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On Exponential Stabilization of a Nonlinear Neutral Wave Equation

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abstract: This work aims to study a nonlinear wave equation subject to a delay of neutral type. The nonlinearity and the delay appear in the second time derivative. In spite of the fact that delays by nature, have an instability effect on the structures, the strong damping is sufficient to allow the system to reach its equilibrium state with an exponential manner. The difficulties arising from the nonlinearity have been overcome by using an inequality due to a Sobolev embedding theorem. The main result has been established without any condition on the coefficient of the neutral delay.

Key Words: Nonlinear equation, neutral delay, exponential stability, strong damping.

Contents

1. Introduction

The main purpose of this work is to study the stability of the following problem of a strongly damped neutral wave equation

$$\begin{cases} \left|(u(t)-pu(t-\tau))_t\right|^{\rho}\left[u(t)-pu(t-\tau)\right]_{tt}-\triangle\left[u(t)-pu(t-\tau)\right]-\triangle u_t \;=0 \\ \text{in } (0,\infty)\times\Omega, \\ u(t,x)=f(t,x),\ t\in[-\tau,0],\ x\in\Omega, \\ u(t,x)=f(t,x)=0,\ t\in(0,\infty),\ x\in\partial\Omega, \\ u(0,x)=u_0(x),\ u_t(0,x)=u_1(x), f(0,x)=u_0(x),\ x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded regular domain of $\mathbb{R}^n$. The delay that is acting in the second time derivative is of neutral type. The functions $u_0(x)$ and $u_1(x)$ are given and p,τ are positive constants such that $0<p<1$. The parameter ρ is called the density dependence coefficient, which is constant here. To determine the future evolution of the system, we need to specify the initial state variables $u(t,x)$ over the time interval $[-\tau,0]$, *i.e.*, $u(t,x)=f(t,x)$, $t\in[-\tau,0]$, $x\in\Omega$, where f is a given function that will be specified later.

Stability of time-delay systems became a main subject of study over the last three decades. There has been a remarkable rise in research activities, creating a diversity in term of tools and results. This study is not only for theoretical reasons, but also due to the appearance of such systems in engineering problems and in ecology [13,14].

Neutral Delay Differential Equations (NDDES) form a part of the larger class of FDEs, see for instance [1,4,5,12,15,21,22,24]. From this class, it has been derived some models of the neutral delay wave equation

$$\begin{aligned} \left[u(t)-pu(t-\tau)\right]_{tt} &= \triangle u(t)+g(t,u(t),u(t-\tau)), \\ \left[u(t)-pf(u(t-\tau))\right]_{tt} &= \triangle u(t)+g(t,u(t)), \\ \left[u(t)-pf(u(t-\tau(t)))\right]_{tt} &= g(\triangle u(t))+h(t,\triangle u(\tau(t)),u(\tau(t))), \end{aligned}$$

When a time-delay system is defined, it is the characteristic that the evolution of systems in the future in addition of its current state, depends also on some time phase of its past history. This particular close relationship can be modeled by functional differential equations (FDEs), or, in particular, by differential-difference equations (DDEs). It was shown, throughout many studies, that the delay term, which arises

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in many practical problems, produces some instability effects. This is proved in [6,7] where small delays can upset the stability of a system which was already stable. The presence of delays is usually considered undesirable, which is a source of perturbations, defect and may causes a dysfonctionnement or destruction of systems. It may be produced for different reasons: delayed measurements, intrinsic property of the system, feedback control action, etc. For these reasons, researchers have been derived a stability result namely (exponential) by considering this effect which is not controllable in most practical situations.

Different methods and control strategies have been adopted to deal with this situation. These investigations have created a variety of results. We cite briefly some interesting ones, Xu *et al.* [27], proved that the stability of the system depends on the delay coefficient using the spectral analysis approach. Nicaise *et al.* [20] examined the case where the delay is a time-varying function. Nicaise and Pignotti [18] obtained an exponential stabilization result using Carleman estimates combined with compactness-uniqueness arguments. The case of dynamic boundary conditions was also investigated by Nicaise and Pignotti [19,16] and Gerbi and Said-Houari [11]. Note that Nicaise and Pignotti in [17] extended these results for other systems (elasticity system and the Petrovsky system) by considering a nonlinear abstract second-order evolution equations. Other results have been obtained when the delay is of witching or intermittent type. (see [10,23].)

