



Supercyclicity of Multiplication on Banach Ideal of Operators

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ABSTRACT: Let X be a complex Banach space with $\dim X > 1$ such that its topological dual X^* is separable and $\mathcal{B}(X)$ the algebra of all bounded linear operators on X . In this paper, we study the passage of property of being supercyclic from $T \in \mathcal{B}(X)$ to the left and the right multiplication induced by T on an admissible Banach ideal of $\mathcal{B}(X)$. Also, we give a sufficient conditions for the tensor product $T \widehat{\otimes} R$ of two operators on $\mathcal{B}(X)$ to be supercyclic.

Key Words: Orbit, Hypercyclicity, Supercyclicity, Left multiplication, Right multiplication, Tensor product, Banach ideal of operators.

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1. Introduction and Preliminary

Throughout this paper, let X be a Banach space with $\dim X > 1$ such that its topological dual X^* is separable, $\mathcal{B}(X)$ be the algebra of all bounded linear operators on X and $\mathcal{K}(X)$ be the algebra of all compact operators on X . For $T \in \mathcal{B}(X)$, the orbit of $x \in X$ under T is the set

$$Orb(T, x) := \{T^n x : n \in \mathbb{N}\}.$$

An operator $T \in \mathcal{B}(X)$ is said to be hypercyclic if there is some vector $x \in X$ such that $Orb(T, x)$ is dense in X ; such a vector x is called a hypercyclic vector for T . Similarly, $T \in \mathcal{B}(X)$ is said to be supercyclic if there is some vector $x \in X$ such that

$$COrb(T, x) := \{\alpha T^n x : \alpha \in \mathbb{C}, n \in \mathbb{N}\}$$

is dense in X ; such a vector x is called supercyclic vector for T . From [7], $T \in \mathcal{B}(X)$ is supercyclic if and only if for each pair (U, V) of nonempty open subsets of X there exist $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$ such that

$$\alpha T^n(U) \cap V \neq \emptyset.$$

We say that an operator $T \in \mathcal{B}(X)$ satisfies the hypercyclicity criterion, if there exist two dense subsets $D_1, D_2 \subset X$, a strictly increasing sequence $(n_k)_{k \geq 0}$ of positive integers and a sequence of maps $S_{n_k} : D_2 \rightarrow X$ such that :

- (i) $T^{n_k} x \rightarrow 0$ for any $x \in D_1$;
- (ii) $S_{n_k} y \rightarrow 0$ for any $y \in D_2$;
- (iii) $T^{n_k} S_{n_k} y \rightarrow y$ for any $y \in D_2$.

The hypercyclicity criterion was introduced in [15] and studied in [3,8,15]. In [21], Salas gave a characterization of the supercyclic bilateral backward weighted shifts in terms of the supercyclicity criterion that ensure supercyclicity. We say that $T \in \mathcal{B}(X)$ satisfies the supercyclicity criterion, if there exist two dense subsets D_1 and D_2 in X , a sequence $(n_k)_{k \geq 0}$ of positive integers and a sequence of maps $S_{n_k} : D_2 \rightarrow X$ such that :

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- (i) $\|T^{n_k}x\| \|S_{n_k}y\| \longrightarrow 0$ for every $x \in D_1$ and $y \in D_2$;
- (ii) $T^{n_k}S_{n_k}y \longrightarrow y$ for every $y \in D_2$.

For a more general overview of hypercyclicity, supercyclicity and related properties in linear dynamics, we refer to [1,2,7,12,18,19].

For $T \in \mathcal{B}(X)$, denote by L_T and R_T the left multiplication operator defined by $L_T(S) = TS$, for $S \in \mathcal{B}(X)$ and the right multiplication defined by $R_T(S) = ST$, for $S \in \mathcal{B}(X)$, respectively. Recall [11] that $(J, \|\cdot\|_J)$ is said to be a Banach ideal of $\mathcal{B}(X)$ if :

- (i) $J \subset \mathcal{B}(X)$ is a linear subspace;
- (ii) The norm $\|\cdot\|_J$ is complete in J and $\|S\| \leq \|S\|_J$ for all $S \in J$;
- (iii) $\forall S \in J, \forall A, B \in \mathcal{B}(X), ASB \in J$ and $\|ASB\|_J \leq \|A\| \|S\|_J \|B\|$;
- (iv) The one-rank operators $x \otimes x^* \in J$ and $\|x \otimes x^*\|_J = \|x\| \|x^*\|$ for all $x \in X$ and $x^* \in X^*$.

A one-rank operator defined on X as $(x \otimes x^*)(z) = \langle z, x^* \rangle x = x^*(z)x$ for all $x \in X, x^* \in X^*$ and any $z \in X$. The space of finite rank operators $\mathcal{F}(X)$ is defined as the linear span of the one-rank operators, that is $\mathcal{F}(X) = \left\{ \sum_{i=1}^m x_i \otimes x_i^*, x_i \in X, x_i^* \in X^*, m \geq 1 \right\}$. A Banach ideal $(J, \|\cdot\|_J)$ of $\mathcal{B}(X)$ is

said to be admissible if $\mathcal{F}(X)$ is dense in J with respect to the norm $\|\cdot\|_J$. Let $T \in \mathcal{B}(X)$, if $(J, \|\cdot\|_J)$ is an admissible Banach ideal of $\mathcal{B}(X)$, we denote by $L_{J,T}$ and $R_{J,T}$ the left multiplication operator defined by $L_{J,T}(S) = TS$, for $S \in J$ and the right multiplication defined by $R_{J,T}(S) = ST$, for $S \in J$, respectively. Similarly, we denote by $L_{\mathcal{K},T}$ and $R_{\mathcal{K},T}$ the left and the right multiplication operators induced by $T \in \mathcal{B}(X)$ on $\mathcal{K}(X)$ endowed with the norm operator topology, respectively.

