Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 ON LINE
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–18.
ISSN-0037-8712 IN PRESS
doi:10.5269/bspm.52175

General Decay for Semilinear Abstract Second-order Viscoelastic Equation in Hilbert Spaces with Time Delay

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ABSTRACT: The paper is concerned with semilinear abstract second-order viscoelastic equation with time delay and a relaxation function satisfying $h'(t) \leq -\zeta(t)G(h(t))$. Under suitable conditions, we establish explicit and general decay rate results of the energy by introducing a suitable Lyaponov functional and some proprieties of the convex functions. Finally, some applications are given. This work generalizes the previous results without time delay term to those with delay.

Key Words: Abstract viscoelastic equation, general decay, Hilbert spaces, time delay.

Contents

1. Introduction

Let H be a real Hilbert space with inner product and related norm denoted by $\langle .,. \rangle$ and $\| \,.\, \|$, respectively. Let $A : D(A) \longrightarrow H$ and $B : D(B) \longrightarrow H$ be a self-adjoint linear positive operator with domains $D(A) \subset D(B) \subset H$ such that the embeddings are dense and compact. $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is the kernel of the memory term, $\tau > 0$ represents a time delay and $F \ : D(A^{\frac{1}{2}}) \to H$ is function satisfying some conditions to be specified later.

In this work, we consider the following semilinear abstract second-order evolution equation

$$\begin{cases} u_{tt}(t) + Au(t) - \int_0^t h(t-s)Bu(s)ds + \mu_1 u_t(t) + \mu_2 u_t(t-\tau) = F(u(t)), & t \in (0, +\infty), \\ u_t(t-\tau) = f_0(t-\tau) & t \in (0, \tau), \\ u(0) = u_0, \quad u_t(0) = u_1, \end{cases} \tag{1.1}$$

where the initial datum (u_0, u_1, f_0) belongs to suitable spaces, μ_1 is a positive constant and μ_2 is a real number such that

$$|\mu_2| \leq \mu_1. \tag{1.2}$$

In the absence of delay ($\mu_2 = 0$), there exist in the literature different stability results for this type of problems. Dafermos in [14] studied the system (1.1) where $\mu_1 = 0$. He showed that the energy tends asymptotically to zero, but he didn't give the decay rate.
Under the following condition on h

$$\exists \delta > 0 : \quad h'(t) \leq -\delta h(t), \quad \forall t \in \mathbb{R}_+. \tag{1.3}$$

2010 *Mathematics Subject Classification:* 35L90, 35B40, 26A51, 93D20.
Submitted February 12, 2020. Published June 13, 2021

Many authors have established the exponential decay of solutions of this system, see [24,25,16,33] and references therein.

Messaoudi in [22] gave a general decay rate of which the exponential and the polynomial decay rates are special cases. Precisely, he considered relaxation functions satisfying

$$h'(t) \leq -\zeta(t)h(t), \quad \forall t \in \mathbb{R}_+, \tag{1.4}$$

where ζ is a nonincreasing positive differentiable function. After that, Alabau-Boussouira et al. [2] introduced the following condition

$$h'(t) \leq -G(h(t)), \quad \forall t \in \mathbb{R}_+,$$

where G is a convex function which appeared in many papers, see [11,20,34,26].

Recently, Mustafa in [28] established an explicit energy decay result where the exponential and the polynomial decay rates are recovered, under a general condition on the relaxation function, $h'(t) \leq -\zeta(t)h^p(t)$, with $1 \leq p < 2$. In [29], the same author established an optimal explicit and general decay result when the relaxation function h satisfy

$$h'(t) \leq -\zeta(t)G(h(t)), \quad \forall t \in \mathbb{R}_+, \tag{1.5}$$

where G is an increasing and convex function. For some works used (1.5), we refer to read [30,23,6,17]

Time delays arises in many applications and practical problems and in many cases, even small delay may destabilize a system which is asymptotically stable in the absence of delay, in this sense, see [15,5,31]. A large part in the literature is available addressing the stability, instability and the connection between the memory term, the frictional damping and the delay terms. In particular, for wave equation with constant or variable delay, we refer to read [3,31,32]. They showed that the frictional damping term is strong enough to stabilize the system when the weight of the delay be sufficiently small. In [19], Kirane and Said-Houari considered the following wave equation

$$u_{tt}(t) - \Delta u(t) + \int_0^t h(t-s)\Delta u(s)ds + \mu_1 u_t(t) + \mu_2 u_t(t-\tau) = 0,$$

where μ_1 and μ_2 are positive constants. They established the energy decay under the condition $\mu_2 \leq \mu_1$ in the case of relaxation functions satisfy (1.4). Recently, there are different results according to the general decay for several problems with internal or boundary feedback and for constant or variable delay. For instance, Chellaoua and Boukhatem in [13] considered the following abstract viscoelastic equation with time-varying delay

$$u_{tt}(t) + Au(t) - \int_0^t h(t-s)Bu(s)ds + \mu_1 u_t(t) + \mu_2 u_t(t-\tau(t)) = 0.$$