In the absence of the neutral delay (*i.e.* $p = 0$), we shall start by citing the first investigation due to Ferreira and Pereira [8] concerning the equation

$$K(x,t)u_{tt} - \triangle u - \triangle u_t + F(u) = 0.$$

They studied the existence of global weak solutions where K may vanish. The result was generalised later by Rojas Medar [9] in non-cylindrical domains. The equation in (1.1) is of the general form

$$f(u_t)u_{tt} - \triangle u - \triangle u_t = 0$$

where in our case $f(u_t) = |u_t|^{\rho}$. The case $f(u_t)$ is not constant indicates that the density of the materials used depends on the velocity u_t. If $p \neq 0$ the equation models the phenomenon of vibrating masses attached to an elastic long rod.

For the linear version of the problem (1.1) (*i.e.* $\rho = 0$), an exponential stability result was established by Tatar [25] considering the model

$$\begin{cases} [u(t) - pu(t-\tau)]_{tt} - \triangle u(t) - \triangle u_t(t) = 0 \text{ in } (0,\infty)\times\Omega, \\ u(t,x) = f(t,x),\ t \in [-\tau, 0],\ x \in \Omega, \\ u(t,x) = f(t,x) = 0,\ t \in (0,\infty),\ x \in \partial\Omega, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), f(0,x) = u_0(x),\ x \in \Omega. \end{cases} \tag{1.2}$$

In [26], the same author considered a viscoelastic string subject to a delay of neutral type. The delay occurs in the second time derivative. The author proved an exponential decay result using the multiplier method. This work lead to an appropriate differential inequality which allows to conclude the desired result. Here the delay acts only on the second time derivative. The result was obtained under the condition that $p < p_*$ where p_* is the positive root of

$$16C_p x^2 + 2(7 + 8C_p)x - 3$$

where C_p is the constant of Poincaré. To the best of our knowledge, the question of the stabilization of neutral delay wave equation likes (1.1) is a new subject and does not attract the attention of researchers and scientists until now comparatively to (NDDES). The major difficulties encountered in our case are the presence of nonlinearity and the delay in the second time derivative. These obstacles were surmounted by using an inequality due to a sobolev embedding theorem.

The content of the remaining parts of this paper is ordered as follows. In section 1, we collect some preliminary results (Lemmas, inequalities, etc.) and introduce our assumptions needed in our analysis. In section 2, we state and prove our main result by making use of the multiplier method combined with some arguments derived from an inequality due to Sobolev embedding theorem.

2. Preliminaries

In this section, we introduce our assumptions and lemmas required in the proof of our main result. We need to use the standard Lebesgue space $L^p(\Omega)$ and the Sobolev space $H_0^1(\Omega)$ equipped with their usual products and norms. We denote by $\|.\|_p$the norm of $L^p(\Omega)$. For simplicity we denote by $\|.\|$ the norm of $L^2(\Omega)$.

Lemma 2.1. *(Sobolev-Poincaré inequality) Let $w \in H_0^1(\Omega)$ and let q a real number such that*

$$\begin{cases} 2 \le q \le \frac{2n}{n-2} \text{ if } n \ge 3, \\ q > 0 \text{ if } n = 1, 2, \end{cases}$$

then the following inequality hold

$$\|w\|_q \le C_s \|w\| .$$

Lemma 2.2. *(Holder inequality) Let $(p, q) \in \Re^2$ such that $1 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $v \in L^p(\Omega)$ and $w \in L^q(\Omega)$, then $vw \in L^1(\Omega)$ and*

$$\|vw\|_1 \le \|v\|_p \|w\|_q .$$

Lemma 2.3. *(Minkowski inequality) Let $1 \le p \le \infty$ and let $v, w \in L^p(\Omega)$. Then $v + w \in L^p(\Omega)$ and we have the triangle inequality*

$$\|v + w\|_p \le \|v\|_p + \|w\|_p .$$

Lemma 2.4. *(Young inequality) Let $(a, b) \in \Re^2$, for any $\eta > 0$, it holds*

$$ab \le \eta a^2 + \frac{b^2}{4\eta} .$$

Our assumption on the density dependence coefficient ρ is such that

$$\begin{cases} 0 < \rho \le \frac{4}{n-2} \text{ if } n \ge 3, \\ \rho > 0 \text{ if } n = 1, 2. \end{cases} \tag{2.1}$$

The existence and uniqueness for this type of problems can be established by the Faedo-Galerkin methods, for further informations and the way of the proof we refer the reader to [2,3]. According to those, we state

Theorem 2.5. *Let $(u_0, u_1) \in \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H^1(\Omega)$ and let the assumption (2.1). Then there exists at least one solution u of problem (1.1) such that*

$$u \in L^\infty \left([0, T), H^2(\Omega) \cap H_0^1(\Omega)\right),\ u_t \in L^\infty \left([0, T), L^2(\Omega)\right),\ u_{tt} \in L^2 \left([0, T), L^2(\Omega)\right)$$

for any $T > 0$.