Let Y and Z be two normed linear spaces. The projective tensor norm on $Y \otimes Z$ is the function $\Pi : Y \otimes Z \longrightarrow [0, +\infty[$ defined for all $z \in Y \otimes Z$ by :

$$\Pi(z) := \inf \left\{ \sum_{j=1}^n \|x_j\| \|y_j\| : z = \sum_{j=1}^n x_j \otimes y_j \right\}.$$

For $z = x \otimes y$, we have $\Pi(z) = \|x\| \|y\|$, with this topology the space is denoted by $Y \otimes_{\pi} Z$ and its completion by $Y \widehat{\otimes}_{\pi} Z$. For more general information about the projective tensor norm and its related properties we refer to [6,14,20].

In the setting of Banach ideals, J. Bonet et al in [4], use tensor product techniques to characterize the hypercyclicity of the left and the right multiplication operators on $(J, \|\cdot\|_J)$. Subsequently, Yousefi et al in [22], characterized the supercyclicity of the left multiplication using Hilbert Schmidt operators. In [13], Gupta et al gave a sufficient criterion for the map $C_{A,B}(S) = ASB$ to be supercyclic on certain algebras of operators on Banach space. Recently, Gilmore et al in [9,10], investigate the study of the hypercyclicity properties of the commutator maps $L_T - R_T$ and the generalized derivations $L_A - R_B$ on $(J, \|\cdot\|_J)$. In the present work, we will characterize the supercyclicity of the left and the right multiplication on $(J, \|\cdot\|_J)$, also we give a sufficient conditions for the tensor product $T \widehat{\otimes} R$ of two operators to be supercyclic. As a consequence, we give some equivalent conditions for the supercyclicity criterion.

In section 2, we study the passage of the property of being supercyclic from an operator $T \in \mathcal{B}(X)$ to $L_{J,T}$ and $R_{J,T}$. So, we prove that :

- (i) T satisfies the supercyclicity criterion on X if and only if $L_{J,T}$ is supercyclic.
- (ii) T^* satisfies the supercyclicity criterion on X^* if and only if $R_{J,T}$ is supercyclic.

In section 3, we give a sufficient conditions for the tensor product $T \widehat{\otimes} R$ of two operators to be supercyclic. As a consequence, we prove that if Y and Z are two separable Banach spaces such that $\dim Z > 1$ then $T \in \mathcal{B}(Y)$ satisfies the supercyclicity criterion if and only if $T \widehat{\otimes} I : Y \widehat{\otimes}_{\pi} Z \longrightarrow Y \widehat{\otimes}_{\pi} Z$ is supercyclic for the projective tensor norm π .

2. Supercyclicity of the left and the right multiplication on Banach ideal of operators

We begin this section with the following lemma which will be used in the sequel.

Lemma 2.1. *Let $(J, \|\cdot\|_J)$ be an admissible Banach ideal of $\mathcal{B}(X)$. If D and Φ are a countable dense subsets of X and X^* , respectively. Then the set*

$$\mathcal{X} := \text{span}\{x \otimes \phi/x \in D, \phi \in \Phi\}$$

is a dense subset of J with respect to $\|\cdot\|_J$ -topology.

Proof. Let $T \in J$. If $\epsilon > 0$ is arbitrary, then there is a finite rank operator F such that $\|T - F\|_J < \frac{\epsilon}{2}$. Let $F = \sum_{i=1}^N \alpha_i a_i \otimes \varphi_i$, where $a_i \in X$, $\varphi_i \in X^*$ and $\alpha_i \in \mathbb{C}$ for $i = 1, 2, \dots, N$. For every $i \in \{1, 2, \dots, N\}$, there exist some $\phi_i \in \Phi$ such that $\|\varphi_i - \phi_i\| < \frac{\epsilon}{4N|\alpha_i|\|a_i\|}$ and there exist $x_i \in D$ such that $\|a_i - x_i\| < \frac{\epsilon}{4N|\alpha_i|\|\phi_i\|}$. Therefore

$$\begin{aligned} \|F - \sum_{i=1}^N \alpha_i x_i \otimes \phi_i\|_J &= \left\| \sum_{i=1}^N \alpha_i a_i \otimes \varphi_i - \sum_{i=1}^N \alpha_i x_i \otimes \phi_i \right\|_J \\ &= \left\| \sum_{i=1}^N \alpha_i (a_i \otimes \varphi_i - x_i \otimes \phi_i) \right\|_J \\ &\leq \sum_{i=1}^N |\alpha_i| \|a_i \otimes \varphi_i - x_i \otimes \phi_i\|_J \\ &= \sum_{i=1}^N |\alpha_i| \|a_i \otimes \varphi_i - a_i \otimes \phi_i + a_i \otimes \phi_i - x_i \otimes \phi_i\|_J \\ &= \sum_{i=1}^N |\alpha_i| \|a_i \otimes (\varphi_i - \phi_i) + (a_i - x_i) \otimes \phi_i\|_J \\ &\leq \sum_{i=1}^N |\alpha_i| (\|a_i \otimes (\varphi_i - \phi_i)\|_J + \|(a_i - x_i) \otimes \phi_i\|_J) \\ &= \sum_{i=1}^N |\alpha_i| (\|a_i\| \|\varphi_i - \phi_i\| + \|a_i - x_i\| \|\phi_i\|) \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} \\ &= \frac{\epsilon}{2}. \end{aligned}$$