They established optimal decay results of the stability of energy for a wider class of kernel memory functions; condition (1.5), under the condition $|\mu_2| \leq \frac{2(1-d)}{2-d}\mu_1$, where the constant d satisfies $\tau'(t) \leq d < 1$, for all $t > 0$. In [12], the same results have been established by the previous authors for problem (1.1) in the absence of source term ($F = 0$) and infinite memory. For more papers have been concerned with the study of general decay results in the case of constant or varying time delay, see [27,7,21,9,18,8] and references therein.

In this work, we are interested in giving optimal, explicit and general decay rates of solution of problem (1.1) under some suitable assumptions. More precisely, we are intending to extend the results of Messaoudi [23] and Mustafa [30] to the abstract viscoelastic equation with time delay in Hilbert spaces; the system (1.1). To the best of our knowledge, there is no decay result for problems with delay where the relaxation functions satisfy (1.5) in the abstract form and the presence of source term. Moreover, our problem generalizes the earlier problems without time delay term to those with time delay.

The paper is organized as follows. In Section 2, we state and prove some preliminary results under suitable hypothesis. In Section 3, we present some technical lemmas needed for our work. Then, we establish the decay results of the energy by using the energy method to produce a suitable Lyapunov functional in the Section 4. Finally, Section 5 is devoted to give some concrete applications to illustrate our abstract result.

2. Preliminary results

In this section, we present some material that we shall use in order to present our results and we state the existence result of problem (1.1). Let us consider the following assumptions:

(A1) There exist positive constants a, b and d satisfying

$$b\|u\|^2 \leq \left\|B^{\frac{1}{2}}u\right\|^2 \leq a\left\|A^{\frac{1}{2}}u\right\|^2, \quad \forall u \in D(A^{\frac{1}{2}}), \tag{2.1}$$

$$\left\|A^{\frac{1}{2}}u\right\|^2 \leq d\left\|B^{\frac{1}{2}}u\right\|^2, \quad \forall u \in D(A^{\frac{1}{2}}). \tag{2.2}$$

(A2) The kernel memory function $h:\mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is nonincreasing function of class C^1 satisfying

$$h(0)>0, \quad h_0=\int_0^{+\infty} h(s)ds < \frac{1}{a}. \tag{2.3}$$

Moreover, there exists a C^1 function $G:[0,+\infty)\to[0,+\infty)$ which is linear or it is strictly increasing and strictly convex C^2 function on $(0,r]$, $r\leq h(0)$, with $G(0)=G'(0)=0$, such that

$$h'(t) \leq -\zeta(t)G(h(t)), \quad \forall t \geq 0, \tag{2.4}$$

where $\zeta:\mathbb{R}_+\to\mathbb{R}_+$ is a nonincreasing differentiable function.

(A3) The functions F is locally lipschitz mapping and there exist a continuous and differentiable mapping $\mathscr{F}: D(A^{\frac{1}{2}}) \to [0,+\infty)$ satisfying $D\mathscr{F}=F$ and

$$\langle F(u),u\rangle \geq \mathscr{F}(u), \quad \forall u \in D(A^{\frac{1}{2}}). \tag{2.5}$$

Moreover, there exists an increasing continuous function $\psi:[0,+\infty)\to[0,+\infty)$, with $\psi(0)=0$, such that

$$|\langle F(u),v\rangle| \leq \psi\left(\left\|A^{\frac{1}{2}}u\right\|\right)\left\|A^{\frac{1}{2}}u\right\|\left\|A^{\frac{1}{2}}v\right\|, \quad \forall u,v \in D(A^{\frac{1}{2}}). \tag{2.6}$$

Remark 2.1. *If G is a strictly increasing and strictly convex C^2 function on $(0,r]$, with $G(0)=G'(0)=0$, then G has an extension $\overline{G}$ which is a strictly increasing and strictly convex C^2 function on $[0,+\infty)$. Moreover, we can define $\overline{G}$ by*

$$\overline{G}(t)=\frac{c}{2}t^2+(b-cr)t+\left(a+\frac{c}{2}r^2-br\right), \quad \text{for} \quad t>r, \tag{2.7}$$

where $a=G(r)$, $b=G'(r)$ and $c=G''(r)$.