3. Decay of the Solution Energy

In this section, we shall state and prove our main result. We first define the classical energy for problem (1.1) as the continuous function given by

$$E(t) := \frac{1}{\rho + 2} \|u_t(t)\|_{\rho+2}^{\rho+2} + \frac{1}{2} \|\nabla u(t)\|^2 ,\ t \ge 0 \tag{3.1}$$

which should be modified to

$$\mathcal{E}(t) := \frac{1}{\rho + 2} \|u_t(t) - p u_t(t - \tau)\|_{\rho+2}^{\rho+2} + \frac{1}{2} \|\nabla (u(t) - p u(t - \tau))\|^2 ,\ t \ge 0. \tag{3.2}$$

Let us begin by estimating the derivative of $\mathcal{E}(t)$

Lemma 3.1. *Let u be a solution of problem (1.1), then the energy $\mathcal{E}(t)$ satisfies*

$$\frac{d}{dt}\mathcal{E}(t) \leq -(1-\eta_1 p)\left\|\nabla u_t(t)\right\|^2 + \frac{p}{4\eta_1}\left\|\nabla u_t(t-\tau)\right\|^2,\ t \geq 0.$$

Proof. Multiplying the equation in (1.1) by the term $u_t(t) - pu_t(t-\tau)$ and integrating over Ω, we find, for $t \geq 0$

$$\begin{aligned} &\frac{1}{\rho+2}\frac{d}{dt}\left\|u_t(t) - pu_t(t-\tau)\right\|_{\rho+2}^{\rho+2} \\ = \quad & -\frac{1}{2}\frac{d}{dt}\left\|\nabla\left(u(t) - pu(t-\tau)\right)\right\|^2 - \left\|\nabla u_t(t)\right\|^2 + p\int_\Omega \nabla u_t(t-\tau)\nabla u_t(t)dx. \end{aligned}$$

Next, by Young inequality, we see that for $\eta_1 > 0$

$$\begin{aligned} &\frac{1}{\rho+2}\frac{d}{dt}\left\|u_t(t) - pu_t(t-\tau)\right\|_{\rho+2}^{\rho+2} + \frac{1}{2}\frac{d}{dt}\left\|\nabla\left(u(t) - pu(t-\tau)\right)\right\|^2 \\ \leq \quad & -(1-\eta_1 p)\left\|\nabla u_t(t)\right\|^2 + \frac{p}{4\eta_1}\left\|\nabla u_t(t-\tau)\right\|^2,\ t \geq 0. \end{aligned}$$

□

Observe that, the derivative of the energy can not respond to the question concerning the dissipativity of the system. Next, the energy functional should be modified to achieve this aim. For positive constant λ to be specified later, we introduce the functional

$$\mathcal{E}_\lambda(t) = \mathcal{E}(t) + \lambda\Psi(t),\ t \geq 0$$

where

$$\Psi(t) := \int_\Omega e^{-\gamma t}\int_{t-\tau}^{t} e^{\gamma(s+\tau)}|\nabla u_t|^2(s)dsdx,\ t \geq 0.$$

We now proceed to evaluate the derivative of $\mathcal{E}_\lambda(t)$.

Lemma 3.2. *Let u be a solution of problem (1.1), then the functional $\mathcal{E}_\lambda(t)$ satisfies*

$$\frac{d}{dt}\mathcal{E}_\lambda(t) \leq -C_1\left\|\nabla u_t(t)\right\|^2 - C_2\left\|\nabla u_t(t-\tau)\right\|^2 - \lambda\gamma\Psi(t),\ t \geq 0$$

where $C_1 = 1 - \frac{p}{2} - \frac{p}{2}(2-p)e^{\gamma\tau}$ and $C_2 = p(1-p)/2$.

Proof. A direct differentiation of $\Psi(t)$ shows that $\mathcal{E}_\lambda(t)$ satisfies along solutions of problem (1.1) the assertion of Lemma 3.2

$$\frac{d}{dt}\mathcal{E}_\lambda(t) = \frac{d}{dt}\mathcal{E}(t) - \lambda\gamma\Psi(t) + \lambda e^{\gamma\tau}\left\|\nabla u_t(t)\right\|^2 - \lambda\left\|\nabla u_t(t-\tau)\right\|^2,\ t \geq 0.$$

