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A New Class of Laguerre-based Frobenius Type Eulerian Numbers and Polynomials

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ABSTRACT: In this article, we introduce a new class of generalized Laguerre-based Frobenius-type Eulerian polynomials and then derive diverse explicit and implicit summation formulae and symmetric identities by using series manipulation techniques. Multifarious summation formulas and identities are given earlier for some well-known polynomials such as Eulerian polynomials and Frobenius type Eulerian polynomials are generalized.

Key Words: Hermite polynomials, Laguerre polynomials, Frobenius-type Eulerian polynomials, Laguerre-based Frobenius-type Eulerian polynomials, summation formulae, symmetric identities.

Contents

1. Introduction

The generalized and multi-variable forms of the special functions of mathematical physics have witnessed a significant evolution during the recent years. In particular, the special polynomials of two variables provided new means of analysis for the solution of large classes of partial differential equations often encountered in physical problems. Also, these polynomials allow the derivation of a number of useful identities in a fairly straight forward way and help in introducing new families of special polynomials. The two-variable forms of the Hermite, Laguerre and truncated exponential polynomials as well as their generalizations are considered by several authors, see for example [1-19].

The generating function for 3-variable Laguerre-Hermite polynomials (3VLHP) $_LH_n(x,y,z)$ [7, 8, 9] are defined by

$$\frac{1}{(1-zt)}\exp\left(\frac{-xt}{1-zt}+\frac{yt^2}{1-zt^2}\right)=\sum_{n=0}^{\infty}{}_LH_n(x,y,z)t^n. \tag{1.1}$$

It is equivalent to

$$\exp(yt+zt^2)C_0(xt)=\sum_{n=0}^{\infty}{}_LH_n(x,y,z)\frac{t^n}{n!}, \tag{1.2}$$

where $C_0(x)$ denotes the 0^{th} order Tricomi function. The n^{th} order Tricomi functions $C_n(x)$ are defined as:

$$C_n(x)=\sum_{n=0}^{\infty}\frac{(-1)^r x^r}{r!(n+r)!},(n\in\mathbb{N}_0) \tag{1.3}$$

with the following generating function

$$\exp\left(t-\frac{x}{t}\right)=\sum_{n=0}^{\infty}C_n(x)t^n, \tag{1.4}$$

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for $t \neq 0$ and for all finite x.

Currently, Dattoli *et al.* [10, p.241] introduced the 3-variable Laguerre-Hermite polynomials (3VLHP) ${}_LH_n(x,y,z)$ which is defined as:

$$ {}_LH_n(x,y,z) = n! \sum_{k=0}^{[n/2]} \frac{z^k L_{n-2k}(x,y)}{k!(n-2k)!}. \tag{1.5}$$

Obviously, we have

$$ {}_LH_n(x,y,-\frac{1}{2}) = {}_LH_n(x,y), $$

$$ {}_LH_n(x,1,-1) = {}_LH_n(x), $$

where ${}_LH_n(x,y)$ denotes the 2-variable Laguerre-Hermite polynomials (2VLHP) [7] and ${}_LH_n(x)$ denotes the Laguerre-Hermite polynomials (LHP) (see [8]), respectively.

For $y=0$ in (1.1), we get the two variable Laguerre polynomials (2-VLP) $Ł_n(x,y)$ [7] are defined by

$$ \frac{1}{(1-zt)} \exp\left(\frac{-xt}{1-zt}\right) = \sum_{n=0}^{\infty} Ł_n(x,z)t^n, (\mid yt \mid < 1), \tag{1.6}$$

which is equivalently [6] given by

$$ \exp(zt)C_0(xt) = \sum_{n=0}^{\infty} Ł_n(x,z)\frac{t^n}{n!}, \tag{1.7}$$

From (1.6) and (1.7), we have

$$ L_n(x,z) = n! \sum_{s=0}^{n} \frac{(-1)^s x^s z^{n-s}}{(s!)^2(n-s)!} = y^n L_n(x/z). \tag{1.8}$$

Thus, we have

$$ Ł_n(x,z) = \frac{(-1)^n x^n}{n!}, Ł_n(0,z) = z^n, Ł_n(x,1) = Ł_n(x), \tag{1.9}$$

where $L_n(x)$ are the ordinary Laguerre polynomials [1].

The Hermite Kampé de Fériet polynomials of 2-variables (2VHKdFP) $H_n(x,y)$ [2, 6] are defined by

$$ H_n(x,y) = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!}, \tag{1.10}$$

and

$$ e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}. \tag{1.11}$$

Note that

$$ H_n(2x,-1) = H_n(x) $$

and

$$ H_n(x,-\frac{1}{2}) = He_n(x), H_n(x,0) = x^n, $$

where $H_n(x)$ and $He_n(x)$ being ordinary Hermite polynomials.

The Frobenius-type Eulerian polynomials $A_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$ are defined by means of the following generating function as follows (see [11, 21, 29]):

$$\left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} A_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}, \tag{1.12}$$

where λ is a complex number with $\lambda \neq 1$. The number A_n is given by

$$A_n^{(\alpha)}(\lambda) = A_n^{(\alpha)}(0;\lambda)$$

are called the Frobenius-type Eulerian numbers (see [5, 6]). Clearly, we have

$$A_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array}\right) A_m^{(\alpha)}(\lambda)x^{n-m}. \tag{1.13}$$

The classical Eulerian polynomials $A_n(\lambda)$ given by

$$A_n(\lambda) = A_n^{(1)}(0;\lambda),$$

are defined by the following generating function:

$$\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda} = \sum_{n=0}^{\infty} A_n(\lambda)\frac{t^n}{n!}, \tag{1.14}$$

and can be computed inductively as follows:

$$A_0(\lambda) = 1$$

and

$$A_n(\lambda) = \sum_{m=0}^{n-1} \left(\begin{array}{c} n \\ m \end{array}\right) A_m(\lambda)(\lambda-1)^{n-m-1}, n \geq 1.$$

These numbers play an important role in combinatorial and number theory. Many authors investigated the Frobenius-type Eulerian polynomials (see [3, 4, 5, 11, 21, 22, 29]). An application to the normal ordering of expressions involving bosonic creation and annihilation operators in [5].

