|^{2\gamma+2} - \int_\Omega u^p \ln |u| \, dx. \tag{3.2}$$

Obviously

$$J(u)=\frac{k}{p}I(u)+\beta_1\left(\frac{p-2k}{2p}\right)\|\Delta u\|^2+\beta_2\left(\frac{p-2\gamma k-2k}{p(2\gamma+2)}\right)\|\Delta u\|^{2\gamma+2}+\frac{k}{p^2}\|u\|_p^p \tag{3.3}$$

and

$$E(t)=\frac{1}{2}\|u_t\|^2+J(u). \tag{3.4}$$

The potential well depth is defined as

$$W=\left\{u\in H_0^2(\Omega)\mid J(u)<d, I(u)>0\right\}\cup\{0\} \tag{3.5}$$

and the outer space of the potential well

$$V=\left\{u\in H_0^2(\Omega)\mid J(u)<d, I(u)<0\right\}. \tag{3.6}$$

Now, we establish some properties of the $J(u)$ and $I(u)$.

Lemma 3.1. *For any $u\in H_0^2(\Omega)$, $\|\Delta u\|\neq 0$ and let $g(\lambda)=J(\lambda u)$. Then we have*

i) $\lim\limits_{\lambda\to 0}g(\lambda)=0,\ \lim\limits_{\lambda\to\infty}g(\lambda)=-\infty$,

ii) there is a unique λ_1 such that $g'(\lambda)=0$,

iii) $g(\lambda)$ is strictly decreasing on $\lambda_1<\lambda$, strictly increasing on $0\leq\lambda\leq\lambda_1$ and takes the maximum at $\lambda=\lambda_1$; $I(\lambda u)=\lambda g'(\lambda)$ and

$$I(\lambda u)=\lambda g'(\lambda)\begin{cases}>0, & 0\leq\lambda\leq\lambda_1,\\ =0, & \lambda=\lambda_1,\\ <0, & \lambda_1\leq\lambda.\end{cases}$$

Proof. i) By the definition of $J(u)$, we obtain

$$\begin{aligned} g(\lambda) &= J(\lambda u)\\ &= \frac{\beta_1}{2}\|\lambda\Delta u\|^2+\frac{\beta_2}{2\gamma+2}\|\lambda\Delta u\|^{2\gamma+2}-\frac{k}{p}\int_\Omega(\lambda u)^p\ln|\lambda u|\,dx+\frac{k}{p^2}\int_\Omega|\lambda u|^p\,dx\\ &= \frac{\beta_1}{2}\lambda^2\|\Delta u\|^2+\frac{\beta_2}{2\gamma+2}\lambda^{2\gamma+2}\|\Delta u\|^{2\gamma+2}\\ &\quad -\frac{k}{p}\lambda^p\int_\Omega u^p\ln|u|\,dx-\frac{k}{p}\lambda^p\ln\lambda\int_\Omega u^p+\frac{k}{p^2}\lambda^p\|u\|_p^p, \end{aligned} \tag{3.7}$$

which means $\lim\limits_{\lambda\to 0}g(\lambda)=0,\ \lim\limits_{\lambda\to\infty}g(\lambda)=-\infty$.

ii) Now, differentiating $g(\lambda)$ with respect to λ, we have

$$\begin{aligned} \frac{d}{d\lambda}J(\lambda u) &= g'(\lambda)=\beta_1\lambda\|\Delta u\|^2+\beta_2\lambda^{2\gamma+1}\|\Delta u\|^{2\gamma+2}\\ &\quad -k\lambda^{p-1}\int_\Omega u^p\ln|u|\,dx-k\lambda^{p-1}\|u\|_p^p\\ &= \lambda\left(\beta_1\|\Delta u\|^2+\beta_2\lambda^{2\gamma}\|\Delta u\|^{2\gamma+2}-k\lambda^{p-2}\int_\Omega u^p\ln|u|\,dx-k\lambda^{p-2}\|u\|_p^p\right)\\ &= \lambda\left(\beta_1\|\Delta u\|^2+\psi(\lambda)\right), \end{aligned} \tag{3.8}$$

where

$$\psi(\lambda) = \beta_2 \lambda^{2\gamma} \|\Delta u\|^{2\gamma+2} - k\lambda^{p-2} \int\limits_{\Omega} u^p \ln|u|\, dx - k\lambda^{p-2} \|u\|_p^p.$$