This gives after inserting of the total derivative of $\mathcal{E}(t)$ from Lemma 3.1

$$\frac{d}{dt}\mathcal{E}_\lambda(t) \leq -(1-\eta_1 p - \lambda e^{\gamma\tau})\left\|\nabla u_t(t)\right\|^2 - \left(\lambda - \frac{p}{4\eta_1}\right)\left\|\nabla u_t(t-\tau)\right\|^2 - \lambda\gamma\Psi(t),\ t \geq 0.$$

We request all the coefficients to be negative. For this, we set $\eta_1 = 1/2$ and $\lambda = (2-p)p/2$, we get

$$\frac{d}{dt}\mathcal{E}_\lambda(t) \leq -C_1\left\|\nabla u_t(t)\right\|^2 - C_2\left\|\nabla u_t(t-\tau)\right\|^2 - \lambda\gamma\Psi(t),\ t \geq 0$$

where $C_1 = 1 - \frac{p}{2} - \frac{p}{2}(2-p)e^{\gamma\tau}$ and $C_2 = p(1-p)/2$. Note that C_1 is positive, if we choose γ small such that $pe^{\gamma\tau} < 1$. □

Theorem 3.3. *Let* $(u_0, u_1) \in \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H^1(\Omega)$, $f_t(-\tau) \in L^2(\Omega), \nabla f(-\tau) \in L^2(\Omega)$ *and* $f \in H^1\left([-\tau, 0], H^1(\Omega)\right)$ *under the assumption (2.1) the solution of problem (1.1) satisfies*

$$\mathcal{E}_\lambda(t) \leq \mathcal{E}_\lambda(0) = \mathcal{E}(0) + \lambda \Psi(0) \tag{3.3}$$

where

$$\Psi(0) := \int_\Omega \int_{-\tau}^0 e^{\gamma(s+\tau)} |\nabla f_t|^2(s) ds dx$$

and

$$\mathcal{E}(0) := \frac{1}{\rho+2} \|u_1 - p f_t(-\tau)\|_{\rho+2}^{\rho+2} + \frac{1}{2} \|\nabla u_0 - p \nabla f(-\tau)\|^2.$$

Proof. Notice that $\mathcal{E}_\lambda(0)$ is well-defined as $(u_0, u_1) \in \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega)$, $f_t(-\tau) \in L^2(\Omega)$, $\nabla f(-\tau) \in L^2(\Omega)$ and $f \in H^1\left([-\tau, 0], H^1(\Omega)\right)$. Integrating the relation in Lemma 3.2 over $(0, t)$, the result follows immediately. □

We introduce for a positive constant ϑ, the functional

$$\mathcal{L}(t) = \mathcal{E}_\lambda(t) + \vartheta \Phi(t), \; t \geq 0$$

where

$$\Phi(t) := \frac{1}{\rho+1} \int_\Omega [u(t) - pu(t-\tau)] \, |u_t(t) - pu_t(t-\tau)|^\rho \, [u_t(t) - pu_t(t-\tau)] \, dx, \; t \geq 0.$$

The next result shows an equivalence result between $\mathcal{L}(t)$ and $\mathcal{E}_\lambda(t)$.

Proposition 3.4. *There exist two positive constants* β_1 *and* β_2 *such that the relation*

$$\beta_1 \mathcal{E}_\lambda(t) \leq \mathcal{L}(t) \leq \beta_2 \mathcal{E}_\lambda(t), \; t \geq 0$$

is satisfied for small ϑ.

Proof. Lemma 2.2 (with: $p = \rho + 2$ and $q = (\rho + 2) / (\rho + 1)$) entails

$$|\Phi(t)| \leq \frac{1}{\rho+1} \|u(t) - pu(t-\tau)\|_{\rho+2} \|u_t(t) - pu_t(t-\tau)\|_{\rho+2}^{\frac{\rho+1}{\rho+2}}, \; t \geq 0$$

which implies by Lemma 2.4 with $\eta = 1/2$ and the relation (3.3) that

$$\begin{aligned} |\Phi(t)| &\leq \frac{1}{2(\rho+1)} \left(\|u(t) - pu(t-\tau)\|_{\rho+2}^2 + \|u_t(t) - pu_t(t-\tau)\|_{\rho+2}^{2(\rho+1)} \right) \\ &\leq \frac{1}{2(\rho+1)} \left(\|u(t) - pu(t-\tau)\|_{\rho+2}^2 + C_3 \|u_t(t) - pu_t(t-\tau)\|_{\rho+2}^{\rho+2} \right), \; t \geq 0 \end{aligned}$$

where $C_3 = \left[(\rho+2) \mathcal{E}_\lambda(0)\right]^{\frac{\rho}{\rho+2}}$. Since $0 < \rho + 2 \leq \frac{2n}{n-2}$ if $n \geq 3$ and $\rho + 2 > 0$ if $n = 1, 2.$, then by applying Lemma 2.1 the previous identity becomes

$$|\Phi(t)| \leq \frac{1}{2(\rho+1)} \left(C_s^2 \|\nabla (u(t) - pu(t-\tau))\|^2 + C_3 \|u_t(t) - pu_t(t-\tau)\|_{\rho+2}^{\rho+2} \right), \; t \geq 0.$$