The Stirling numbers of the second kind is defined by means of the following generating function (see [11, 30]):

$$(e^t-1)^n = n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!}, \tag{1.17}$$

and the Stirling numbers of the first kind is given by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(l,n)x^l, \text{(see [1-30])}. \tag{1.18}$$

The remainder of the paper is organized as follows. We give a brief review of Laguerre-based Frobenius-type Eulerian polynomials ${}_LA_n^{(\alpha)}(x,y,z;\lambda)$ and their properties. Some implicit summation formulae and general symmetry identities are derived by using different analytical means and applying generating functions.

2. Laguerre-based Frobenius-type Eulerian polynomials ${}_LA_n^{(\alpha)}(x,y,z;\lambda)$

In this section, we define Laguerre-based Frobenius-type Eulerian polynomials (LbFtEp) ${}_LA_n^{(\alpha)}(x,y,z;\lambda)$ and explicit formula for the Frobenius-type Eulerian polynomials and investigate its properties. Now we start at the following definition.

Definition 2.1. For $\lambda \in \mathbb{C}$, the generalized Laguerre-based Frobenius-type Eulerian polynomials ${}_LA_n^{(\alpha)}(x,y,z;\lambda)$ of order α are defined by means of the following generating function:

$$\left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^{\alpha} C_0(xt)\exp(yt+zt^2) = \sum_{n=0}^{\infty} {}_LA_n^{(\alpha)}(x,y,z;\lambda)\frac{t^n}{n!}. \tag{2.1}$$

When $x=y=z=0$ in (2.1), ${}_LA_n^{(\alpha)}(0,0,0;\lambda) = A_n^{(\alpha)}(\lambda)$ are called the n^{th} Frobenius-type Eulerian numbers of order α.

For $x=y=0$ in (2.1), we get

$${}_LA_n^{(\alpha)}(0,0,z;\lambda) = \sum_{m=0}^{[\frac{n}{2}]} A_{n-2m}^{(\alpha)}(\lambda) z^m \frac{n!}{(n-2m)!}.$$

Remark 2.1. On setting $x=0$ in (2.1), the result reduces to the known result of Khan *et al.* [11], we have

$$\left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^{\alpha} e^{yt+zt^2} = \sum_{n=0}^{\infty} {}_HA_n^{(\alpha)}(y,z;\lambda)\frac{t^n}{n!}. \tag{2.2}$$

Theorem 2.1. The following series representation for the Laguerre-based Frobenius-type Eulerian polynomials ${}_LA_n^{(\alpha)}(x,y,z;\lambda)$ of order α holds true:

$$(\lambda-1)^{-n} {}_LA_n^{(\alpha)}\left((\lambda-1)x, (\lambda-1)y, (\lambda-1)^2 z;\lambda\right) = {}_LE_n^{(\alpha)}(x,y,z;\lambda). \tag{2.3}$$

Proof. Consider the generating function (2.1), we have

$$\sum_{n=0}^{\infty} \frac{{}_LA_n^{(\alpha)}\left((\lambda-1)x, (\lambda-1)y, (\lambda-1)^2 z;\lambda\right)}{n!} \frac{t^n}{(\lambda-1)^n} = \left(\frac{1-\lambda}{e^t-\lambda}\right)^{\alpha} C_0(xt)\exp(yt+zt^2)$$

$$= \sum_{n=0}^{\infty} {}_LE_n^{(\alpha)}(x,y,z;\lambda)\frac{t^n}{n!},$$

where ${}_LE_n^{(\alpha)}(x,y,z;\lambda)$ is called the Laguerre-based Frobenius polynomials and comparing the coefficients of t^n, we arrive at the required result (2.3).

Theorem 2.2. The following series representation for the Laguerre-based Frobenius-type Eulerian polynomials ${}_LA_n^{(\alpha)}(x,y,z;\lambda)$ of order α holds true:

$${}_LA_n^{(\alpha)}(x,y,z;\lambda) = \sum_{m=0}^{n} \binom{n}{m} A_{n-m}^{(\alpha)}(\lambda) {}_LH_m(x,y,z). \tag{2.4}$$

Proof. Using equation (1.1) and (1.12) in the l.h.s. of equation (2.1) and then applying the Cauchy product rule and equating the coefficients of same powers of t in both sides of resultant equation, we get representation (2.4).

Theorem 2.3. The following summation formula for the Laguerre-based Frobenius-type Eulerian polynomials ${}_LA_n^{(\alpha)}(x,y,z;\lambda)$ of order α holds true:

$$ {}_LA_n^{(\alpha+\beta)}(x,y+w,z+u;\lambda)=\sum_{m=0}^{n}\binom{n}{m}{}_HA_m^{(\beta)}(u,w;\lambda){}_LA_{n-m}^{(\alpha)}(x,y,z;\lambda). \tag{2.5}$$

$$ {}_LA_n^{(\alpha)}(x,y+u,z;\lambda)=\sum_{m=0}^{n}\binom{n}{m}{}_LA_{n-m}^{(\alpha)}(x,y;\lambda)H_m(u,z). \tag{2.6}$$

Proof. Applying Definition (2.1), we have

$$\sum_{n=0}^{\infty}{}_LA_n^{(\alpha+\beta)}(x,x+y,z+w;\lambda)\frac{t^n}{n!}=\left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^{\alpha+\beta}C_0(xt)\exp\left((y+w)t+(z+u)t^2\right)$$

$$=\sum_{n=0}^{\infty}{}_LA_n^{(\alpha)}(x,y,z;\lambda)\frac{t^n}{n!}\sum_{m=0}^{\infty}{}_HA_m^{(\beta)}(u,w;\lambda)\frac{t^m}{m!}.$$

Replacing n by $n-m$ in above equation, we get

$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}{}_HA_m^{(\beta)}(u,w;\lambda){}_LA_{n-m}^{(\alpha)}(x,y,z;\lambda)\right)\frac{t^n}{n!}.$$

Now equating the coefficients of the like powers of t in the above equation, we get the result (2.5). Again by Definition (2.1) of generalized Laguerre based Frobenius type Eulerian polynomials, we have

$$\left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^{\alpha}C_0(xt)\exp\left((y+u)t+zt^2\right)=\sum_{n=0}^{\infty}{}_LA_n^{(\alpha)}(x,y+u,z;\lambda)\frac{t^n}{n!}, \tag{2.7}$$

which can be written as

$$\left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^{\alpha}C_0(xt)e^{yt}e^{ut+zt^2}=\sum_{n=0}^{\infty}{}_LA_n^{(\alpha)}(x,y;\lambda)\frac{t^n}{n!}\sum_{m=0}^{\infty}H_m(u,z)\frac{t^m}{m!}. \tag{2.8}$$

Replacing n by $n-m$ in the r.h.s. of above equation, we have

$$\sum_{n=0}^{\infty}{}_LA_n^{(\alpha)}(x,y+u,z;\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}{}_LA_{n-m}^{(\alpha)}(x,y;\lambda)H_m(u,z)\frac{t^n}{n!}.$$

Equating their coefficients of t^n leads to formula (2.6).