Hence

$$\begin{aligned} \|T - \sum_{i=1}^N \alpha_i x_i \otimes \phi_i\|_J &= \|T - F + F - \sum_{i=1}^N \alpha_i x_i \otimes \phi_i\|_J \\ &\leq \|T - F\|_J + \|F - \sum_{i=1}^N \alpha_i x_i \otimes \phi_i\|_J \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus \mathcal{X} is a dense subset of J with respect to $\|\cdot\|_J$ -topology. □

In the setting of Banach ideals, J. Bonet et al [4] prove that :

(i) T satisfies the hypercyclicity criterion on X if and only if $L_{J,T}$ is hypercyclic.

(ii) T^* satisfies the hypercyclicity criterion on X^* if and only if $R_{J,T}$ is hypercyclic.

In the following, we prove that this result holds for supercyclicity.

Theorem 2.2. *Let $T \in \mathcal{B}(X)$ and $(J, \|\cdot\|_J)$ an admissible Banach ideal of $\mathcal{B}(X)$. Then T satisfies the supercyclicity criterion on X if and only if $L_{J,T}$ is supercyclic.*

Proof. \Rightarrow) Assume that T satisfies the supercyclicity criterion on X , then there exist a strictly increasing sequence $(n_k)_k$ of positive integers, two dense subsets D_1, D_2 of X and a sequence of maps $S_{n_k} : D_2 \rightarrow X$ such that for all $x \in D_1$ and $y \in D_2$

$$\text{a) } \|T^{n_k}x\| \|S_{n_k}y\| \rightarrow 0, \text{ as } k \rightarrow +\infty;$$

$$\text{b) } T^{n_k}S_{n_k}y \rightarrow y, \text{ as } k \rightarrow +\infty.$$

Let Φ be a dense subset of X^* and consider the sets $X_0 := \text{span}\{x \otimes \varphi/x \in D_1, \varphi \in \Phi\}$ and $Y_0 := \text{span}\{y \otimes \phi/y \in D_2, \phi \in \Phi\}$ and the maps $Q_{n_k} : Y_0 \rightarrow J$ define by

$$Q_{n_k} \left(\sum_{j=1}^N \beta_j y_j \otimes \phi_j \right) = \sum_{j=1}^N \beta_j (S_{n_k} y_j \otimes \phi_j).$$

By Lemma 2.1, X_0 and Y_0 are subsets of J which are $\|\cdot\|_J$ -dense in J . Let $A = \sum_{i=1}^{N_1} \alpha_i x_i \otimes \varphi_i \in X_0$ and

$$B = \sum_{j=1}^{N_2} \beta_j y_j \otimes \phi_j \in Y_0, \text{ then}$$

$$\begin{aligned} \|(L_{J,T})^{n_k} A\|_J \|Q_{n_k} B\|_J &= \|(L_{J,T})^{n_k} \left(\sum_{i=1}^{N_1} \alpha_i x_i \otimes \varphi_i \right)\|_J \|Q_{n_k} \left(\sum_{j=1}^{N_2} \beta_j y_j \otimes \phi_j \right)\|_J \\ &= \left\| \sum_{i=1}^{N_1} \alpha_i T^{n_k} x_i \otimes \varphi_i \right\|_J \left\| \sum_{j=1}^{N_2} \beta_j (S_{n_k} y_j \otimes \phi_j) \right\|_J \\ &\leq \left(\sum_{i=1}^{N_1} |\alpha_i| \|T^{n_k} x_i \otimes \varphi_i\|_J \right) \left(\sum_{j=1}^{N_2} |\beta_j| \|S_{n_k} y_j \otimes \phi_j\|_J \right) \\ &= \left(\sum_{i=1}^{N_1} |\alpha_i| \|T^{n_k} x_i\| \|\varphi_i\| \right) \left(\sum_{j=1}^{N_2} |\beta_j| \|S_{n_k} y_j\| \|\phi_j\| \right) \\ &= \sum_{i \leq N_1; j \leq N_2} |\alpha_i| |\beta_j| \|T^{n_k} x_i\| \|S_{n_k} y_j\| \|\varphi_i\| \|\phi_j\|. \end{aligned}$$