Consequently, we have for $t \geq 0$

$$\begin{aligned} |\mathcal{L}(t)| \leq\; & \frac{1}{\rho+2} \left(1 + \frac{\vartheta C_3}{2}\right) \|u_t(t) - pu_t(t-\tau)\|_{\rho+2}^{\rho+2} \\ & + \frac{1}{2} \left(1 + \frac{\vartheta C_s^2}{\rho+2}\right) \|\nabla (u(t) - pu(t-\tau))\|^2 + \lambda \Psi(t). \end{aligned}$$

One can take $\beta_2 = \max\left\{\left(1+\frac{\vartheta C_3}{2}\right), \left(1+\frac{C_s^2}{\rho+2}\right)\right\}$. For β_1, we infer that for $t \geq 0$

$$\begin{aligned}\mathcal{L}(t) &\geq \frac{1}{\rho+2}\left(1-\frac{\vartheta C_3}{2}\right)\|u_t(t)-pu_t(t-\tau)\|_{\rho+2}^{\rho+2} \\ &+\frac{1}{2}\left(1-\frac{\vartheta C_s^2}{\rho+2}\right)\|\nabla\left(u(t)-pu(t-\tau)\right)\|^2+\lambda\Psi(t)\end{aligned}$$

which permits to choose $\beta_1 = \min\left\{\left(1-\frac{\vartheta C_3}{2}\right), \left(1-\frac{\vartheta C_s^2}{\rho+2}\right)\right\}$ provided that $\vartheta < \min\left\{\frac{2}{C_3}, \frac{\rho+2}{C_s^2}\right\}$. □

Lemma 3.5. *Let u be a solution of problem (1.1), then the functional $\Phi(t)$ satisfies*

$$\begin{aligned}\frac{d}{dt}\Phi(t) &\leq -\left(1-\eta_2\right)\|\nabla\left(u(t)-pu(t-\tau)\right)\|^2+\left(\frac{1}{4\eta_2}+C_4\right)\|\nabla u_t(t)\|^2 \\ &+C_4p^2\|\nabla u_t(t-\tau)\|_2^2, \ t\geq 0.\end{aligned}$$

for $\eta_2 > 0$, where $C_4 = \frac{2}{\rho+1}\left[(\rho+2)\mathcal{E}_\lambda(0)\right]^{\frac{\rho}{\rho+2}} C_s^{\rho+2}$.

Proof. A differentiation of $\Phi(t)$ gives for $t \geq 0$

$$\begin{aligned}\frac{d}{dt}\Phi(t) &:= \int_\Omega \left[u(t)-pu(t-\tau)\right]|u_t(t)-pu_t(t-\tau)|^\rho\left[u_{tt}(t)-pu_{tt}(t-\tau)\right]dx \\ &+\frac{1}{\rho+1}\int_\Omega\left(u_t(t)-pu_t(t-\tau)\right)|u_t(t)-pu_t(t-\tau)|^\rho\left[u_t(t)-pu_t(t-\tau)\right]dx.\end{aligned}$$

Using now the first equation in (1.1) to replace the term $\left[u_{tt}(t)-pu_{tt}(t-\tau)\right]$, we get for $t \geq 0$

$$\begin{aligned}\frac{d}{dt}\Phi(t) &:= -\|\nabla\left(u(t)-pu(t-\tau)\right)\|^2-\int_\Omega\nabla\left(u(t)-pu(t-\tau)\right)\nabla u_t(t)dx \\ &+\frac{1}{\rho+1}\int_\Omega\left(u_t(t)-pu_t(t-\tau)\right)|u_t(t)-pu_t(t-\tau)|^\rho\left[u_t(t)-pu_t(t-\tau)\right]dx.\end{aligned}$$

Applying Lemma 2.4 with $\eta = \eta_2$, we infer that

$$\begin{aligned}\frac{d}{dt}\Phi(t) &\leq -\left(1-\eta_2\right)\|\nabla\left(u(t)-pu(t-\tau)\right)\|^2+\frac{1}{4\eta_2}\|\nabla u_t(t)\|^2 \\ &+\frac{1}{\rho+1}\|u_t(t)-pu_t(t-\tau)\|_{\rho+2}^{\rho+2}, \ t\geq 0.\end{aligned} \tag{3.4}$$