Theorem 2.4. The following recursive formulas for the Laguerre based Frobenius-type Eulerian polynomials of order α holds true:

$$\frac{\partial_LA_n^{(\alpha)}(x,y,z;\lambda)}{\partial y}=n_LA_{n-1}^{(\alpha)}(x,y;\lambda)\ and\ \frac{\partial_LA_n^{(\alpha)}(x,y;\lambda)}{\partial z}=n(n-1)_LA_{n-2}^{(\alpha)}(x,y;\lambda). \tag{2.9}$$

Proof. The proof follows from (2.3). So we omit them.

Theorem 2.5. The following relation for the Laguerre-based Frobenius-type Eulerian polynomials ${}_LA_n^{(\alpha)}(x,y;;\lambda)$ of order α holds true:

$$(2\lambda-1)\sum_{k=0}^{n}\binom{n}{k}{}_LA_k(x;\lambda){}_HA_{n-k}(y,z;1-\lambda)$$

$$= \lambda {}_LA_n(x,y,z;\lambda) - (1-\lambda){}_LA_n(x,y,z;1-\lambda). \tag{2.10}$$

Proof. We set

$$\frac{(2\lambda-1)}{(e^{t(\lambda-1)}-\lambda)(e^{t(\lambda-1)}-(1-\lambda))} = \frac{1}{e^{t(\lambda-1)}-\lambda} - \frac{1}{e^{t(\lambda-1)}-(1-\lambda)}.$$

From the above equation, we see that

$$(2\lambda-1)\frac{(1-\lambda)C_0(xt)(1-(1-\lambda))e^{yt+zt^2}}{(e^{t(\lambda-1)}-\lambda)(e^{t(\lambda-1)}-(1-\lambda))}$$

$$= \frac{(1-\lambda)e^{yt+zt^2}\lambda C_0(xt)}{e^{t(\lambda-1)}-\lambda} - \frac{(1-\lambda)e^{yt+zt^2}C_0(xt)(1-(1-\lambda))}{e^{t(\lambda-1)}-(1-\lambda)},$$

which on using equations (2.1) and (2.2) in both sides, we have

$$(2\lambda-1)\left(\sum_{k=0}^{\infty}{}_LA_k(x;\lambda)\frac{t^k}{k!}\right)\left(\sum_{n=0}^{\infty}{}_HA_n(y,z;1-\lambda)\frac{t^n}{n!}\right)$$

$$= \lambda\sum_{n=0}^{\infty}{}_LA_n(x,y,z;\lambda)\frac{t^n}{n!} - (1-\lambda)\sum_{n=0}^{\infty}{}_LA_n(x,y,z;1-\lambda)\frac{t^n}{n!}.$$

Applying the Cauchy product rule in the above equation and then equating the coefficients of like powers of t in both sides of the resultant equation, assertion (2.10) follows.

Theorem 2.6. The following relation for the Laguerre-based Frobenius-type Eulerian polynomials ${}_HA_n^{(\alpha)}(x,y;\lambda)$ of order α holds true:

$$\lambda {}_LA_n(x,y,z;\lambda) = \sum_{k=0}^{n}\binom{n}{k}{}_LA_{n-k}(x,y,z;\lambda)(1-\lambda)^k - (1-\lambda){}_LH_n(x,y,z). \tag{2.11}$$

Proof. Consider the following identity

$$\frac{\lambda}{(e^{t(\lambda-1)}-\lambda)e^{t(\lambda-1)}} = \frac{1}{(e^{t(\lambda-1)}-\lambda)} - \frac{1}{e^{t(\lambda-1)}}.$$

Evaluating the following fraction using above identity, we find

$$\frac{\lambda(1-\lambda)C_0(xt)e^{yt+zt^2}}{(e^{t(\lambda-1)}-\lambda)e^{t(\lambda-1)}} = \frac{(1-\lambda)C_0(xt)e^{yt+zt^2}}{(e^{t(\lambda-1)}-\lambda)} - \frac{(1-\lambda)C_0(xt)e^{yt+zt^2}}{e^{t(\lambda-1)}}$$

$$\lambda\sum_{n=0}^{\infty}{}_LA_n(x,y,z;\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty}{}_LA_n(x,y,z;\lambda)\frac{t^n}{n!}\sum_{k=0}^{\infty}(1-\lambda)^k\frac{t^k}{k!} - (1-\lambda)\sum_{n=0}^{\infty}{}_LH_n(x,y,z)\frac{t^n}{n!}.$$

Applying the Cauchy product rule in the above equation and then equating the coefficients of like powers of t in both sides of the resultant equation, assertion (2.11) follows.

Theorem 2.7. The following relation for the Laguerre-based Frobenius-type Eulerian polynomials ${}_LA_n^{(\alpha)}(x,y,z;\lambda)$ of order α holds true:

$${}_LA_n^{(\alpha)}(x,y,z;\lambda)$$

$$= \frac{1}{1-\lambda}\sum_{k=0}^{n}\binom{n}{k}\left[A_{n-k}(\lambda){}_LA_k^{(\alpha)}(x,(1-\lambda)y,z;\lambda) - \lambda A_{n-k}(\lambda){}_LA_k^{(\alpha)}(x,y,z;\lambda)\right]. \tag{2.12}$$

Proof. Consider generating function (2.1), we have

$$\sum_{n=0}^{\infty} {}_LA_n^{(\alpha)}(x,y,z;\lambda)\frac{t^n}{n!} = \left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)\left(\frac{e^{t(\lambda-1)}-\lambda}{1-\lambda}\right)\left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^{\alpha} C_0(xt)e^{yt+zt^2}$$

$$= \frac{1}{1-\lambda}\left[\left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)e^{(\lambda-1)t}\left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^{\alpha} C_0(xt)e^{yt+zt^2} -\lambda\left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)\left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^{\alpha} C_0(xt)e^{yt+zt^2}\right]$$

$$= \frac{1}{1-\lambda}\left[\sum_{n=0}^{\infty} A_n(\lambda)\frac{t^n}{n!}\sum_{k=0}^{\infty} {}_LA_k^{(\alpha)}(x,(\lambda-1)y,z;\lambda)\frac{t^k}{k!} - \lambda\sum_{n=0}^{\infty} A_n(\lambda)\frac{t^n}{n!}\sum_{k=0}^{\infty} {}_LA_k^{(\alpha)}(x,y,z;\lambda)\frac{t^k}{k!}\right].$$

Applying the Cauchy product rule in the above equation and then equating the coefficients of like powers of t in both sides of the resultant equation, assertion (2.12) follows.