We observe from $2\gamma \le p-2$ that $\lim\limits_{\lambda\to\infty} \psi(\lambda) = -\infty$. Then we can see $\psi(\lambda)$ is monotone decreasing when $\lambda > \lambda^*$ and there exists a unique λ^* such that $\psi(\lambda^*) = 0$. Then we obtain there is a $\lambda_1 > \lambda^*$ such that $\beta_1 \lambda \|\Delta u\|^2 + \psi(\lambda) = 0$, which means $g'(\lambda_1) = 0$.

iii) The conclusion (iii) directly follows from

$$I(\lambda u) = \lambda \frac{dJ(\lambda u)}{d\lambda} = \lambda g'(\lambda). \tag{3.9}$$

□

Lemma 3.2. *i) The definition the depth of potential well*

$$d = \inf_{u\in N} J(u), \tag{3.10}$$

where

$$N = \left\{u \in H_0^2(\Omega) \setminus \{0\} : I(u) = 0\right\},$$

is equivalent to

$$d = \inf \left\{ \sup_{\lambda\ge 0} J(\lambda u) \mid u \in H_0^2(\Omega),\ \|\Delta u\|^2 \ne 0 \right\}. \tag{3.11}$$

ii) d is defined as

$$d = \frac{\beta_1 (p-2k)}{2p} \left(\frac{\beta_1}{C_*^{p+1}} \right)^{\frac{2}{p-1}}.$$

Proof. i) On one hand from (iii) of Lemma 3.1 it implies that for any $u \in H_0^2(\Omega)$, there exist a λ_1 such that $I(\lambda_1 u) = 0$, that is $\lambda_1 u \in N$. By the definition of d we obtain

$$J(\lambda_1 u) \ge d \text{ for any } u \in H_0^2(\Omega) \setminus \{0\}, \tag{3.12}$$

and because of Lemma 3.1 of property (iii), this λ_1 is also the maximizer of $J(\lambda u)$ such that

$$\sup_{\lambda\ge 0} J(\lambda u) = J(\lambda_1 u),$$

which by virtue of (3.12) means

$$\inf_{u\in H_0^2(\Omega)} \sup_{\lambda\ge 0} J(\lambda u) = \inf_{u\in H_0^2(\Omega)} J(\lambda_1 u) \ge d. \tag{3.13}$$

As $u \in H_0^2(\Omega) \setminus \{0\}$, we obtain d is not equavilent to 0, which gives (3.11). But then, from the definition of d given by (3.11) it implies that there exists λ^* such that

$$\sup_{\lambda\ge 0} J(\lambda u) = J(\lambda^* u).$$

Then from Lemma 3.1 we can deduce $\lambda^* = \lambda_1$. Again from Lemma 3.1 of property (iii) it shows that

$$I(\lambda^* u) = I(\lambda_1 u) = 0,$$

which means $\lambda^* u \in N$. By the definition of d we get

$$d = \inf_{\lambda^* u\in N} J(\lambda^* u),$$

that is

$$d=\inf_{u\in N}J(u). \tag{3.14}$$

This complete our proof for (i).

ii) By virtue of $I(u)=0$, definition of $I(u)$ and embedding theorems we obtain

$$\begin{aligned}\beta_1\|\Delta u\|^2+\beta_2\|\Delta u\|^{2\gamma+2} &= \int_\Omega u^p\ln|u|\,dx\\ \beta_1\|\Delta u\|^2 &\le \int_\Omega u^p\ln|u|\,dx\\ &\le \|u\|_{p+1}^{p+1}\\ &\le C_*^{p+1}\|\Delta u\|^{p+1},\end{aligned}\tag{3.15}$$

which means

$$\|\Delta u\|\ge\left(\frac{\beta_1}{C_*^{p+1}}\right)^{\frac{1}{p-1}}. \tag{3.16}$$