We see from (3.3) that

$$\begin{aligned}\|u_t(t)-pu_t(t-\tau)\|_{\rho+2}^{\rho+2} &= \|u_t(t)-pu_t(t-\tau)\|_{\rho+2}^{\rho}\|u_t(t)-pu_t(t-\tau)\|_{\rho+2}^{2} \\ &\leq \left[(\rho+2)\mathcal{E}_\lambda(0)\right]^{\frac{\rho}{\rho+2}}\|u_t(t)-pu_t(t-\tau)\|_{\rho+2}^2, \ t\geq 0.\end{aligned}$$

Applying Lemma 2.1 with $q = \rho + 2$, this leads for $t \geq 0$ to

$$\begin{aligned}\|u_t(t)-pu_t(t-\tau)\|_{\rho+2}^{\rho+2} &\leq \left[(\rho+2)\mathcal{E}_\lambda(0)\right]^{\frac{\rho}{\rho+2}}C_s^{\rho+2}\|\nabla\left(u_t(t)-pu_t(t-\tau)\right)\|_2^2 \\ &\leq 2\left[(\rho+2)\mathcal{E}_\lambda(0)\right]^{\frac{\rho}{\rho+2}}C_s^{\rho+2}\left(\|\nabla u_t(t)\|_2^2+p^2\|\nabla u_t(t-\tau)\|_2^2\right).\end{aligned} \tag{3.5}$$

The conclusion follows by replacing the last identity in (3.4). □

We are now ready to state the main result.

Theorem 3.6. *Let* $(u_0,u_1)\in\left(H^2(\Omega)\cap H_0^1(\Omega)\right)\times H_0^1(\Omega)$, $f_t(-\tau)\in L^2(\Omega)$, $\nabla f(-\tau)\in L^2(\Omega)$ *and* $f_t\in H^1\left([-\tau,0];H^1(\Omega)\right)$, *then there exist two positive constants* M *and* m *such that the classical energy satisfies*

$$E(t):=\frac{1}{\rho+2}\left\|u_t(t)\right\|_{\rho+2}^{\rho+2}+\frac{1}{2}\left\|\nabla u(t)\right\|^2\leq Me^{-mt},\ t\geq 0.$$

Proof. First, we estimate the derivative of $\mathcal{L}(t)$, from Lemma 3.2 and Lemma 3.5, we get for $t\geq 0$

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t)\quad &\leq\quad -\left[C_1-\vartheta\left(\frac{1}{4\eta_2}+C_4\right)\right]\left\|\nabla u_t(t)\right\|^2-\left(C_2-\vartheta C_4p^2\right)\left\|\nabla u_t(t-\tau)\right\|^2\\ &\quad\quad -\vartheta\left(1-\eta_2\right)\left\|\nabla\left(u(t)-pu(t-\tau)\right)\right\|^2-\lambda\gamma\Psi(t).\end{aligned}$$

Adding and subtracting the term $\frac{\vartheta}{\rho+2}\left\|u_t(t)-pu_t(t-\tau)\right\|_{\rho+2}^{\rho+2}$. In view of the identity (3.5), we infer for $t\geq 0$ that

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t)\quad &\leq\quad -\left[C_1-\vartheta\left(\frac{1}{4\eta_2}+C_4+C_5\right)\right]\left\|\nabla u_t\right\|^2-\left[C_2-\vartheta p^2\left(C_4+C_5\right)\right]\left\|\nabla u_t(t-\tau)\right\|^2\\ &\quad\quad -\frac{\vartheta}{(\rho+2)}\left\|u_t(t)-pu_t(t-\tau)\right\|_{\rho+2}^{\rho+2}-\vartheta\left(1-\eta_2\right)\left\|\nabla\left(u(t)-pu(t-\tau)\right)\right\|^2\\ &\quad\quad -\lambda\gamma\Psi(t).\end{aligned}$$

where $C_5=\frac{2}{\rho+2}\left[(\rho+2)\,\mathcal{E}_\lambda(0)\right]^{\frac{\rho}{\rho+2}}C_s^{\rho+2}$. Taking $\eta_2=1/2$ and choosing ϑ so small that

$$\vartheta<\min\left\{C_1/\left(\frac{1}{4\eta_2}+C_4+C_5\right),C_2/\left[p^2\left(C_4+C_5\right)\right],\gamma\right\}$$

to arrive at

$$\frac{d}{dt}\mathcal{L}(t)\leq-\vartheta\mathcal{E}_\lambda(t),\ t\geq 0.$$

The equivalence result in proposition 3.4 implies

$$\frac{d}{dt}\mathcal{L}(t)\leq-\frac{\vartheta}{\beta_2}\mathcal{L}(t),\ t\geq 0. \tag{3.6}$$