Lemma 2.1. *For δ and t_1 be positive constants, we have*

$$h'(t) \leq -\delta h(t), \quad \forall t \in [0,t_1].$$

Proof: Similarly to [30], from assumption (A2), we clearly deduce that $\lim_{t\to+\infty} h(s)=0$. Therefore, there exists $t_1>0$ large enough such that

$$h(t_1)=r \quad \text{and} \quad h(t)\leq r, \quad \forall t \geq t_1. \tag{2.8}$$

By using the fact that h and ζ are positive nonincreasing continuous and G is a positive continuous function, we get, for all $t\in[0,t_1]$,

$$\begin{cases} 0<h(t_1)\leq h(t)\leq h(0) \\ 0<\zeta(t_1)\leq\zeta(t)\leq\zeta(0), \end{cases}$$

which gives, for two positive constants δ_1 and δ_2,

$$\delta_1 \leq \zeta(t)G(h(t)) \leq \delta_2.$$

Consequently, for all $t \in [0, t_1]$,

$$h'(t) \leq -\zeta(t)G(h(t)) \leq -\frac{\delta_1}{h(0)}h(0) \leq -\frac{\delta_1}{h(0)}h(t). \tag{2.9}$$

□

In the following remark, we present the inequality of Jensen which will be used in establishing our main result.

Remark 2.2. *If Q is a convex function on $[a,b]$, $f : \Omega \to [a,b]$ and h are integrable functions on Ω, $h(x) \geq 0$, and $\int_\Omega h(x)dx = k > 0$, then Jensen's inequality states that*

$$Q\left[\frac{1}{k}\int_\Omega f(x)h(x)dx\right] \leq \frac{1}{k}\int_\Omega Q[f(x)]h(x)dx.$$

In the following result, we state, without proof, the local existence, uniqueness and regularity of (2.10), see [7,10].

Proposition 2.1. *Under the assumptions (A1)-(A3), for an initial datum $(u_0, u_1) \in D(A^{\frac{1}{2}}) \times H$, the system (2.10) has a unique local mild solution u such that*

$$u \in C\left(0, T; D(A^{\frac{1}{2}})\right) \cap C^1(0, T; H).$$

Moreover, if $(u_0, u_1) \in D(A^{\frac{1}{2}}) \times D(A^{\frac{1}{2}})$, then the solution of (2.10) satisfies (classical solution)

$$u \in C\left(0, T; D(A^{\frac{1}{2}})\right) \cap C^1\left(0, T; D(A^{\frac{1}{2}})\right) \cap C^2(0, T; H).$$

In order to state and prove the desired results, as in [31], we introduce the variable z by

$$z(\rho, t) = u_t(t - \rho\tau), \quad \rho \in (0,1),\ t > 0,$$

therefore, the problem (1.1) takes the following form

$$\begin{cases} u_{tt}(t) + Au(t) - \int_0^t h(t-s)Bu(s)ds + \mu_1 u_t(t) + \mu_2 z(1,t) = F(u(t)), & t \in (0, +\infty), \\ \tau z_t(\rho, t) + z_\rho(\rho, t) = 0, & \rho \in (0,1),\ t > 0, \\ z(\rho, 0) = f_0(-\rho\tau), & \rho \in (0,1), \\ z(0,t) = u_t(t) & t \geq 0, \\ u(0) = u_0, \quad u_t(0) = u_1, & t \geq 0. \end{cases} \tag{2.10}$$

Now, let us define the modified energy functional E associated with problem (2.10) by

$$E(t) = \frac{1}{2}\left(\left\|A^{\frac{1}{2}}u\right\|^2 - \int_0^t h(s)ds\left\|B^{\frac{1}{2}}u\right\|^2 + \|u_t\|^2 + (h \diamond B^{\frac{1}{2}}u)(t) - 2\mathscr{F}(u) + \xi\tau\int_0^1 \|z(\rho,t)\|^2 d\rho\right), \tag{2.11}$$

for all $t \in \mathbb{R}_+$ and the initial energy is given by

$$E(t) = \frac{1}{2}\left(\left\|A^{\frac{1}{2}}u_0\right\|^2 + \|u_1\|^2 - 2\mathscr{F}(u_0) + \xi\tau\int_{-\tau}^0 \|f_0(s)\|^2 ds\right), \tag{2.12}$$