Using the assumption a), we show that $\|(L_{J,T})^{n_k} A\|_J \|Q_{n_k} B\|_J \rightarrow 0$, as $k \rightarrow +\infty$. In the other hand

we have :

$$\begin{aligned}
 \|L_{J,T}^{n_k} Q_{n_k}(B) - B\|_J &= \|L_{J,T}^{n_k} Q_{n_k}(\sum_{j=1}^{N_2} \beta_j y_j \otimes \phi_j) - \sum_{j=1}^{N_2} \beta_j y_j \otimes \phi_j\|_J \\
 &= \|L_{J,T}^{n_k}(\sum_{j=1}^{N_2} \beta_j S_{n_k} y_j \otimes \phi_j) - \sum_{j=1}^{N_2} \beta_j y_j \otimes \phi_j\|_J \\
 &= \|\sum_{j=1}^{N_2} \beta_j T^{n_k} S_{n_k} y_j \otimes \phi_j - \sum_{j=1}^{N_2} \beta_j y_j \otimes \phi_j\|_J \\
 &= \|\sum_{j=1}^{N_2} \beta_j (T^{n_k} S_{n_k} y_j \otimes \phi_j - y_j \otimes \phi_j)\|_J \\
 &\leq \sum_{j=1}^{N_2} |\beta_j| \| (T^{n_k} S_{n_k} y_j - y_j) \otimes \phi_j \|_J \\
 &= \sum_{j=1}^{N_2} |\beta_j| \| T^{n_k} S_{n_k} y_j - y_j \| \| \phi_j \|_J.
 \end{aligned}$$

Using the assumption *b*), we prove that $\|L_{J,T}^{n_k} Q_{n_k}(B) - B\|_J \rightarrow 0$, as $k \rightarrow +\infty$. Hence $L_{J,T}$ satisfies the supercyclicity criterion. Thus $L_{J,T}$ is supercyclic.

\Leftarrow) Suppose that $L_{J,T}$ is supercyclic. Assume that $x_1, x_2 \in X$ are linearly independent and define

$$\begin{aligned}
 \varphi : J &\longrightarrow X \oplus X \\
 R &\longmapsto Rx_1 \oplus Rx_2,
 \end{aligned}$$

then φ is surjective. Indeed, let $y_1, y_2 \in X$, by using the Hahn-Banach theorem, there exist $x_1^*, x_2^* \in X^*$ such that $x_1^*(x_1) = x_2^*(x_2) = 1$ and $x_1^*(x_2) = x_2^*(x_1) = 0$. Let $R = y_1 \otimes x_1^* + y_2 \otimes x_2^* \in J$, then $\varphi(R) = Rx_1 \oplus Rx_2 = y_1 \oplus y_2$.

For $A \in J$, we have

$$\begin{aligned}
 (\varphi \circ L_{J,T})A &= \varphi(TA) \\
 &= (TA)x_1 \oplus (TA)x_2 \\
 &= T(Ax_1) \oplus T(Ax_2) \\
 &= (T \oplus T)(Ax_1 \oplus Ax_2) \\
 &= (T \oplus T) \circ \varphi(A) \\
 &= ((T \oplus T) \circ \varphi)A.
 \end{aligned}$$

Therefore, $\varphi \circ L_{J,T} = (T \oplus T) \circ \varphi$ on J . Thus $T \oplus T$ is supercyclic on $X \oplus X$ by quasi-similarity. Hence, [2, Lemma 3.1], implies that T satisfies the supercyclicity criterion. \square

It was shown in [17,21] that the supercyclicity of L_T on $\mathcal{B}(X)$ with the strong operator topology provided that T satisfies the supercyclicity criterion. In the following corollary, we give a simple and different proof for supercyclicity of L_T from what has been proven in [17,21].

Corollary 2.3. *If $\mathcal{K}(X)$ is an admissible Banach ideal of $\mathcal{B}(X)$, then for all $T \in \mathcal{B}(X)$, the following statements are equivalent :*

- (i) T satisfies the supercyclicity criterion on X .
- (ii) $L_{\mathcal{K},T}$ is supercyclic.

(iii) L_T is supercyclic on $\mathcal{B}(X)$ in the strong operator topology.

Proof. (i) \Leftrightarrow (ii) Consequence of Theorem 2.2, since $\mathcal{K}(X)$ is an admissible Banach ideal of $\mathcal{B}(X)$.

(i) \Rightarrow (iii) Suppose that T satisfies the supercyclicity criterion on X . Let U and V be two non-empty open subsets of $\mathcal{B}(X)$ in the strong operator topology. Since $\mathcal{K}(X)$ is dense in $\mathcal{B}(X)$ with the strong operator topology [5, Corollary 3], there exist $A_1, A_2 \in \mathcal{K}(X)$ such that $A_1 \in U$ and $A_2 \in V$. Thus we can find $x_1, x_2 \in X \setminus \{0\}$ and $\epsilon_1, \epsilon_2 > 0$ such that

$$\{A \in \mathcal{B}(X) : \|(A - A_1)x_1\| < \epsilon_1\} \subset U$$

and

$$\{A \in \mathcal{B}(X) : \|(A - A_2)x_2\| < \epsilon_2\} \subset V.$$

Let

$$U_i = \{A \in \mathcal{K}(X) : \|A - A_i\| < \frac{\epsilon_i}{\|x_i\|}\}, \quad i = 1, 2.$$

U_i is a non-empty open subset of $\mathcal{K}(X)$ with the norm operator topology. By Theorem 2.2 with $J = \mathcal{K}(X)$, $L_{\mathcal{K}, T}$ is supercyclic, so there is some $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$ such that

$$\alpha(L_{\mathcal{K}, T})^n U_1 \cap U_2 \neq \emptyset.$$

Hence, it follows that $\alpha(L_T)^n U \cap V \neq \emptyset$. Thus, L_T is supercyclic on $\mathcal{B}(X)$ in the strong operator topology.