We deduce from (3.6) that

$$\mathcal{L}(t)\leq\mathcal{L}(0)e^{-\frac{\vartheta}{\beta_2}t},\ t\geq 0.$$

Since $(u_0,u_1)\in H^1(\Omega)\times H^1(\Omega)$, $f_t(-\tau)\in L^2(\Omega),\nabla f(-\tau)\in L^2(\Omega)$ and $f\in H^1\left([-\tau,0],H^1(\Omega)\right)$ then $\mathcal{L}(0)$ is well defined. Moreover, using again the equivalence result in proposition 3.4, it holds

$$\mathcal{E}_\lambda(t)\leq\frac{\mathcal{L}(0)}{\beta_1}e^{-\frac{\vartheta}{\beta_2}t},\ t\geq 0.$$

The expression of $\mathcal{E}_\lambda(t)$ and Lemma 2.3 (with: $v=u_t(t)-pu_t(t-\tau)$, $w=pu_t(t-\tau)$, $p=\rho+2$) imply

$$\begin{aligned}\left\|u_t(t)\right\|_{\rho+2}\quad &\leq\quad \left\|u_t(t)-pu_t(t-\tau)\right\|_{\rho+2}+p\left\|u_t(t-\tau)\right\|_{\rho+2}\\ &\leq\quad \left(\frac{\mathcal{L}(0)\,(\rho+2)}{\beta_1}\right)^{\frac{1}{\rho+2}}e^{-\frac{\vartheta}{\beta_2(\rho+2)}t}+p\left\|u_t(t-\tau)\right\|_{\rho+2},\ t\geq 0.\end{aligned} \tag{3.7}$$

We claim that

$$\left\|u_t(t)\right\|_{\rho+2}<Ae^{-\frac{\vartheta}{\beta_2(\rho+2)}t}+e^{-bt}\sup_{-\tau\leq s\leq 0}\left\|u_t(s)\right\|_{\rho+2},\ t\geq 0 \tag{3.8}$$

for some $A\geq\left(\frac{\mathcal{L}(0)(\rho+2)}{\beta_1}\right)^{\frac{1}{\rho+2}}$ to be determined and for some $b>0$. For $t=0$, since $p<1$

$$\left\|u_t(0)\right\|_{\rho+2}\leq\left(\frac{\mathcal{L}(0)\,(\rho+2)}{\beta_1}\right)^{\frac{1}{\rho+2}}+p\left\|u_t(-\tau)\right\|_{\rho+2}<A+\sup_{-\tau\leq s\leq 0}\left\|u_t(s)\right\|_{\rho+2}.$$

That is, the claim is valid for $t=0$. Let us suppose, for contradiction, that there exists a $\bar{t}>0$ such that

$$\left\|u_t(\bar{t})\right\|_{\rho+2} = Ae^{-\frac{\vartheta}{\beta_2(\rho+2)}\bar{t}} + e^{-b\bar{t}} \sup_{-\tau\leq s\leq 0} \|u_t(s)\|_{\rho+2} \tag{3.9}$$

and

$$\|u_t(t)\|_{\rho+2} < Ae^{-\frac{\vartheta}{\beta_2(\rho+2)}\bar{t}} + e^{-b\bar{t}} \sup_{-\tau\leq s\leq 0} \|u_t(s)\|_{\rho+2}\,,\ t\in[0,\bar{t}).$$

The relation (3.7) at $t=\bar{t}$ gives us

$$\left\|u_t(\bar{t})\right\|_{\rho+2} \leq C_6 e^{-\frac{\vartheta}{\beta_2(\rho+2)}\bar{t}} + p\left\|u_t(\bar{t}-\tau)\right\|_{\rho+2}$$

where $C_6 = \left(\frac{\mathcal{L}(0)(\rho+2)}{\beta_1}\right)^{\frac{1}{\rho+2}}$, which implies by virtue of (3.9)

$$\begin{aligned}
\left\|u_t(\bar{t})\right\|_{\rho+2} &\leq C_6 e^{-\frac{\vartheta}{\beta_2(\rho+2)}\bar{t}} + pAe^{-\frac{\vartheta}{\beta_2(\rho+2)}(\bar{t}-\tau)} + pe^{-b(\bar{t}-\tau)} \sup_{-\tau\leq s\leq 0}\|u_t(s)\|_{\rho+2} \\
&\leq \left[C_6 + pAe^{\frac{\vartheta}{\beta_2(\rho+2)}\tau}\right] e^{-\frac{\vartheta}{\beta_2(\rho+2)}\bar{t}} + pe^{b\tau}e^{-b\bar{t}} \sup_{-\tau\leq s\leq 0}\|u_t(s)\|_{\rho+2}\,.
\end{aligned}$$