where

$$(h \diamond B^{\frac{1}{2}}u)(t) = \int_0^t h(t-s)\left\|B^{\frac{1}{2}}u(t) - B^{\frac{1}{2}}u(s)\right\|^2 ds \tag{2.13}$$

and ξ is a positive constant (note that ξ exists according to (1.2)) such that

$$|\mu_2| \leq \xi \leq 2\mu_1 - |\mu_2|. \tag{2.14}$$

Lemma 2.2. *Assume that (A1)-(A3) hold. Then, the energy functional defined by (2.11) satisfies*

$$E'(t) \leq \frac{1}{2}(h' \diamond B^{\frac{1}{2}}u)(t) \leq 0, \quad \forall t \in \mathbb{R}_+. \tag{2.15}$$

Proof: By using the first equation of (2.10), we get

$$\frac{1}{2}\frac{d}{dt}\left(\|u_t\|^2 + \left\|A^{\frac{1}{2}}u\right\|^2 - 2\mathscr{F}(u)\right) + \mu_1\|u_t\|^2 + \mu_2 \langle z(1,t), u_t\rangle = \int_0^t h(t-s)\langle Bu(s), u_t(t)\rangle \, ds. \tag{2.16}$$

On the other hand, we can easily check that

$$2\int_0^t h(t-s)\langle Bu(s), u_t(t)\rangle \, ds = \frac{d}{dt}\left[\int_0^t h(s)ds \left\|B^{\frac{1}{2}}u\right\|^2 - (h \diamond B^{\frac{1}{2}}u)(t)\right] + (h' \diamond B^{\frac{1}{2}}u)(t) - h(t)\left\|B^{\frac{1}{2}}u\right\|^2. \tag{2.17}$$

Similarly, by the second equation of (2.10), we have

$$\xi\tau\frac{d}{dt}\|z(\rho,t)\|^2 + \xi\frac{\partial}{\partial\rho}\|z(\rho,t)\|^2 = 0.$$

Integration over $(0,1)$, with respect to ρ, yields

$$\xi\tau\int_0^1 \frac{d}{dt}\|z(\rho,t)\|^2 d\rho = \xi\left(\|u_t(t)\|^2 - \|z(1,t)\|^2\right). \tag{2.18}$$

Then, by using Cauchy-Schwarz's and Young's inequalities and inserting (2.17) and (2.18) in (2.16), we get

$$\begin{aligned} E'(t) &\leq \frac{1}{2}(h' \diamond B^{\frac{1}{2}}u)(t) - \frac{1}{2}h(t)\left\|B^{\frac{1}{2}}u\right\|^2 + \left(\frac{|\mu_2|}{2} - \mu_1 + \frac{\xi}{2}\right)\|u_t(t)\|^2 + \left(\frac{|\mu_2|}{2} - \frac{\xi}{2}\right)\|z(1,t)\|^2 \\ &\leq \frac{1}{2}(h' \diamond B^{\frac{1}{2}}u)(t) - \frac{1}{2}h(t)\left\|B^{\frac{1}{2}}u\right\|^2 - C\left(\|u_t(t)\|^2 + \|z(1,t)\|^2\right), \end{aligned}$$

where

$$C = \min\left\{\mu_1 - \frac{|\mu_2|}{2} - \frac{\xi}{2}, \frac{\xi}{2} - \frac{|\mu_2|}{2}\right\},$$

which is positive by (2.14). This completes the proof of the Lemma. □

By using Lemma 2.2, we prove the global existence of solution of problem (2.10) under small initial conditions.

Theorem 2.1. *Assume that (A1)-(A3) hold and there exist a positive constant ρ_0 such that for any $(u_0, u_1, f_0) \in D(A^{\frac{1}{2}}) \times H \times L^2(-\tau, 0; H)$ satisfying*

$$\left(\|A^{\frac{1}{2}}u_0\| + \|u_1\|^2 + \int_{-\tau}^0 \|f_0(s)\|^2 ds\right)^{\frac{1}{2}} < \rho_0,$$

The problem (2.10) admits a unique mild solution u on $[0, +\infty)$.