Theorem 2.8. For $n \geq 0$, $p, q \in \mathbb{R}$, the following formula for Laguerre-based Frobenius-type Eulerian polynomials ${}_LA_n^{(\alpha)}(x,y,z;\lambda)$ of order α holds true:

$$ {}_LA_n^{(\alpha)}(x,py,qz;\lambda) = n!\sum_{k=0}^{n}\sum_{j=0}^{[\frac{k}{2}]} {}_LA_{n-k}^{(\alpha)}(x,y,z;\lambda)((p-1)y)^{k-2j}((q-1)z)^j \frac{t^n}{(n-k-2j)!j!k!}. \tag{2.13}$$

Proof. Rewrite the generating function (2.1), we have

$$\sum_{n=0}^{\infty} {}_LA_n^{(\alpha)}(x,py,qz;\lambda)\frac{t^n}{n!} = \left(\frac{1-\lambda}{e^{(\lambda-1)t}-\lambda}\right)^{\alpha} C_0(xt)e^{yt+zt^2}e^{(p-1)yt}c^{(q-1)zt^2}$$

$$= \left(\sum_{n=0}^{\infty} {}_LA_n^{(\alpha)}(x,y,z;\lambda)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}((p-1)y)^k\frac{t^k}{k!}\right)\left(\sum_{j=0}^{\infty}((q-1)z)^j\frac{t^{2j}}{j!}\right)$$

$$= \left(\sum_{n=0}^{\infty} {}_LA_n^{(\alpha)}(x,y,z;\lambda)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}((p-1)y)^k((q-1)z)^j\frac{t^{k+2j}}{n!k!j!}\right).$$

Replacing k by $k-2j$ in above equation, we have

$$L.H.S. = \left(\sum_{n=0}^{\infty} {}_LA_n^{(\alpha)}(x,y,z;\lambda)\frac{t^n}{n!}\right)\left(\sum_{k=2j}^{\infty}((p-1)y)^{k-2j}((q-1)z)^j\frac{t^k}{(k-2j)!j!}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{k=2j}^{\infty} {}_LA_n^{(\alpha)}(x,y,z;\lambda)((p-1)y)^{k-2j}((q-1)z)^j\frac{t^{n+k}}{(k-2j)!j!n!}.$$

Again replacing n by $n-k$ in the above equation, we have

$$L.H.S. = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{[\frac{k}{2}]} {}_LA_{n-k}^{(\alpha)}(x,y,z;\lambda)((p-1)y)^{k-2j}((q-1)z)^j\frac{t^n}{(n-k-2j)!j!k!}.$$

Finally, equating the coefficients of t^n on both sides, we acquire the result (2.13).

Theorem 2.9. For $n \geq 0$, $p, q \in \mathbb{R}$ and $x, y \in \mathbb{C}$, then we have

$$ {}_LA_n^{(\alpha)}(x, py, qz; \lambda) $$
$$ = \sum_{k=0}^{n} \binom{n}{k} {}_LA_{n-k}^{(\alpha)}(x, y, z; \lambda) H_k((p-1)x, (q-1)y). \tag{2.14} $$

3. Summation formulae for Laguerre-based Frobenius-type Eulerian polynomials

In this section, we provide implicit formulae, Stirling numbers of the second kind and some relationships for Laguerre-based Frobenius-type Eulerian polynomials of order α related to Apostol type Bernoulli polynomials, Apostol type Euler polynomials and Apostol type Genocchi polynomials. We now begin with the following theorem.

Theorem 3.1. The following summation formulae for Laguerre based Frobenius-type Eulerian polynomials ${}_LA_n^{(\alpha)}(x, y, z; \lambda)$ of order α holds true:

$$ {}_LA_{k+l}^{(\alpha)}(x, u, z; \lambda) $$
$$ = \sum_{n,m=0}^{k,l} \binom{l}{m} \binom{k}{n} (u-y)^{n+m} {}_LA_{k+l-n-m}^{(\alpha)}(x, y, z; \lambda). \tag{3.1} $$

Proof. We replace t by $t + w$ and rewrite the generating function (2.1) as

$$ \left(\frac{1-\lambda}{e^{(\lambda-1)(t+w)} - \lambda} \right)^{\alpha} C_0(x(t+w)) e^{z(t+w)^2} $$
$$ = e^{-y(t+w)} \sum_{k,l=0}^{\infty} {}_LA_{k+l}^{(\alpha)}(x, y, z; \lambda) \frac{t^k}{k!} \frac{w^l}{l!}. \tag{3.2} $$

Replacing y by u in the above equation and equating the resulting equation to the above equation, we get

$$ e^{(u-y)(t+w)} \sum_{k,l=0}^{\infty} {}_LA_{k+l}^{(\alpha)}(x, y, z; \lambda) \frac{t^k}{k!} \frac{w^l}{l!} $$
$$ = \sum_{k,l=0}^{\infty} {}_LA_{k+l}^{(\alpha)}(x, u, z; \lambda) \frac{t^k}{k!} \frac{w^l}{l!}. \tag{3.3} $$