(iii) \Rightarrow (i) By the same technique as in the proof of Theorem 2.2. \square

Theorem 2.4. *Let $T \in \mathcal{B}(X)$ and $(J, \|\cdot\|_J)$ an admissible Banach ideal of $\mathcal{B}(X)$. Then T^* satisfies the supercyclicity criterion on X^* if and only if $R_{J, T}$ is supercyclic.*

Proof. \Rightarrow Assume that T^* satisfies the supercyclicity criterion on X^* , then there exist a strictly increasing sequence $(n_k)_k$ of positive integers, two dense subsets Φ_1, Φ_2 of X^* and a sequence of maps $M_{n_k} : \Phi_2 \rightarrow X^*$ such that for all $\varphi \in \Phi_1$ and $\phi \in \Phi_2$

a) $\|(T^*)^{n_k} \varphi\| \|M_{n_k} \phi\| \rightarrow 0$, as $k \rightarrow +\infty$,

b) $(T^*)^{n_k} M_{n_k} \phi \rightarrow \phi$, as $k \rightarrow +\infty$.

Let D be a dense subset of X and consider the sets $\Phi_0 := \text{span}\{x \otimes \varphi / x \in D, \varphi \in \Phi_1\}$ and $\Psi_0 := \text{span}\{y \otimes \phi / y \in D, \phi \in \Phi_2\}$ and the maps $N_{n_k} : \Psi_0 \rightarrow J$ define by

$$N_{n_k} \left(\sum_{j=1}^N \beta_j y_j \otimes \phi_j \right) = \sum_{j=1}^N \beta_j y_j \otimes M_{n_k} \phi_j.$$

By Lemma 2.1, Φ_0 and Ψ_0 are subsets of J which are $\|\cdot\|_J$ -dense in J . Let $A = \sum_{i=1}^{N_1} \alpha_i x_i \otimes \varphi_i \in \Phi_0$ and

$B = \sum_{j=1}^{N_2} \beta_j y_j \otimes \phi_j \in \Psi_0$, then

$$\begin{aligned}
 \|(R_{J,T})^{n_k} A\|_J \|N_{n_k} B\|_J &= \|(R_{J,T})^{n_k} \left(\sum_{i=1}^{N_1} \alpha_i x_i \otimes \varphi_i \right)\|_J \|N_{n_k} \left(\sum_{j=1}^{N_2} \beta_j y_j \otimes \phi_j \right)\|_J \\
 &= \left\| \sum_{i=1}^{N_1} \alpha_i x_i \otimes (T^*)^{n_k} \varphi_i \right\|_J \left\| \sum_{j=1}^{N_2} \beta_j y_j \otimes M_{n_k} \phi_j \right\|_J \\
 &\leq \left(\sum_{i=1}^{N_1} |\alpha_i| \|x_i \otimes (T^*)^{n_k} \varphi_i\|_J \right) \left(\sum_{j=1}^{N_2} |\beta_j| \|y_j \otimes M_{n_k} \phi_j\|_J \right) \\
 &= \left(\sum_{i=1}^{N_1} |\alpha_i| \|x_i\| \| (T^*)^{n_k} \varphi_i \| \right) \left(\sum_{j=1}^{N_2} |\beta_j| \|y_j\| \|M_{n_k} \phi_j\| \right) \\
 &= \sum_{i \leq N_1; j \leq N_2} |\alpha_i| |\beta_j| \| (T^*)^{n_k} \varphi_i \| \|M_{n_k} \phi_j\| \|x_i\| \|y_j\|.
 \end{aligned}$$

Using the assumption $a)$, we show that $\|(R_{J,T})^{n_k} A\|_J \|N_{n_k} B\|_J \rightarrow 0$, as $k \rightarrow +\infty$. In the other hand we have

$$\begin{aligned}
 \|(R_{J,T})^{n_k} N_{n_k}(B) - B\|_J &= \|(R_{J,T})^{n_k} N_{n_k} \left(\sum_{j=1}^{N_2} \beta_j y_j \otimes \phi_j \right) - \sum_{j=1}^{N_2} \beta_j y_j \otimes \phi_j\|_J \\
 &= \|(R_{J,T})^{n_k} \left(\sum_{j=1}^{N_2} \beta_j y_j \otimes M_{n_k} \phi_j \right) - \sum_{j=1}^{N_2} \beta_j y_j \otimes \phi_j\|_J \\
 &= \left\| \sum_{j=1}^{N_2} \beta_j y_j \otimes (T^*)^{n_k} M_{n_k} \phi_j - \sum_{j=1}^{N_2} \beta_j y_j \otimes \phi_j \right\|_J \\
 &= \left\| \sum_{j=1}^{N_2} \beta_j (y_j \otimes (T^*)^{n_k} M_{n_k} \phi_j - y_j \otimes \phi_j) \right\|_J \\
 &\leq \sum_{j=1}^{N_2} |\beta_j| \|y_j \otimes ((T^*)^{n_k} M_{n_k} \phi_j - \phi_j)\|_J \\
 &= \sum_{j=1}^{N_2} |\beta_j| \|y_j\| \| (T^*)^{n_k} M_{n_k} \phi_j - \phi_j \|
 \end{aligned}$$

Using the assumption $b)$, we prove that $\|(R_{J,T})^{n_k} N_{n_k}(B) - B\|_J \rightarrow 0$, as $k \rightarrow +\infty$. Hence $R_{J,T}$ satisfies the supercyclicity criterion. Thus $R_{J,T}$ is supercyclic.