Proof: From the proposition 2.1, the problem admits a unique local solution u in a maximal time interval $[0, T)$. Now, similarly to [1] and by using (2.5), (2.6) and (2.12), we have

$$E(0) \geq \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\left\|A^{\frac{1}{2}}u_0\right\|^2 - \mathscr{F}(u_0) \geq \frac{1}{2}\|u_1\|^2 + \frac{l}{4}\left\|A^{\frac{1}{2}}u_0\right\|^2 \geq 0,$$

if $\psi\left(\left\|A^{\frac{1}{2}}u\right\|\right) < \frac{l}{4}$ where $l = (1 - ah_0)$. Furthermore, we show that if

$$\psi\left(\left\|A^{\frac{1}{2}}u\right\|\right) < \frac{l}{4} \quad \text{and} \quad \psi\left(2\left(\frac{E(0)}{l}\right)^{\frac{1}{2}}\right) < \frac{l}{4}, \tag{2.19}$$

then

$$\begin{aligned} E(t) &\geq \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\left\|A^{\frac{1}{2}}u(t)\right\|^2 - \frac{\int_0^t h(s)ds}{2}\left\|B^{\frac{1}{2}}u(t)\right\|^2 - \mathscr{F}(u(t)) \\ &\geq \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\left\|A^{\frac{1}{2}}u(t)\right\|^2 - \frac{h_0}{2}\left\|B^{\frac{1}{2}}u(t)\right\|^2 - \mathscr{F}(u(t)) \\ &\geq \frac{1}{2}\|u_t(t)\|^2 + \frac{l}{4}\left\|A^{\frac{1}{2}}u(t)\right\|^2, \quad \forall t \in [0,T). \end{aligned} \tag{2.20}$$

Now, let consider r the supremum of all $s \in [0,T)$ such that (2.20) holds true for any $t \in [0,s]$. Suppose $r < T$. By continuity of the function E, we obtain

$$E(r) \geq \frac{1}{2}\|u_t(r)\|^2 + \frac{l}{4}\left\|A^{\frac{1}{2}}u(r)\right\|^2 \geq 0. \tag{2.21}$$

Hence, from (2.21), we have

$$\psi\left(\left\|A^{\frac{1}{2}}u(r)\right\|\right) \leq \psi\left(2\left(\frac{E(r)}{l}\right)^{\frac{1}{2}}\right) \leq \psi\left(2\left(\frac{E(0)}{l}\right)^{\frac{1}{2}}\right) < \frac{l}{4},$$

which gives

$$\begin{aligned} E(r) &\geq \frac{1}{2}\|u_t(r)\|^2 + \frac{1-ah_0}{2}\left\|A^{\frac{1}{2}}u(r)\right\|^2 - \mathscr{F}(u(r)) \\ &\geq \frac{1}{2}\|u_t(r)\|^2 + \left(\frac{l}{2} - \frac{l}{4}\right)\left\|A^{\frac{1}{2}}u(r)\right\|^2 = \frac{1}{2}\|u_t(r)\|^2 + \frac{l}{4}\left\|A^{\frac{1}{2}}u(r)\right\|^2. \end{aligned}$$

This contradicts the maximality of r. Let

$$\rho_0 = \frac{\sqrt{l}}{2}\psi^{-1}\left(\frac{l}{4}\right) > 0.$$

then $\psi\left(\left\|A^{\frac{1}{2}}u\right\|\right) < \frac{l}{4}$. For any $u_0 \in D(A^{\frac{1}{2}})$, $u_1 \in H$ and $f_0) \in L^2(-\tau, 0; H)$ such that

$$\left(\left\|A^{\frac{1}{2}}u_0\right\|^2 + \|u_1\|^2 + \int_{-\tau}^0 \|f_0(s)\|^2 ds\right)^{\frac{1}{2}} < \rho_0. \tag{2.22}$$

This assumption implies that $\left\|A^{\frac{1}{2}}u_0\right\| < \rho_0$, so, we have

$$\psi\left(\left\|A^{\frac{1}{2}}u_0\right\|\right) < \psi(\rho_0) = \psi\left(\frac{\sqrt{l}}{2}\psi^{-1}\left(\frac{l}{4}\right)\right).$$

Moreover, by using (2.5) and (2.6), we obtain

$$\begin{aligned} E(0) &\leq \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\left\|A^{\frac{1}{2}}u_0\right\| + \frac{1}{2}\int_{-\tau}^0 \|f_0(s)\|^2 ds \\ &\leq \left(\left\|A^{\frac{1}{2}}u_0\right\| + \|u_1\|^2 + \int_{-\tau}^0 \|f_0(s)\|^2 ds\right) < \rho_0^2, \end{aligned}$$

and, by definition of ρ_0, we deduce that

$$\psi\left(2\left(\frac{E(0)}{l}\right)^{\frac{1}{2}}\right) < \psi\left(\psi^{-1}\left(\frac{l}{4}\right)\right) = \frac{l}{4}$$