On expanding exponential function (3.3) gives

$$ \sum_{N=0}^{\infty} \frac{[(u-y)(t+w)]^N}{N!} \sum_{k,l=0}^{\infty} {}_LA_{k+l}^{(\alpha)}(x, y, z; \lambda) \frac{t^k}{k!} \frac{w^l}{l!} $$
$$ = \sum_{k,l=0}^{\infty} {}_LA_{k+l}^{(\alpha)}(x, u, z; \lambda) \frac{t^k}{k!} \frac{w^l}{l!}, \tag{3.4} $$

which on using formula [28,p.52(2)]

$$ \sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m) \frac{x^n}{n!} \frac{y^m}{m!}, \tag{3.5} $$

in the left hand side becomes

$$ \sum_{n,m=0}^{\infty} \frac{(u-y)^{n+m} t^n w^m}{n!m!} \sum_{k,l=0}^{\infty} {}_LA_{k+l}^{(\alpha)}(x, y, z; \lambda) \frac{t^k}{k!} \frac{w^l}{l!} $$

$$= \sum_{k,l=0}^{\infty} {}_LA_{k+l}^{(\alpha)}(x,u,z;\lambda)\frac{t^k}{k!}\frac{w^l}{l!}. \tag{3.6}$$

Now replacing k by $k-n$, and l by $l-m$ in the left hand side of (3.6), we get

$$\sum_{k,l=0}^{\infty}\sum_{n,m=0}^{k,l}\frac{(u-y)^{n+m}}{n!m!}{}_LA_{k+l-n-m}^{(\alpha)}(x,y,z;\lambda)\frac{t^k}{(k-n)!}\frac{w^l}{(l-m)!}$$

$$= \sum_{k,l=0}^{\infty} {}_LA_{k+l}^{(\alpha)}(x,u,z;\lambda)\frac{t^k}{k!}\frac{w^l}{l!}. \tag{3.7}$$

Finally on equating the coefficients of the like powers of t and w in the above equation, we get the required result.

Remark 3.1. By taking $l=0$ in Eq. (3.1), we immediately deduce the following result.

Corollary 3.1. The following summation formula for Laguerre based Frobenius-type Eulerian polynomials ${}_LA_n^{(\alpha)}(x,y,z;\lambda)$ of order α holds true:

$${}_LA_k^{(\alpha)}(x,u,z;\lambda) = \sum_{n=0}^{k}\binom{k}{n}(u-y)^n\,{}_LA_{k-n}^{(\alpha)}(x,y,z;\lambda). \tag{3.8}$$

Remark 3.2. On replacing u by $u+y$ and setting $z=0$ in Theorem (3.1), we get the following result involving Laguerre based Frobenius type Eulerian polynomials ${}_LA_n^{(\alpha)}(x,y,z;\lambda)$ of one variable

$${}_LA_{k+l}^{(\alpha)}(x,u+y;\lambda)$$

$$= \sum_{n,m=0}^{k,l}\binom{l}{m}\binom{k}{n}u^{n+m}\,{}_LA_{k+l-n-m}^{(\alpha)}(x,y;\lambda), \tag{3.9}$$

whereas by setting $u=0$ in Theorem 3.1, we get another result involving Laguerre based Frobenius-type Eulerian polynomials ${}_LA_n^{(\alpha)}(x,y,z;\lambda)$ of one and two variables

$${}_LA_{k+l}^{(\alpha)}(x,y,z;\lambda)$$

$$= \sum_{n,m=0}^{k,l}\binom{l}{m}\binom{k}{n}(-y)^{n+m}\,{}_LA_{k+l-n-m}^{(\alpha)}(x,y,z;\lambda). \tag{3.10}$$

Theorem 3.2. The following summation formulae for Laguerre based Frobenius-type Eulerian polynomials ${}_LA_n^{(\alpha)}(x,y,z;\lambda)$ of order α holds true:

$${}_LA_n^{(\alpha+1)}(x,y,z;\lambda) = \sum_{m=0}^{n}\binom{n}{m}A_{n-m}(\lambda)\,{}_LA_m^{(\alpha)}(x,y,z;\lambda). \tag{3.11}$$

Proof. From (2.1), we have

$$\frac{1-\lambda}{e^{(\lambda-1)t}-\lambda}\left(\frac{1-\lambda}{e^{(\lambda-1)t}-\lambda}\right)^{\alpha}C_0(xt)e^{yt+zt^2} = \frac{1-\lambda}{e^{(\lambda-1)t}-\lambda}\sum_{m=0}^{\infty}{}_LA_m^{(\alpha)}(x,y,z;\lambda)\frac{t^m}{m!}$$

$$\left(\frac{1-\lambda}{e^{(\lambda-1)t}-\lambda}\right)^{\alpha+1}C_0(xt)e^{yt+zt^2} = \frac{1-\lambda}{e^{(\lambda-1)t}-\lambda}\sum_{m=0}^{\infty}{}_LA_m^{(\alpha)}(x,y,z;\lambda)\frac{t^m}{m!}$$

$$= \sum_{n=0}^{\infty} A_n(\lambda)\frac{t^n}{n!}\sum_{m=0}^{\infty} {}_LA_m^{(\alpha)}(x,y,z;\lambda)\frac{t^m}{m!}.$$

Now replacing n by $n-m$ and equating the coefficients of t^n leads to formula (3.11).

Theorem 3.3. The following summation formulae for Laguerre based Frobenius-type Eulerian polynomials ${}_LA_n^{(\alpha)}(x,y,z;\lambda)$ of order α holds true:

$$ {}_LA_n^{(\alpha)}(x,y+1,z;\lambda) = \sum_{k=0}^{n}\binom{n}{k} {}_LA_k^{(\alpha)}(x,y,z;\lambda). \tag{3.12}$$

Proof. Using definition (2.1), we have

$$\sum_{n=0}^{\infty} {}_LA_n^{(\alpha)}(x,y+1,z;\lambda)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_LA_n^{(\alpha)}(x,y,z;\lambda)\frac{t^n}{n!}$$

$$= \left(\frac{1-\lambda}{e^{(\lambda-1)t}-\lambda}\right)^{\alpha} C_0(xt)e^{yt+zt^2}(e^t-1)$$

$$= \left(\sum_{k=0}^{\infty} {}_LA_k^{(\alpha)}(x,y,z;\lambda)\frac{t^k}{k!}\right)\left(\sum_{n=0}^{\infty}\frac{t^n}{n!}\right) - \sum_{n=0}^{\infty} {}_LA_n^{(\alpha)}(x,y,z;\lambda)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} {}_LA_k^{(\alpha)}(x,y,z;\lambda)\frac{t^n}{(n-k)!} - \sum_{n=0}^{\infty} {}_LA_n^{(\alpha)}(x,y,z;\lambda)\frac{t^n}{n!}.$$

Finally, equating the coefficients of the like powers of t^n, we get (3.12).