From the definition of d, we have $u\in N$. By (3.15) and $I(u)=0$, we get

$$\begin{aligned}J(u) &= \frac{k}{p}I(u)+\beta_1\left(\frac{p-2k}{2p}\right)\|\Delta u\|^2+\beta_2\left(\frac{p-2\gamma-2k}{p(2\gamma+2)}\right)\|\Delta u\|^{2\gamma+2}+\frac{k}{p^2}\|u\|_p^p\\ &\ge \beta_1\left(\frac{p-2k}{2p}\right)\|\Delta u\|^2\\ &\ge \beta_1\left(\frac{p-2k}{2p}\right)\left(\frac{\beta_1}{C_*^{p+1}}\right)^{\frac{2}{p-1}},\end{aligned}$$

we take $2\gamma\le p-2k$ and since $\beta_1>1$, $\beta_2>0$ and k is a positive constant. Combining of (3.14) and (3.16), we can see clearly that

$$d=\beta_1\left(\frac{p-2k}{2p}\right)\left(\frac{\beta_1}{C_*^{p+1}}\right)^{\frac{2}{p-1}}.$$

□

Definition 3.3. *A function $u(t)$ is called a weak solution to problem (1.1) on $\Omega\times[0,T)$, if*

$$u\in C\left((0,T);H_0^2(\Omega)\right)\cap C^1\left((0,T);L^2(\Omega)\right)\cap C^2\left((0,T);H^{-2}\right)$$

and

$$u_t\in L^\infty\left((0,T);L^2(\Omega)\cap L^{r+1}(\Omega)\right)$$

satisfy

$$\begin{cases}\int_\Omega u_{tt}(x,t)\,w(x)\,dx+\int_\Omega M\left(\|\Delta u\|^2\right)\Delta u\Delta w(x)\,dx++\int_\Omega a_0u_tw(x)\,dx\\ +\int_\Omega a_1|u_t|^{r-1}u_tw(x)\,dx=k\int_\Omega\ln|u(x,t)|\,u^{p-1}(x,t)\,w(x)\,dx.\end{cases}$$

Lemma 3.4. *Let $u(t)$ be a weak solution problem of (1.1) and $u_0(t)\in H_0^2(\Omega), u_1(t)\in L^2(\Omega)$. Suppose that $E(0)<d$.*

i) If $u_0\in W$, then $u\in W$ for $0\le t\le T$;

ii) If $u_0\in V$, then $u\in V$ for $0\le t\le T$,

where T is the maximum existence time of $u(t)$.

Proof. i) If $u(t)$ is a weak solution problem of (1.1) under the conditions $E(0) < d$, $u_0 \in W$, then by (2.4) says that

$$E(u(t)) < E(0) < d.$$

We shall prove $I(u(t)) > 0$ for $0 < t < T$. We will use conradiction and we suppose that; there is a $t_1 \in (0,T)$ such that $I(u(t_1)) < 0$. Observe by the continuity of $I(u(t))$ in t that there exists a $t^* \in (0,T)$ such that $I(u(t^*)) = 0$.Then by (3.10), we get

$$d > E(0) \geq E(u(t^*)) \geq J(u(t^*)) \geq d,$$

which is a contradiction.

ii) The proof of case (ii) is similar. □

Lemma 3.5. *Under the conditions of Lemma 3.4 in (i) , we have*

$$E(0) \geq E(u) \geq J(u) > \beta_1 \left(\frac{p-2k}{2p}\right) \|\Delta u\|^2 .$$