In addition, under the assumption (2.22) and (2.19), we get

$$0 \leq \frac{1}{2}\|u_t(t)\|^2 + \frac{l}{4}\left\|A^{\frac{1}{2}}u(t)\right\|^2 \leq E(t) \leq E(0) \leq \rho_0^2. \tag{2.23}$$

Thus, the energy function is nonnegative on $[0,T)$ and bounded which means that the solution exists on $[0,+\infty)$ and from (2.23), we have

$$\psi\left(\left\|A^{\frac{1}{2}}u(t)\right\|\right) \leq \psi\left(2\left(\frac{E(t)}{l}\right)^{\frac{1}{2}}\right) \leq \psi\left(2\left(\frac{E(0)}{l}\right)^{\frac{1}{2}}\right) < \frac{l}{4}, \quad \forall t \geq 0. \tag{2.24}$$

This completes the proof of Theorem 2.1. □

3. Technical Lemmas

In the section, we state and prove some technical Lemmas in order to prove the desired results.

Lemma 3.1. *Let u be the solution of (2.10). Then the functional*

$$I_1(t) = \langle u_t(t), u(t)\rangle, \tag{3.1}$$

satisfies, for $\delta_1 > 0$ and for all $t \geq 0$

$$I_1'(t) \leq \left(1 + \frac{\mu_1}{4\delta_1}\right)\|u_t\|^2 - \left(\frac{l}{2} - \frac{a}{b}\delta_1(\mu_1 + |\mu_2|)\right)\left\|A^{\frac{1}{2}}u\right\|^2 + \frac{aC_\alpha}{2l}(k \diamond B^{\frac{1}{2}}u)(t) + \frac{|\mu_2|}{4\delta_1}\|z(1,t)\|^2, \tag{3.2}$$

for any $0 < \alpha < 1$, where

$$C_\alpha = \int_0^{+\infty} \frac{h^2(s)}{\alpha h(s) - h'(s)} ds \quad \textit{and} \quad k(t) = \alpha h(t) - h'(t). \tag{3.3}$$

Proof: Differentiating (3.1) with respect to t, we find

$$I_1'(t) = \|u_t\|^2 + \langle u_{tt}(t), u(t)\rangle.$$

On the other hand, multiplying the first equation of (2.10) by $u(t)$, we have

$$\begin{aligned}&\langle u_{tt}(t), u(t)\rangle + \langle Au(t), u(t)\rangle - \left\langle \int_0^t h(t-s)Bu(s)ds, u(t)\right\rangle \\ &+\mu_1\langle u_t(t), u(t)\rangle + \mu_2\langle z(1,t), u(t)\rangle = \langle F(u(t)), u(t)\rangle,\end{aligned}$$

By the definitions of $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$, we have

$$\begin{aligned} I_1'(t) &= \|u_t\|^2 - \left\|A^{\frac{1}{2}}u\right\|^2 + \int_0^t h(s)ds\left\|B^{\frac{1}{2}}u\right\|^2 + \left\langle \int_0^t h(t-s)B^{\frac{1}{2}}\left(u(s)-u(t)\right)ds, B^{\frac{1}{2}}u(t)\right\rangle \\ &\quad -\mu_1\langle u_t(t), u(t)\rangle - \mu_2\langle z(1,t), u(t)\rangle + \langle F(u(t)), u(t)\rangle. \end{aligned} \tag{3.4}$$

By using Cauchy-Schwarz's and Young's inequalities and (2.1), we have, for $\delta_1 > 0$

$$-\mu_1\langle u_t(t), u(t)\rangle \leq \frac{\mu_1}{4\delta_1}\|u_t(t)\|^2 + \frac{a\mu_1\delta_1}{b}\left\|A^{\frac{1}{2}}u\right\|^2, \tag{3.5}$$

$$-\mu_2\langle z(1,t), u(t)\rangle \leq \frac{|\mu_2|}{4\delta_1}\|z(1,t)\|^2 + \frac{a|\mu_2|\delta_1}{b}\left\|A^{\frac{1}{2}}u\right\|^2, \tag{3.6}$$

$$\left\langle \int_0^t h(t-s)B^{\frac{1}{2}}(u(s)-u(t))ds, B^{\frac{1}{2}}u(t)\right\rangle \leq \frac{l}{4}\left\|A^{\frac{1}{2}}u\right\|^2 + \frac{a}{l}\left\|\int_0^t h(t-s)B^{\frac{1}{2}}(u(s)-u(t))ds\right\|^2 \tag{3.7}$$

and, using (2.6) and (2.24), we obtain

$$\langle F(u(t)), u(t)\rangle \leq \psi_2\left(\left\|A^{\frac{1}{2}}u\right\|\right)\left\|A^{\frac{1}{2}}u\right\|^2 \leq \frac{l}{4}\left\|A^{\frac{1}{2}}u\right\|^2. \tag{3.8}$$