Theorem 3.4. The following relationship holds true:

$$ {}_LA_n^{(\alpha)}(x,y,z;\lambda) = \sum_{j=0}^{n}\sum_{l=j}^{n}\binom{\alpha+j-1}{j} j!\binom{n}{l} S_2(l,j)(\lambda-1)^{l-j} {}_LH_{n-l}(x,y,z). \tag{3.13}$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty} {}_LA_n^{(\alpha)}(x,y,z;\lambda)\frac{t^n}{n!} = \left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^{\alpha} C_0(xt)e^{yt+zt^2}$$

$$= C_0(xt)e^{yt+zt^2}\left(1+\frac{e^{t(\lambda-1)}-1}{1-\lambda}\right)^{-\alpha}$$

$$= \sum_{j=0}^{\infty}\binom{-\alpha}{j}\left(\frac{e^{t(\lambda-1)}-1}{1-\lambda}\right)^{j}\sum_{n=0}^{\infty} {}_LH_n(x,y,z)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{l=j}^{n}\binom{\alpha+j-1}{j} j!\binom{n}{l} S_2(l,j)(\lambda-1)^{l-j} {}_LH_{n-l}(x,y,z)\right)\frac{t^n}{n!}$$

On comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we get (3.13).

Theorem 3.5. The following relation between the Laguerre based Frobenius-type Eulerian polynomials ${}_LA_n^{(\alpha)}(x,y,z;\lambda)$ and Apostol type Bernoulli polynomials $B_n(x;\lambda)$ holds true:

$$ {}_LA_n^{(\alpha)}(x,y,z;\lambda) = \sum_{k=0}^{n+1}\binom{n+1}{k}\left(\lambda\sum_{r=0}^{k}\binom{k}{r}B_{k-r}(y;\lambda) - B_k(y;\lambda)\right) {}_LA_{n-k+1}^{(\alpha)}(x,0,z;\lambda). \tag{3.14}$$

Proof. Consider generating function (2.1), we have

$$\sum_{n=0}^{\infty} {}_LA_n^{(\alpha)}(x,y,z;\lambda)\frac{t^n}{n!} = \left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^{\alpha} C_0(xt)e^{yt+zt^2}\left(\frac{t}{\lambda e^t-1}\right)\left(\frac{\lambda e^t-1}{t}\right)$$

$$= \frac{1}{t}\left(\lambda\sum_{n=0}^{\infty} {}_LA_n^{(\alpha)}(x,0,z;\lambda)\frac{t^n}{n!}\sum_{k=0}^{\infty}B_k(y;\lambda)\frac{t^k}{k!}\sum_{r=0}^{\infty}\frac{t^r}{r!} - \sum_{n=0}^{\infty} {}_LA_n^{(\alpha)}(x,0,z;\lambda)\frac{t^n}{n!}\sum_{k=0}^{\infty}B_k(y;\lambda)\frac{t^k}{k!}\right).. \quad (3.15)$$

On equating the coefficients of same powers of t after using Cauchy product rule in (3.15), assertion (3.14) follows.

Theorem 3.6. The following relation between the Laguerre based Frobenius-type Eulerian polynomials ${}_LA_n^{(\alpha)}(x,y,z;\lambda)$ and Apostol type Euler polynomials $E_n(x;\lambda)$ holds true:

$${}_LA_n^{(\alpha)}(x,y,z;\lambda) = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\left(\lambda\sum_{r=0}^{k}\binom{k}{r}E_{k-r}(y;\lambda)+E_k(y;\lambda)\right){}_LA_{n-k}^{(\alpha)}(x,0,z;\lambda). \quad (3.16)$$

Proof. Consider generating function (2.1), we have

$$\sum_{n=0}^{\infty} {}_LA_n^{(\alpha)}(x,y,z;\lambda)\frac{t^n}{n!} = \left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^{\alpha} C_0 e^{xt+yt^2}\left(\frac{2}{\lambda e^t+1}\right)\left(\frac{\lambda e^t+1}{2}\right)$$

$$= \frac{1}{2}\left(\lambda\sum_{n=0}^{\infty} {}_LA_n^{(\alpha)}(x,0,z;\lambda)\frac{t^n}{n!}\sum_{k=0}^{\infty}E_k(y;\lambda)\frac{t^k}{k!}\sum_{r=0}^{\infty}\frac{t^r}{r!} + \sum_{n=0}^{\infty} {}_LA_n^{(\alpha)}(x,0,z;\lambda)\frac{t^n}{n!}\sum_{k=0}^{\infty}E_k(x;\lambda)\frac{t^k}{k!}\right). \quad (3.17)$$

On equating the coefficients of same powers of t after using Cauchy product rule in (3.17), assertion (3.16) follows.

Theorem 3.7. The following relation between the Laguerre based Frobenius-type Eulerian polynomials ${}_LA_n^{(\alpha)}(x,y,z;\lambda)$ and Apostol type Genocchi polynomials $G_n(y;\lambda)$ holds true:

$${}_LA_n^{(\alpha)}(x,y,z;\lambda) = \frac{1}{2}\sum_{k=0}^{n+1}\binom{n+1}{k}\left(\lambda\sum_{r=0}^{k}\binom{k}{r}G_{k-r}(y;\lambda)+G_k(y;\lambda)\right){}_LA_{n-k+1}^{(\alpha)}(x,0,z;\lambda). \quad (3.18)$$

Proof. Consider generating function (2.1), we have

$$\sum_{n=0}^{\infty} {}_LA_n^{(\alpha)}(x,y,z;\lambda)\frac{t^n}{n!} = \left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^{\alpha} C_0(xt)e^{yt+zt^2}\left(\frac{2t}{\lambda e^t+1}\right)\left(\frac{\lambda e^t+1}{2t}\right)$$

$$= \frac{1}{2t}\left(\lambda\sum_{n=0}^{\infty} {}_LA_n^{(\alpha)}(x,0,z;\lambda)\frac{t^n}{n!}\sum_{k=0}^{\infty}G_k(y;\lambda)\frac{t^k}{k!}\sum_{r=0}^{\infty}\frac{t^r}{r!} + \sum_{n=0}^{\infty} {}_LA_n^{(\alpha)}(x,0,z;\lambda)\frac{t^n}{n!}\sum_{k=0}^{\infty}G_k(y;\lambda)\frac{t^k}{k!}\right).$$

On equating the coefficients of same powers of t after using Cauchy product rule in above equation, we get (3.18).