Proof. By definition of $J(u)$, $I(u)$ and $I(u) > 0$, we get

$$\begin{aligned} J(u) &= \frac{k}{p} I(u) + \beta_1 \left(\frac{p-2k}{2p}\right) \|\Delta u\|^2 + \beta_2 \left(\frac{p-2\gamma-2k}{p(2\gamma+2)}\right) \|\Delta u\|^{2\gamma+2} + \frac{k}{p^2} \|u\|_p^p \\ &> \beta_1 \left(\frac{p-2k}{2p}\right) \|\Delta u\|^2 + \beta_2 \left(\frac{p-2\gamma-2k}{p(2\gamma+2)}\right) \|\Delta u\|^{2\gamma+2} + \frac{k}{p^2} \|u\|_p^p \\ &> \beta_1 \left(\frac{p-2k}{2p}\right) \|\Delta u\|^2 . \end{aligned}$$

Because of (3.4) and (2.4) we can see clearly that

$$E(0) \geq E(u) \geq J(u) > \beta_1 \left(\frac{p-2k}{2p}\right) \|\Delta u\|^2 .$$

□

Theorem 3.6. *Let $u_0(x) \in H_0^2(\Omega), u_1(x) \in L^2(\Omega)$. If $I(u_0) > 0$ and $E(0) < d$ or $\|\Delta u_0\| = 0$, then problem (1.1) admits a global weak solution $u \in L^\infty\left(0,\infty; H_0^2(\Omega)\right)$, $u_t \in L^\infty\left(0,\infty; L^2(\Omega) \cap L^{r+1}(\Omega)\right)$.*

Proof. Let $\{w_j\}_{j=1}^\infty$ be an orthogonal basis of the "separable" space $H_0^2(\Omega)$ which is orthonormal in $L^2(\Omega)$. Set

$$V_m = \text{span}\{w_1, w_2, ..., w_m\}$$

and let the projections of the initial data on the finite dimensional subspace V_m be given by

$$\begin{aligned} u_0^m(x) &= \sum_{j=1}^m a_j w_j(x) \to u_0 \text{ in } H_0^2(\Omega), \\ u_1^m(x) &= \sum_{j=1}^m b_j w_j(x) \to u_1 \text{ in } L^2(\Omega), \end{aligned}$$

for $j = 1, 2, ..., m$.

We look for the approximate solutions

$$u^m(x,t) = \sum_{j=1}^m h_j^m(t) w_j(x),$$

of the approximate problem in V_m

$$\begin{cases} \int\limits_{\Omega}\left(u_{tt}^{m}wdx+M\left(\|\Delta u\|^{2}\right)\triangle u^{m}\Delta w+a_{0}u_{t}^{m}w+a_{1}\left|u_{t}^{m}\right|^{r-1}u_{t}^{m}w\right)dx \\ \qquad =\int\limits_{\Omega}\ln\left|u^{m}\right|^{k}\left|u^{m}\right|^{p-1}wdx, \qquad w\in V_{m}, \\ u^{m}(0)=u_{0}^{m}=\sum\limits_{j=1}^{m}(u_{0},w_{j})w_{j}, \\ u_{t}^{m}(0)=u_{1}^{m}=\sum\limits_{j=1}^{m}(u_{1},w_{j})w_{j}. \end{cases} \tag{3.17}$$

This leads to a system of ordinary differantial equations for unknown functions $h_j^m(t)$. According to the standard existence theory for ordinary differantial equation, one can obtain functions

$$h_j:[0,t_m)\to R,\quad j=1,2,...,m,$$

which satisfy (3.17) in a maximal interval $[0,t_m)$, $0<t_m\le T$. Next, we show that $t_m=T$ and that the local solution is uniformly bounded independent of m and t. For this purpose, let us replace w by u_t^m in (3.17) and integrate by parts, we have

$$\frac{d}{dt}E^{m}(t)=-a_{0}\left\|u_{t}^{m}\right\|^{2}-a_{1}\left\|u_{t}^{m}\right\|_{r+1}^{r+1}\le 0, \tag{3.18}$$

where

$$E^{m}(t)=\frac{1}{2}\left\|u_{t}^{m}\right\|^{2}+\frac{\beta_{1}}{2}\left\|\Delta u^{m}\right\|^{2}+\frac{\beta_{2}}{2\gamma+2}\left\|\Delta u^{m}\right\|^{2\gamma+2}-\frac{k}{p}\int\limits_{\Omega}\left|u^{m}\right|^{p}\ln\left|u^{m}\right|dx+\frac{k}{p^{2}}\left\|u^{m}\right\|_{p}^{p}. \tag{3.19}$$