\Leftarrow) Suppose that $R_{J,T}$ is supercyclic. Let $x_1^*, x_2^* \in X^*$ are linearly independent and define

$$\begin{aligned}
 \phi &: J \longrightarrow X^* \oplus X^* \\
 R &\longmapsto R^* x_1^* \oplus R^* x_2^*.
 \end{aligned}$$

Then ϕ is surjective, indeed, let $y_1^*, y_2^* \in X^*$, we take $x_1^*, x_2^* \in X^*$ such that $x_i^*(x_j) = \delta_{i,j}$ and set $R = x_1 \otimes y_1^* + x_2 \otimes y_2^*$, then $\phi(R) = (R^* x_1^*, R^* x_2^*) = (x_1^* \circ R, x_2^* \circ R) = (y_1^*, y_2^*)$.

For $A \in J$, we have

$$\begin{aligned}
(\phi \circ R_{J,T})A &= \phi(AT) \\
&= (AT)^*x_1^* \oplus (AT)^*x_2^* \\
&= T^*A^*x_1^* \oplus T^*A^*x_2^* \\
&= (T^* \oplus T^*)(A^*x_1^* \oplus A^*x_2^*) \\
&= ((T^* \oplus T^*) \circ \phi)A.
\end{aligned}$$

Therefore, $\phi \circ R_{J,T} = (T^* \oplus T^*) \circ \phi$ on J . Thus $T^* \oplus T^*$ is supercyclic on $X^* \oplus X^*$. Hence, by [2, Lemma 3.1], T^* satisfies the supercyclicity criterion on X^* . \square

Corollary 2.5. *If $\mathcal{K}(X)$ is an admissible Banach ideal of $\mathcal{B}(X)$, then for all $T \in \mathcal{B}(X)$, the following statements are equivalent :*

(i) T^* satisfies the supercyclicity criterion on X^* .

(ii) $R_{\mathcal{K},T}$ is supercyclic.

(iii) $R_{J,T}$ is supercyclic on $\mathcal{B}(X)$ in the strong operator topology.

Proof. (i) \Leftrightarrow (ii) Consequence of Theorem 2.4, since $\mathcal{K}(X)$ is an admissible Banach ideal of $\mathcal{B}(X)$.

(i) \Rightarrow (iii) Suppose that T^* satisfies the supercyclicity criterion on X^* . Let U and V be two non-empty open subsets of $\mathcal{B}(X)$ in the strong operator topology. Since $\mathcal{K}(X)$ is dense in $\mathcal{B}(X)$ with the strong operator topology [5, Corollary 3], there exist $A_1, A_2 \in \mathcal{K}(X)$ such that $A_1 \in U$ and $A_2 \in V$. Thus we can find $x_1, x_2 \in X \setminus \{0\}$ and $\epsilon_1, \epsilon_2 > 0$ such that

$$\{A \in \mathcal{B}(X) : \|(A - A_1)x_1\| < \epsilon_1\} \subset U$$

and

$$\{A \in \mathcal{B}(X) : \|(A - A_2)x_2\| < \epsilon_2\} \subset V.$$

Let

$$U_i = \{A \in \mathcal{K}(X) : \|A - A_i\| < \frac{\epsilon_i}{\|x_i\|}\}, \quad i = 1, 2.$$

U_i is a non-empty open subset of $\mathcal{K}(X)$ with the norm operator topology. By Theorem 2.4 with $J = \mathcal{K}(X)$, $R_{J,T}$ is supercyclic on $(\mathcal{K}(X), \|\cdot\|)$, so there is some $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$ such that

$$\alpha(R_{\mathcal{K},T})^n U_1 \cap U_2 \neq \emptyset.$$

Hence, it follows that $\alpha(R_{J,T})^n U \cap V \neq \emptyset$. Thus $R_{J,T}$ is supercyclic on $\mathcal{B}(X)$ in the strong operator topology.

(iii) \Rightarrow (i) By the same technique as in the proof of Theorem 2.4. \square

3. Tensor stability of supercyclicity

In [16] the authors gave a sufficient conditions for the tensor product $T \widehat{\otimes} R$ of two operators to be hypercyclic. We extend this results to the supercyclic case, we give a sufficient conditions for the tensor product $T \otimes R$ of two operators to be supercyclic.