4. Identities for Laguerre-based Frobenius-type Eulerian polynomials

In this section, we give general symmetry identities for the Laguerre based Frobenius-type Eulerian polynomials ${}_LA_n^{(\alpha)}(x,y,z;\lambda)$ and generalized Frobenius type Eulerian polynomials $A_n^{(\alpha)}(x;\lambda)$ by applying the generating functions (1.12) and (2.1).

Theorem 4.1. Let $a, b, > 0$ and $a \neq b$. For $x, y \in \mathbb{R}$ and $n \geq 0$, the following identity holds true:

$$\sum_{k=0}^{n} \binom{n}{k} b^k a^{n-k} {}_L A_{n-k}^{(\alpha)}(bx, by, b^2 z; \lambda) {}_L A_k^{(\alpha)}(ax, ay, a^2 y; \lambda)$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} {}_L A_{n-k}^{(\alpha)}(ax, ay, a^2 z; \lambda) {}_L A_k^{(\alpha)}(bx, by, b^2 z; \lambda). \tag{4.1}$$

Proof. Let

$$A(t) = \left(\frac{(1-\lambda)^2}{(e^{(\lambda-1)at} - \lambda)(e^{(\lambda-1)bt} - \lambda)} \right)^{\alpha} [C_0(abxt)]^2 e^{abyt + a^2 b^2 z t^2}. \tag{4.2}$$

Then the expression for $A(t)$ is symmetric in a and b, we obtain

$$A(t) = \sum_{n=0}^{\infty} {}_L A_n^{(\alpha)}(bx, by, b^2 z; \lambda) \frac{(at)^n}{n!} \sum_{k=0}^{\infty} {}_L A_k^{(\alpha)}(ax, ay, a^2 z; \lambda) \frac{(bt)^k}{k!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} b^k a^{n-k} {}_L A_{n-k}^{(\alpha)}(bx, by, b^2 z; \lambda) {}_L A_k^{(\alpha)}(ax, ay, a^2 z; \lambda) \right) \frac{t^n}{n!}.$$

Similarly, we can show that

$$A(t) = \sum_{n=0}^{\infty} {}_L A_n^{(\alpha)}(ax, ay, a^2 z; \lambda) \frac{(bt)^n}{n!} \sum_{k=0}^{\infty} {}_L A_k^{(\alpha)}(bx, by, b^2 z; \lambda) \frac{(at)^k}{k!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} {}_L A_{n-k}^{(\alpha)}(ax, ay, a^2 z; \lambda) {}_L A_k^{(\alpha)}(bx, by, b^2 z; \lambda) \right) \frac{t^n}{n!}.$$

On comparing the coefficients of t^n on the right hand sides of the last two equations, we arrive at the desired result (4.1).

Remark 4.1. For $\alpha = 1$ in Theorem 4.1, the result reduces to

$$\sum_{k=0}^{n} \binom{n}{k} b^k a^{n-k} {}_L A_{n-k}(bx, by, b^2 z; \lambda) {}_L A_k(ax, ay, a^2 z; \lambda)$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} {}_L A_{n-k}(ax, ay, a^2 z; \lambda) {}_L A_k(bx, by, b^2 z; \lambda). \tag{4.3}$$

Theorem 4.2. Let $a, b, > 0$ and $a \neq b$. For $x, y \in \mathbb{R}$ and $n \geq 0$, the following identity holds true:

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-\lambda)^{i+j} a^{n-k} b^k {}_L A_{n-k}^{(\alpha)} \left(bx, by + \frac{b}{a} i + j, b^2 z; \lambda \right) A_k^{(\alpha)}(au; \lambda)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (-\lambda)^{i+j} b^{n-k} a^k {}_L A_{n-k}^{(\alpha)} \left(ax, ay + \frac{a}{b} i + j, a^2 z; \lambda \right) A_k^{(\alpha)}(bu; \lambda). \tag{4.4}$$

Proof. Consider the identity

$$B(t) = \left(\frac{(1-\lambda)^2}{(e^{(\lambda-1)at} - \lambda)(e^{(\lambda-1)bt} - \lambda)} \right)^{\alpha} \frac{1 + \lambda(-1)^{a+1} e^{abt}}{(\lambda e^{at} + 1)(\lambda e^{bt} + 1)} C_0(xt) e^{ab(y+u)t + a^2 b^2 z t^2}$$

$$B(t) = \left(\frac{1-\lambda}{e^{(\lambda-1)at} - \lambda} \right)^{\alpha} C_0(abxt) e^{abyt + a^2 b^2 z t^2} \left(\frac{1 - \lambda(-e^{-bt})^a}{\lambda e^{bt} + 1} \right) \left(\frac{1-\lambda}{e^{(\lambda-1)bt} - \lambda} \right)^{\alpha}$$

$$\times e^{abut}\left(\frac{1-\lambda(-e^{-at})^b}{\lambda e^{at}+1}\right)$$

$$=\left(\frac{1-\lambda}{e^{(\lambda-1)at}-\lambda}\right)^{\alpha} C_0(abxt)e^{abyt+a^2b^2zt^2}\sum_{i=0}^{a-1}(-\lambda)^i e^{bti}\left(\frac{1-\lambda}{e^{(\lambda-1)bt}-\lambda}\right)^{\alpha} e^{abut}\sum_{j=0}^{b-1}(-\lambda)^j e^{atj}$$

$$=\left(\frac{1-\lambda}{e^{(\lambda-1)at}-\lambda}\right)^{\alpha} C_0(abxt)e^{a^2b^2zt^2}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{i+j}e^{(by+\frac{b}{a}i+j)at}\sum_{k=0}^{\infty}A_k^{(\alpha)}(au;\lambda)\frac{(bt)^k}{k!}$$

$$=\sum_{n=0}^{\infty}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{i+j}{}_LA_n^{(\alpha)}\left(bx,by+\frac{b}{a}i+j,b^2z;\lambda\right)\frac{(at)^n}{n!}\sum_{k=0}^{\infty}A_k^{\alpha}(au;\lambda)\frac{(bt)^k}{(k)!}$$

$$B(t)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{i+j}a^{n-k}b^k{}_LA_{n-k}^{(\alpha)}\left(bx,by+\frac{b}{a}i+j,b^2z;\lambda\right)$$

$$\times A_k^{(\alpha)}(au;\lambda)\frac{t^n}{n!}. \tag{4.7}$$

On the other hand, we have

$$B(t)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(-\lambda)^{i+j}b^{n-k}a^k{}_LA_{n-k}^{(\alpha)}\left(ax,ay+\frac{a}{b}i+j,a^2z;\lambda\right)$$

$$\times A_k^{(\alpha)}(bu;\lambda)\frac{t^n}{n!}. \tag{4.8}$$

By comparing the coefficients of t^n on the right hand sides of the last two equations, we arrive at the desired result (4.4).