Integrating (3.18) from 0 to t, and using of (3.4), we obtain

$$\frac{1}{2}\left\|u_{t}^{m}\right\|^{2}+J(u^{m})+\int\limits_{0}^{t}\left(a_{0}\left\|u_{s}^{m}\right\|^{2}+a_{1}\left\|u_{s}^{m}\right\|_{r+1}^{r+1}\right)ds=E^{m}(0). \tag{3.20}$$

By virtue problem of (3.17) initial data, while $m\to\infty$ we obtain $E^m(0)\to E(0)$. By choosing of large m we have

$$\frac{1}{2}\left\|u_{t}^{m}\right\|^{2}+J(u^{m})+\int\limits_{0}^{t}\left(a_{0}\left\|u_{s}^{m}\right\|^{2}+a_{1}\left\|u_{s}^{m}\right\|_{r+1}^{r+1}\right)ds<d. \tag{3.21}$$

By $u_0\in W$,

$$\frac{1}{2}\left\|u_{t}^{m}(0)\right\|^{2}+J(u^{m}(0))=E(0),$$

and initial data, for choosing large m and $0\le t<\infty$, we get $u^m(0)\in W$. By (3.21) and an argument similar to Lemma 3.4, by choosing large m and $0\le t<\infty$, we have $u^m(t)\in W$. Therefore, by virtue of (3.21) and (3.1) we get

$$\begin{aligned} &\frac{1}{2}\left\|u_{t}^{m}\right\|^{2}+\frac{\beta_{1}}{2}\left\|\Delta u^{m}\right\|^{2}+\frac{\beta_{2}}{2\gamma+2}\left\|\Delta u^{m}\right\|^{2\gamma+2} \\ &-\frac{k}{p}\int\limits_{\Omega}\left|u^{m}\right|^{p}\ln\left|u^{m}\right|dx+\frac{k}{p^{2}}\left\|u^{m}\right\|_{p}^{p} \\ &+\int\limits_{0}^{t}\left(a_{0}\left\|u_{s}^{m}\right\|^{2}+a_{1}\left\|u_{s}^{m}\right\|_{r+1}^{r+1}\right)ds \\ <\ & d \end{aligned} \tag{3.22}$$

where $0 \le t < \infty$ and choosing k is a positive constant smallest enough and $p \ge 2\gamma + 2k$. For a sufficiently large m and $0 \le t < \infty$, (3.22) gives

$$\|u_t^m\|^2 < 2d,$$

$$\|\Delta u^m\|^2 < \frac{2p}{\beta_1 (p - 2k)} d,$$

$$\|\Delta u^m\|^{2\gamma+2} < \frac{p (2\gamma + 2)}{\beta_2 (p - 2\gamma - 2k)} d,$$

$$\|u^m\|_p^p < \frac{p^2}{k} d,$$

$$\int_0^t \|u_s^m\|^2 \, ds < \frac{d}{\alpha_0},$$

$$\int_0^t \|u_s^m\|_{r+1}^{r+1} \, ds < \frac{d}{\alpha_1}.$$

Then we obtain

$$\begin{cases} u^m, \text{ is uniformly bounded in } L^\infty \left(0, \infty; H_0^2 (\Omega)\right), \\ u_t^m, \text{ is uniformly bounded in } L^\infty \left(0, \infty; L^2 (\Omega) \cap L^{r+1} (\Omega)\right). \end{cases}$$

By using of Sobolev embedding inequality, (3.21) and (3.22) we get

$$\begin{aligned} \int_\Omega |u^m|^p \ln |u^m| \, dx &\le \|u^m\|_{p+1}^{p+1} \\ &\le C_p \|\Delta u^m\|_2^{p+1} \\ &< \left(\frac{2pd}{\beta_1 (p - 2k)}\right)^{\frac{p+1}{2}}, \end{aligned}$$

so that we obtain

$$|u^m|^{p+1}, \text{ is uniformly bounded in } L^\infty \left(0, \infty; L^{p+1} (\Omega)\right).$$