Definition 3.1. *Let $T \in \mathcal{B}(X)$. We say that T satisfies the tensor supercyclicity criterion if there exists two dense subsets $D_1, D_2 \subset X$, an increasing sequence $(n_k)_{k \in \mathbb{N}}$ of positive integers, $(\lambda_{n_k})_{k \in \mathbb{N}} \subset \mathbb{C} \setminus \{0\}$ and a sequence of maps $S_{n_k} : D_2 \rightarrow X$ such that :*

(i) $(\lambda_{n_k} T^{n_k} x)_{k \in \mathbb{N}}$ is bounded for all $x \in D_1$;

(ii) $(\frac{1}{\lambda_{n_k}} S_{n_k} y)_{k \in \mathbb{N}}$ is bounded for all $y \in D_2$;

(iii) $(T^{n_k} S_{n_k} y)_{k \in \mathbb{N}} \longrightarrow y$ for all $y \in D_2$.

Example 3.2. 1. A sequences of operators satisfying the supercyclicity criterion satisfy the tensor supercyclicity criterion .

2. The identity map on X satisfies the tensor supercyclicity criterion .

3. Any isometry on a Banach space satisfies the tensor supercyclicity criterion with respect to the sequence of all positive integers.

Theorem 3.3. Let Y and Z be two separable Banach spaces. If $T_1 \in \mathcal{B}(Y)$ satisfies the supercyclicity criterion and $T_2 \in \mathcal{B}(Z)$ satisfies the tensor supercyclicity criterion with respect to the same sequence $(n_k)_k$ of positive integers, then

$$T_1 \widehat{\otimes}_\pi T_2 : Y \widehat{\otimes}_\pi Z \longrightarrow Y \widehat{\otimes}_\pi Z$$

satisfies the supercyclicity criterion.

Proof. Let $D_1, D_2 \subset Y$, $D_3, D_4 \subset Z$ be two dense subspaces of Y and Z respectively, $(\lambda_{n_k}^1)_{k \in \mathbb{N}}$, $(\lambda_{n_k}^2)_{k \in \mathbb{N}} \subset \mathbb{C} \setminus \{0\}$ and $S_{n_k}^1 : D_2 \longrightarrow Y$, $S_{n_k}^2 : D_4 \longrightarrow Z$, for all $k \in \mathbb{N}$, linear maps satisfying the conditions of supercyclicity criterion and tensor supercyclicity criterion for $T_1 \in \mathcal{B}(Y)$ and $T_2 \in \mathcal{B}(Z)$, respectively. We will see that $\mathcal{X}_1 := D_1 \otimes D_3$, $\mathcal{X}_2 := D_2 \otimes D_4$, $(\lambda_{n_k} := \lambda_{n_k}^1 \cdot \lambda_{n_k}^2)_{k \in \mathbb{N}}$ and the maps $S_{n_k} := S_{n_k}^1 \otimes S_{n_k}^2 : \mathcal{X}_2 \longrightarrow Y \otimes Z$ are such that the conditions of the supercyclicity criterion are satisfied for the operator

$$T := T_1 \widehat{\otimes}_\pi T_2 : Y \widehat{\otimes}_\pi Z \longrightarrow Y \widehat{\otimes}_\pi Z.$$

Indeed, for every $x_1 \in D_1$, $x_2 \in D_3$, $y_1 \in D_2$ and $y_2 \in D_4$:

$$\begin{aligned} \lim_{k \rightarrow +\infty} \Pi((\lambda_{n_k} T^{n_k})(x_1 \otimes x_2)) &= \lim_{k \rightarrow +\infty} \Pi(\lambda_{n_k}^1 \cdot \lambda_{n_k}^2 (T_1^{n_k} \otimes T_2^{n_k})(x_1 \otimes x_2)) \\ &= \lim_{k \rightarrow +\infty} \Pi((\lambda_{n_k}^1 T_1^{n_k} \otimes \lambda_{n_k}^2 T_2^{n_k})(x_1 \otimes x_2)) \\ &= \lim_{k \rightarrow +\infty} \|\lambda_{n_k}^1 T_1^{n_k} x_1\| \|\lambda_{n_k}^2 T_2^{n_k} x_2\| \\ &= 0. \end{aligned}$$

Analogously

$$\begin{aligned} \lim_{k \rightarrow +\infty} \Pi\left(\frac{1}{\lambda_{n_k}} S_{n_k}\right)(y_1 \otimes y_2) &= \lim_{k \rightarrow +\infty} \Pi\left(\frac{1}{\lambda_{n_k}^1 \cdot \lambda_{n_k}^2} (S_{n_k}^1 \otimes S_{n_k}^2)(y_1 \otimes y_2)\right) \\ &= \lim_{k \rightarrow +\infty} \Pi\left(\frac{1}{\lambda_{n_k}^1} S_{n_k}^1 \otimes \frac{1}{\lambda_{n_k}^2} S_{n_k}^2\right)(y_1 \otimes y_2) \\ &= \lim_{k \rightarrow +\infty} \left\| \frac{1}{\lambda_{n_k}^1} S_{n_k}^1 y_1 \right\| \left\| \frac{1}{\lambda_{n_k}^2} S_{n_k}^2 y_2 \right\| \\ &= 0. \end{aligned}$$