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References

[1] Andrews, L.C, Special functions for engineers and mathematicians, Macmillan. Co., New York, 1985.

[2] Bell, E. T, Exponential polynomials, Ann. of Math., 35(1934), 258-277.y

[3] Carlitz, L, Eulerian numbers and polynomials, Math. Mag., 32(1959), 247-260.

[4] Carlitz, L, Eulerian numbers and polynomials of higher order, Duke Math. J., 27(1960), 401-423.

[5] Choi, J, Kim, D. S, Kim, T and Kim, Y. H, A note on some identities of Frobenius-Euler numbers and polynomials, Inter. J. Math. Math. Sci., 2012(2012), 1-9.

[6] Dattoli, G, Lorenzutta, S and Cesarano, C, Finite sums and generalized forms of Bernoulli polynomials Rendiconti di Mathematica, 19(1999), 385-391.

[7] Dattoli, G, Torre, A, Operational methods and two variable Laguerre polynomials, Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur., 132(1998) 3-9.

[8] Dattoli, G, Torre, A and Mancho, A.M, The generalized Laguerre polynomials, the associated Bessel functions and applications to propagation problems, Radiat. Phys. Chem., 59(2000), 229-237.

[9] Dattoli, G, Torre, A and Mazzacurati, G, Monomiality and integrals involving Laguerre polynomials, Rend. Mat., (VII) 18(1998), 565-574.

[10] Dattoli, G, Torre, A, Lorenzutta, S and Cesarano, C, Generalized polynomials and operational identities, Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur., 134(2000), 231-249.

[11] Khan, W. A, Nisar, K. S, Acikgoz, M and Duran, U, A new class of Hermite based Frobenius type Eulerian polynomials, Proceedings of the Jangjeon Mathematical Society, 22(4)(2019), 551-563.

[12] Khan, W. A, Some properties of the generalized Apostol type Hermite-Based polynomials, Kyungpook Math. J., 55(2015), 597-614.

[13] Khan, W. A, On generalized Lagarange-based Apostol type and related polynomials, Kragujevac Journal of Mathematics, (2022), 46-22, 865-882.

[14] Khan, W. A., Nisar, K S., Acikgoz, M., Duran, U., Hassan, A., On unified Gould-Hopper based Apostol type polynomials, Journal of Mathematics and Computer Science, 24(4)(2022), 287-298.

[15] Khan, W. A, Khan, I. A, Duran, U, Acikgoz, M, Apostol type (p, q)-Frobenius Eulerian polynomials and numbers, Afrika Matematika, 32(1-2)(2021), 115-130.

[16] Khan, W. A, Srivastava, D, Certain properties of Apostol type Hermite-based Frobenius Genocchi polynomials, Kragujevac Journal of Mathematics, 45(6) (2021), 859-872.

[17] Khan, W. A, Srivastava, D, On the generalized Apostol type-Frobenious-Genocchi polynomials, Filomat Journal, 33(7) (2019), 1967-1977.

[18] Khan, W. A, Pathan, M. A, On generalized Lagrange-Hermite-Bernoulli and related polynomials, Acta et. Commentationes Universitatis Tartuensis de Mathematica, 23(2) (2019), 211-224.

[19] Khan, W. A, Ahmad, M, Partially degenerate poly-Bernoulli polynomials associated with Hermite polynomials, Advance Studies in Contemporary Mathematics, 28(3) (2018), 487-496.

[20] Kang, J. Y, Khan, W. A, A new class of q-Hermite based Apostol type Frobenius Genocchi polynomials, Communication of the Korean Mathematical Society, 35(3) (2020), 759-771.

[21] Kim, D. S, Kim, T, Kim, Y. H, Dolgy, D. V, A note on Eulerian polynomials associated with Bernoulli and Euler numbers and polynomials, Adv. Stud. Contemp. Math., 22(2012), 379-389.

[22] Kim, D. S, Kim, T, Kim, W. J, Dolgy, D. V, A note on Eulerian polynomials, Abtr. Appl. Anal. (2012) Art. ID 269640, 10pp.

[23] Kim, D. S, Kim, T, Some new identities of Frobenius-Euler numbers and polynomials, J. Ineq. Appl., 2012(2012), 1-10.

[24] Kurt, B and Simsek, Y, On the generalized Apostol type Frobenius Euler polynomials, Advances in Differences equations, (2013), 1-9.

[25] Muhiuddin, G, Khan, W. A, Duran, U, Al-Kadi, D, A new class of higher-order hypergeometric Bernoulli polynomials associated with Lagrange-Hermite polynomials. Symmetry, 13 (648) (2021), 1-11. 20.

[26] Pathan, M. A, Khan, W. A, On the three families of extended Laguerre-based Apostol-type polynomials. Proyecciones Journal of Mathematics, 40(2) (2021), 291-312.

[27] Pathan, M. A, Khan, W. A, A new class of generalized polynomials associated with Hermite and poly-Bernoulli polynomials. Miskolc Mathematical Journal, 22(1) (2021), 317-330.

[28] Srivastava, H. M and Manocha, H. L, A treatise on generating functions, Ellis Horwood Limited, New York, 1984.

[29] Srivastva, H. M, Boutiche, M. A, Rahmani, M, A class of Frobenius-type Eulerian polynomials, Rocky Mountain J. Math., 48(2018), 1003-1013.

[30] Simsek, Y, Generating functions for generalized Stirlings type numbers array type polynomials, Eulerian type polynomials and their application, Fixed Point Theory and Appl., doi:10.1186/1687-1812-2013-87, 2013 .

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