Then integrating (3.17) with respect to t , for $0 \le t < \infty$ we have

$$\begin{aligned} &(u_t, w) + \int_0^t M\left(\|\Delta u\|^2\right) (\triangle u^m, \Delta w) \, d\tau \\ = \; &(u_1, w) + (\triangle u_0, \Delta w) + \int_0^t \left(\ln |u^m|^k |u^m|^{p-1}, w\right) d\tau \\ &-a_0 \int_0^t (u_t^m, w) - a_1 \int_0^t \left(|u_t^m|^{r-1} |u_t^m|, w\right) d\tau. \end{aligned} \tag{3.23}$$

Therefore, up to a subsequence, we may pass to the limit in (3.23), and get a weak solution (u) to problem (1.1) with the above regularity. On the other hand, initial data conditions in (3.17) we may conclude $(u(x, 0)) = (u_0)$ in H_0^2 and $(u_t (x, 0)) = (u_1)$ in $L^2 (\Omega) \cap L^{q+1} (\Omega)$. □

4. Decay Estimates

Theorem 4.1. *Let $u_0(x) \in H_0^2(\Omega)$, $u_1(x) \in L^2(\Omega)$. Assume that $E(0) < d$, $I(u_0) > 0$ or $\|\Delta u_0\| = 0$ and*

$$\begin{aligned} 1 &\leq r \leq \frac{n+2}{n-2}, \quad n \geq 3, \\ 1 &\leq r \leq \infty, \quad n = 1,2, \end{aligned}$$

there exist positive constants N and n such that

$$E(t) \leq Ne^{-nt}, t \geq 0.$$

Proof. Let

$$L(t) = E(t) + \varepsilon \int_\Omega uu_t + \varepsilon \frac{a_0}{2} \|u\|^2, \tag{4.1}$$

then we observe for sufficient small ε that there exist positive constants λ_1, λ_2, such that

$$\lambda_1 E(t) \leq L(t) \leq \lambda_2 E(t), \tag{4.2}$$

and $L(t) > 0$ for any $t \geq 0$.

By multiplying the (1.1) by u and integrating on Ω, we obtain

$$\int_\Omega u_{tt} u dx = k \int_\Omega \ln|u| u^{p-1} w u dx - \int_\Omega M\left(\|\Delta u\|^2\right) \Delta u \Delta u dx - \int_\Omega a_0 u_t u dx - a_1 \int_\Omega |u_t|^{r-1} u_t u dx. \tag{4.3}$$

By derivative of (4.1) and using of (2.3) and (4.3) we obtain

$$\begin{aligned} L'(t) &= -a_0 \|u_t\|^2 - a_1 \|u_t\|_{r+1}^{r+1} + \varepsilon \left(\|u_t\|^2 + \int_\Omega uu_{tt} dx + a_0 \int_\Omega uu_t dx \right) \\ &= -a_0 \|u_t\|^2 - a_1 \|u_t\|_{r+1}^{r+1} + \varepsilon \left(\|u_t\|^2 + \int_\Omega |u|^p \ln|u|^k dx \right) \\ &\quad -\varepsilon \left(M\left(\|\Delta u\|^2\right) \|\Delta u\|^2 + a_1 \int_\Omega |u_t|^{r-1} u_t u dx \right) \\ &= -a_0 \|u_t\|^2 - a_1 \|u_t\|_{r+1}^{r+1} + \varepsilon \left(\|u_t\|^2 + \int_\Omega |u|^p \ln|u|^k dx \right) \\ &\quad -\varepsilon \left(\beta_1 \|\Delta u\|^2 + \beta_2 \|\Delta u\|^{2\gamma+2} + a_1 \int_\Omega |u_t|^{r-1} u_t u dx \right). \end{aligned} \tag{4.4}$$

Now, our aim is to estimate every term of (4.4) severally.