Moreover, we have

$$\begin{aligned}
&\left\|\int_0^t h(t-s)B^{\frac{1}{2}}(u(s)-u(t))ds\right\|^2 \\
\leq\;& \left(\int_0^t h(t-s)\left\|B^{\frac{1}{2}}(u(s)-u(t))\right\| ds\right)^2 \\
\leq\;& \left(\int_0^t \frac{h(t-s)}{\sqrt{\alpha h(t-s)-h'(t-s)}}\sqrt{\alpha h(t-s)-h'(t-s)}\left\|B^{\frac{1}{2}}(u(s)-u(t))\right\| ds\right)^2 \\
\leq\;& \left(\int_0^t \frac{h^2(s)}{\alpha h(s)-h'(s)}ds\right)\int_0^t (\alpha h(t-s)-h'(t-s))\left\|B^{\frac{1}{2}}(u(s)-u(t))\right\|^2 ds \\
\leq\;& C_\alpha(k\diamond B^{\frac{1}{2}}u)(t).
\end{aligned} \tag{3.9}$$

Substituting the inequalities (3.5), (3.6), (3.7) and (3.9) in (3.4), we get (3.2). □

Lemma 3.2. *Let u be the solution of (2.10). Then the functional*

$$I_2(t) = -\left\langle u_t(t), \int_0^t h(t-s)(u(t)-u(s))ds\right\rangle, \tag{3.10}$$

satisfies, for $\varepsilon > 0$ and for all $t \geq 0$

$$\begin{aligned}
I_2'(t) \;\leq\;& (\varepsilon - \int_0^t h(s)ds)\|u_t\|^2 + \varepsilon\left\|A^{\frac{1}{2}}u\right\|^2 + \mu_2\left\langle z(1,t), \int_0^t h(t-s)(u(t)-u(s))ds\right\rangle \\
&+\frac{c(C_\alpha+1)}{\varepsilon}(k\diamond B^{\frac{1}{2}}u)(t),
\end{aligned} \tag{3.11}$$

where $c = \max\left\{\frac{c'}{b}, d+\varepsilon+\frac{l^2}{16b}+\frac{\mu_1^2}{2b}+\frac{ah_0^2}{2}+\frac{\alpha^2}{b}\right\}$.

Proof: Differentiating (3.10) with respect to t, we find

$$I_2'(t) = -\left\langle u_{tt}(t), \int_0^t h(t-s)(u(t)-u(s))ds\right\rangle - \left\langle u_t(t), \int_0^t h'(t-s)(u(t)-u(s))ds\right\rangle - \int_0^t h(s)ds\|u_t\|^2.$$

Then, using the first equation of (2.10) and using the definitions of $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$, we get

$$\begin{aligned}
I_2'(t) \;=\;& -\int_0^t h(s)ds\|u_t\|^2 + \mu_1\left\langle u_t(t), \int_0^t h(t-s)(u(t)-u(s))ds\right\rangle \\
&+\mu_2\left\langle z(1,t), \int_0^t h(t-s)(u(t)-u(s))ds\right\rangle - \left\langle F(u(t)), \int_0^t h(t-s)(u(t)-u(s))ds\right\rangle \\
&-\left\langle u_t(t), \int_0^t h'(t-s)(u(t)-u(s))ds\right\rangle + \left\langle A^{\frac{1}{2}}u(t), \int_0^t h(t-s)A^{\frac{1}{2}}(u(t)-u(s))ds\right\rangle \\
&-\int_0^t h(s)ds\left\langle B^{\frac{1}{2}}u(t), \int_0^t h(t-s)B^{\frac{1}{2}}(u(t)-u(s))ds\right\rangle \qquad (3.12)\\
&+\left\|\int_0^t h(t-s)B^{\frac{1}{2}}(u(s)-u(t))ds\right\|^2. \qquad (3.13)
\end{aligned}$$