Now to achieve our aim, we may assume that

$$\begin{cases} pe^{b\tau} \leq 1, \\ C_6 \leq A\left(1 - pe^{\frac{\vartheta}{\beta_2(\rho+2)}\tau}\right). \end{cases}$$

That is, we need to consider b small enough and ϑ so small that $pe^{\frac{\vartheta}{\beta_2(\rho+2)}\tau} \leq 1$. This leads to a contradiction with the assumption (3.9) and proves (3.8).

Next, we repeat the proof for $\|\nabla u(t)\|^2$. The expression of $\mathcal{E}_\lambda(t)$ and Lemma 2.3 (with: $v = \nabla\left(u_t(t) - pu_t(t-\tau)\right)$, $w = \nabla u_t(t-\tau)$, $p=2$) imply

$$\begin{aligned}
\|\nabla u(t)\| &\leq \|\nabla\left(u_t(t) - pu_t(t-\tau)\right)\| + p\,\|\nabla u_t(t-\tau)\| \\
&\leq \left(\frac{2\mathcal{L}(0)}{\beta_1}\right)^{\frac{1}{2}} e^{-\frac{\vartheta}{2\beta_2}t} + p\,\|\nabla u_t(t-\tau)\|\,,\ t\geq 0.
\end{aligned} \tag{3.10}$$

Our claim by this time is that

$$\|\nabla u(t)\| < Be^{-\frac{\vartheta}{2\beta_2}t} + e^{-ct} \sup_{-\tau\leq s\leq 0}\|\nabla u(s)\|\,,\ t\geq 0 \tag{3.11}$$

for some $B \geq \left(\frac{2\mathcal{L}(0)}{\beta_1}\right)^{\frac{1}{2}}$ to be determined and for some $c>0$. For $t=0$, since $p<1$

$$\|\nabla u(0)\| \leq \left(\frac{2\mathcal{L}(0)}{\beta_1}\right)^{\frac{1}{2}} + p\,\|\nabla u(-\tau)\| < B + \|\nabla u(-\tau)\|\,.$$

This validate the claim at $t=0$. For contradiction, let us suppose that there exists a $t^*>0$ such that

$$\|\nabla u(t^*)\| = Be^{-\frac{\vartheta}{2\beta_2}t^*} + e^{-ct^*} \sup_{-\tau\leq s\leq 0}\|\nabla u(s)\| \tag{3.12}$$

and

$$\|\nabla u(t)\| < Be^{-\frac{\vartheta}{2\beta_2}t} + e^{-ct} \sup_{-\tau\leq s\leq 0}\|\nabla u(s)\|\,,\ t\in[0,\bar{t}).$$

The relation (3.10) at $t=t^*$ gives us

$$\|\nabla u(t^*)\| \leq C_7 e^{-\frac{\vartheta}{2\beta_2}t^*} + p\,\|\nabla u(t^*-\tau)\|$$

where $C_7 = \left(\frac{2\mathcal{L}(0)}{\beta_1}\right)^{\frac{1}{2}}$, which implies by means of (3.12)

$$\begin{aligned} \|\nabla u(t^*)\| &\leq C_7 e^{-\frac{\vartheta}{2\beta_2}t^*} + pBe^{-\frac{\vartheta}{2\beta_2}(t^*-\tau)} + pe^{-c(t^*-\tau)} \sup_{-\tau \leq s \leq 0} \|\nabla u(s)\| \\ &\leq \left[C_7 + pBe^{\frac{\vartheta}{2\beta_2}\tau}\right] e^{-\frac{\vartheta}{2\beta_2}t^*} + pe^{c\tau}e^{-bt^*} \sup_{-\tau \leq s \leq 0} \|\nabla u(s)\|. \end{aligned}$$

Now, to reach our aim, we can assume that

$$\begin{cases} pe^{c\tau} \leq 1, \\ C_7 \leq B\left(1 - pe^{\frac{\vartheta}{\beta_2(\rho+2)}\tau}\right). \end{cases}$$

Arguining as for $\|u_t(t)\|_{\rho+2}$ we need to consider c small enough and ϑ so small that $pe^{\frac{\vartheta}{2\beta_2}\tau} \leq 1$. This leads to a contradiction with the assumption (3.12) and proves (3.11). Finally, the proof of the theorem is established by combining the relations (3.8) and (3.11). □

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