By Sobolev embedding inequality and Lemma 3.5, we conclude

$$\begin{aligned} \int_\Omega |u|^p \ln|u|^k dx &\leq \|u\|_{p+1}^{p+1} \\ &\leq C_*^{p+1} \|\Delta u\|^{p+1} \\ &\leq C_*^{p+1} \left(\frac{2p}{\beta_1 (p-2k)} E(0) \right)^{\frac{p-1}{2}} \|\Delta u\|^2 \\ &= \Psi \|\Delta u\|^2, \end{aligned} \tag{4.5}$$

where $\Psi = C_*^{p+1}\left(\frac{2p}{\beta_1(p-2k)}E(0)\right)^{\frac{p-1}{2}}$.

By using Young's inequality for the last term in the right term-hand side, for $\delta > 0$, we conclude

$$\left| a_1 \int_\Omega |u_t|^{r-1} u_t u dx \right| \le a_1 C(\delta) \|u\|_{r+1}^{r+1} + a_1 \delta \|u_t\|_{r+1}^{r+1}, \tag{4.6}$$

where $C(\delta) = \frac{\delta^{-(r+1)}}{r+1}$ and $\delta = \frac{r}{r+1}\delta^{\frac{r+1}{r}}$.

Then by Lemma 3.5 and embedding inequality

$$\begin{aligned} \|u\|_{r+1}^{r+1} &\le C_2^{r+1} \|\Delta u\|_{r+1}^{r+1} \\ &\le C_2^{r+1}\left(\frac{2p}{\beta_1(p-2k)}E(0)\right)^{\frac{r-1}{2}} \|\Delta u\|^2 \\ &= \xi \|\Delta u\|^2, \end{aligned} \tag{4.7}$$

where $\xi = C_2^{r+1}\left(\frac{2p}{\beta_1(p-2k)}E(0)\right)^{\frac{r-1}{2}} > 0$. So that we can write (4.6)

$$\left| a_1 \int_\Omega |u_t|^{r-1} u_t u dx \right| \le a_1 C(\delta)\xi \|\Delta u\|^2 + a_1 \delta \|u_t\|_{r+1}^{r+1}. \tag{4.8}$$

Inserting (4.8) and (4.5) into (4.4), yields

$$\begin{aligned} L'(t) \le\ & (\varepsilon - a_0)\|u_t\|^2 + a_1(\varepsilon\delta - 1)\|u_t\|_{r+1}^{r+1} - \varepsilon\beta_2 \|\Delta u\|^{2\gamma+2} \\ & + \varepsilon(\Psi - \beta_1 + a_1 C(\delta)\xi)\|\Delta u\|^2. \end{aligned} \tag{4.9}$$

By the definition of $E(t)$,which combining (4.5) makes (4.9) to be

$$\begin{aligned} L'(t) \le\ & (\varepsilon - a_0)\|u_t\|^2 + a_1(\varepsilon\delta - 1)\|u_t\|_{r+1}^{r+1} - \varepsilon\beta_2 \|\Delta u\|^{2\gamma+2} \\ & + \varepsilon(\Psi - \beta_1 + a_1 C(\delta)\xi)\|\Delta u\|^2 - \varepsilon\omega E(t) + \frac{\varepsilon\omega}{2}\|u_t\|^2 \\ & + \frac{\beta_1\varepsilon\omega}{2}\|\Delta u\|^2 + \frac{\beta_2\varepsilon\omega}{2\gamma+2}\|\Delta u\|^{2\gamma+2} - \frac{k\varepsilon\omega}{p}\int_\Omega u^p \ln|u|\, dx + \frac{k\varepsilon\omega}{p^2}\|u\|_p^p \\ \le\ & \left(\varepsilon\left(1+\frac{\omega}{2}\right) - a_0\right)\|u_t\|^2 + a_1(\varepsilon\delta - 1)\|u_t\|_{r+1}^{r+1} + \beta_2\varepsilon\left(\frac{\omega}{2\gamma+2} - 1\right)\|\Delta u\|^{2\gamma+2} \\ & + \frac{k\varepsilon\omega}{p^2}\|u\|_p^p + \varepsilon\left(\Psi - \beta_1 + a_1 C(\delta)\xi + \frac{\beta_1\omega}{2} + \frac{k\omega}{p}\Psi\right)\|\Delta u\|^2 - \varepsilon\omega E(t), \end{aligned} \tag{4.10}$$