Finally,

$$\begin{aligned} \lim_{k \rightarrow +\infty} \Pi[(T^{n_k} S_{n_k})(y_1 \otimes y_2) - (y_1 \otimes y_2)] &= \lim_{k \rightarrow +\infty} \Pi[(T_1^{n_k} \otimes T_2^{n_k})(S_{n_k}^1 \otimes S_{n_k}^2)(y_1 \otimes y_2) - (y_1 \otimes T_2^{n_k} S_{n_k}^2 y_2) + \\ &\quad (y_1 \otimes T_2^{n_k} S_{n_k}^2 y_2) - (y_1 \otimes y_2)] \\ &= \lim_{k \rightarrow +\infty} \Pi[(T_1^{n_k} S_{n_k}^1 y_1 - y_1) \otimes T_2^{n_k} S_{n_k}^2 y_2 + y_1 \otimes (T_2^{n_k} S_{n_k}^2 y_2 - y_2)] \\ &\leq \lim_{k \rightarrow +\infty} \{ \Pi[(T_1^{n_k} S_{n_k}^1 y_1 - y_1) \otimes T_2^{n_k} S_{n_k}^2 y_2] + \Pi[y_1 \otimes (T_2^{n_k} S_{n_k}^2 y_2 - y_2)] \} \\ &= \lim_{k \rightarrow +\infty} \{ \|T_1^{n_k} S_{n_k}^1 y_1 - y_1\| \|T_2^{n_k} S_{n_k}^2(y_2)\| + \|y_1\| \|T_2^{n_k} S_{n_k}^2 y_2 - y_2\| \} \\ &= 0, \end{aligned}$$

which completes the proof by taking a linear combinations of elementary tensors. \square

The following Corollary is an immediate consequence of the above Theorem.

Corollary 3.4. *Let $T \in \mathcal{B}(X)$. If T satisfies the supercyclicity criterion, then*

$$T \widehat{\otimes}_{\pi} T : X \widehat{\otimes}_{\pi} X \longrightarrow X \widehat{\otimes}_{\pi} X$$

satisfies the supercyclicity criterion. Accordingly, it is supercyclic.

In the following Corollary, we show the connection between supercyclicity of tensor products and supercyclicity of direct sum, and yields another equivalent formulation in the context of tensor products of the supercyclicity criterion.

Corollary 3.5. *Let Y and Z be two separable Banach spaces with $\dim Z \geq 2$ and $T \in \mathcal{B}(Y)$. The following are equivalent :*

- (i) *T satisfies the supercyclicity criterion.*
- (ii) *$T \widehat{\otimes} I : Y \widehat{\otimes}_{\pi} Z \longrightarrow Y \widehat{\otimes}_{\pi} Z$ is supercyclic for the projective tensor norm π .*
- (iii) *$T \oplus T : Y \oplus Y \longrightarrow Y \oplus Y$ is supercyclic.*

Proof. (i) \Rightarrow (ii) This is a consequence of Theorem 3.3 by taking $T_2 = I$.

(iii) \Rightarrow (i) See [2, Lemma 3.1].

(ii) \Rightarrow (iii) Since $\dim Z \geq 2$, let $x_1^*, x_2^* \in Z^*$ and consider the following commutative diagram

$$\begin{array}{ccc} Y \widehat{\otimes}_{\pi} Z & \xrightarrow{T \widehat{\otimes} I} & Y \widehat{\otimes}_{\pi} Z \\ \varphi \downarrow & & \downarrow \varphi \\ Y \oplus Y & \xrightarrow{T \oplus T} & Y \oplus Y \end{array}$$

where $\varphi(\sum_{i \leq N} e_i \otimes x_i) := (\sum_{i \leq N} \langle x_i, x_1^* \rangle e_i, \sum_{i \leq N} \langle x_i, x_2^* \rangle e_i)$. φ is surjective. Indeed, let $e_1, e_2 \in Y$, we take $x_1, x_2 \in Z$ such that $x_i^*(x_j) = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker symbol, so we have $\varphi(e_1 \otimes x_1 + e_2 \otimes x_2) = (e_1, e_2)$.

Let $u = \sum_{i \leq N} e_i \otimes x_i \in Y \widehat{\otimes}_{\pi} Z$, then

$$\begin{aligned} (T \oplus T \circ \varphi)(u) &= (T \oplus T \circ \varphi)\left(\sum_{i \leq N} e_i \otimes x_i\right) \\ &= (T \oplus T)\left(\sum_{i \leq N} \langle x_i, x_1^* \rangle e_i, \sum_{i \leq N} \langle x_i, x_2^* \rangle e_i\right) \\ &= \left(\sum_{i \leq N} \langle x_i, x_1^* \rangle T e_i, \sum_{i \leq N} \langle x_i, x_2^* \rangle T e_i\right) \\ &= \left(\sum_{i \leq N} (T e_i \otimes x_1^*) x_i, \sum_{i \leq N} (T e_i \otimes x_2^*) x_i\right) \\ &= \left(\sum_{i \leq N} (T \widehat{\otimes} I)(e_i \otimes x_1^*) x_i, \sum_{i \leq N} (T \widehat{\otimes} I)(e_i \otimes x_2^*) x_i\right) \\ &= \varphi \circ (T \widehat{\otimes} I)\left(\sum_{i \leq N} e_i \otimes x_i\right) \\ &= \varphi \circ (T \widehat{\otimes} I)(u). \end{aligned}$$

Thus $T \widehat{\otimes} I$ is quasi-similar to $T \oplus T$, hence $T \oplus T$ is supercyclic on $Y \oplus Y$. \square

Remark 3.6. *All the results in this section are valid for any tensor norm a see [6], since the π -topology is the finest one.*

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