where $\omega > 0$, which will be determined later. By again using of sobolev embedding inequality and Lemma 3.5, we estimate $\|u\|_p^p$ as follows

$$\begin{aligned} \|u\|_p^p &\le C_3^p \|\Delta u\|_p^p \\ &\le C_3^p\left(\frac{2p}{\beta_1(p-2k)}E(0)\right)^{\frac{p-2}{2}} \|\Delta u\|^2 \\ &\le \varrho \|\Delta u\|^2, \end{aligned} \tag{4.11}$$

where $\varrho = C_3^p\left(\frac{2p}{\beta_1(p-2k)}E(0)\right)^{\frac{p-2}{2}} > 0$.

Then by combining (4.11) and (4.10), we have

$$\begin{aligned} L'(t) &\leq \left(\varepsilon\left(1+\frac{\omega}{2}\right)-a_0\right)\|u_t\|^2 + a_1(\varepsilon\delta-1)\|u_t\|_{r+1}^{r+1} + \beta_2\varepsilon\left(\frac{\omega}{2\gamma+2}-1\right)\|\Delta u\|^{2\gamma+2} \\ &\quad +\varepsilon\left(\Psi-\beta_1+a_1C(\delta)\xi+\frac{\beta_1\omega}{2}+\frac{k\omega\Psi}{p}+\frac{k\varrho\omega}{p^2}\right)\|\Delta u\|^2-\varepsilon\omega E(t) \\ &= \mu_1\|u_t\|^2+\mu_2\|u_t\|_{r+1}^{r+1}+\mu_3\|\Delta u\|^{2\gamma+2}+\mu_4\|\Delta u\|^2-\varepsilon\omega E(t). \end{aligned} \tag{4.12}$$

Firstly, combining $E(0)<d$ and definition of d we conclude

$$\begin{aligned} E(0) &< d=\beta_1\left(\frac{p-2k}{2p}\right)\left(\frac{\beta_1}{C_*^{p+1}}\right)^{\frac{2}{p-1}} \\ \beta_1 &> C_*^{p+1}\left(\frac{2p}{\beta_1(p-2k)}E(0)\right)^{\frac{p-1}{2}}, \end{aligned}$$

which means

$$\Psi=C_*^{p+1}\left(\frac{2p}{\beta_1(p-2k)}E(0)\right)^{\frac{p-1}{2}}<\beta_1. \tag{4.13}$$

We can take ε small enough such that $\mu_1<0$ and $\mu_2<0$. In order to ensure $\mu_3\leq 0$ and $\mu_4\leq 0$ for $\Psi<\beta_1$,we need

$$\omega=\min\left\{(2\gamma+2),\frac{\beta_1-\Psi-a_1C(\delta)\xi}{\frac{\beta_1}{2}+\frac{k\Psi}{p}+\frac{k\varrho}{p^2}}\right\}>0. \tag{4.14}$$

By choosing small enough $C(\delta)>0$ such that (4.14) makes sense.

Hence, by the above arguments we can write (4.12) such that

$$L'(t)\leq -\varepsilon\omega E(t). \tag{4.15}$$

Therefore, by using of (4.2), we can write (4.15) as follows

$$L'(t)\leq \mu_5\frac{L(t)}{\lambda_2}.$$

Then by Gronwall inequality we get

$$L(t)\leq L(0)e^{-nt},\ n=\frac{\mu_5}{\lambda_2}.$$

Again using of (4.2) we obtain

$$E(t)\leq Ne^{-nt},$$

where $N=\frac{\lambda_2E(0)}{\lambda_1}$. This completed our proof. □

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