By using Cauchy-Schwarz's and Young's inequalities, (2.1), (2.2), (2.3) and (3.9) , we get, for $\varepsilon > 0$

$$\begin{aligned}\mu_1 \left\langle u_t(t), \int_0^t h(t-s)(u(t)-u(s))ds \right\rangle &\leq \frac{\varepsilon}{2}\|u_t(t)\|^2 + \frac{\mu_1^2}{2\varepsilon}\left\|\int_0^t h(t-s)(u(t)-u(s))ds\right\|^2 \\ &\leq \frac{\varepsilon}{2}\|u_t(t)\|^2 + \frac{\mu_1^2 C_\alpha}{2b\varepsilon}(k\diamond B^{\frac{1}{2}}u)(t),\end{aligned}$$

$$\begin{aligned}\left\langle A^{\frac{1}{2}}u(t), \int_0^t h(t-s)A^{\frac{1}{2}}(u(t)-u(s))ds \right\rangle &\leq \frac{\varepsilon}{4}\left\|A^{\frac{1}{2}}u\right\|^2 + \frac{1}{\varepsilon}\left\|\int_0^t h(t-s)A^{\frac{1}{2}}(u(t)-u(s))ds\right\|^2 \\ &\leq \frac{\varepsilon}{4}\left\|A^{\frac{1}{2}}u\right\|^2 + \frac{dC_\alpha}{\varepsilon}(k\diamond B^{\frac{1}{2}}u)(t)\end{aligned}$$

and

$$-\left\langle B^{\frac{1}{2}}u(t), \int_0^t h(t-s)B^{\frac{1}{2}}(u(t)-u(s))ds \right\rangle \leq \frac{\varepsilon}{2h_0}\left\|A^{\frac{1}{2}}u\right\|^2 + \frac{ah_0C_\alpha}{2\varepsilon}(k\diamond B^{\frac{1}{2}}u)(t).$$

Then, by using (2.6) and (2.24), we get

$$\left\langle F(u(t)), \int_0^t h(t-s)(u(t)-u(s))ds \right\rangle \leq \frac{\varepsilon}{4}\left\|A^{\frac{1}{2}}u\right\|^2 + \frac{l^2C_\alpha}{16b\varepsilon}(k\diamond B^{\frac{1}{2}}u)(t). \tag{3.14}$$

On other hand, we have

$$\begin{aligned}&-\left\langle u_t(t), \int_0^t h'(t-s)(u(t)-u(s))ds \right\rangle \\ =\ & \left\langle u_t(t), \int_0^t k(t-s)(u(t)-u(s))ds \right\rangle - \left\langle u_t(t), \int_0^t \alpha h(t-s)(u(t)-u(s))ds \right\rangle \\ \leq\ & \frac{\varepsilon}{2}\|u_t(t)\|^2 + \frac{1}{\varepsilon}\left(\int_0^t \sqrt{k(t-s)}\,\sqrt{k(t-s)}\|u(t)-u(s)\|ds\right)^2 + \frac{\alpha^2}{\varepsilon}\left(\int_0^t h(t-s)\|u(t)-u(s)\|ds\right)^2 \\ \leq\ & \frac{\varepsilon}{2}\|u_t(t)\|^2 + \left(\frac{\int_0^t k(s)ds}{\varepsilon b} + \frac{\alpha^2 C_\alpha}{\varepsilon b}\right)(k\diamond B^{\frac{1}{2}}u)(t) \leq \frac{\varepsilon}{2}\|u_t(t)\|^2 + \left(\frac{c'}{\varepsilon b} + \frac{\alpha^2 C_\alpha}{\varepsilon b}\right)(k\diamond B^{\frac{1}{2}}u)(t),\end{aligned}$$

where $c' = \alpha h_0 + h(0)$. Then, inserting these five inequalities and the inequality (3.9) in (3.13), we get (3.11). □

Lemma 3.3. *Let u be the solution of (2.10). Then the functional*

$$I_3(t) = \tau e^{2\tau}\int_0^1 e^{-2\tau\rho}\|z(\rho,t)\|^2 ds, \tag{3.15}$$

satisfies, for all $t \geq 0$

$$I_3'(t) \leq -2\tau\int_0^1 \|z(\rho,t)\|^2 ds + e^{2\tau}\|u_t\|^2 - \|z(1,t)\|^2. \tag{3.16}$$

Proof: By using the second equation of (2.10), we get

$$\begin{aligned}I_3'(t) &= 2\tau e^{2\tau}\int_0^1 e^{-2\tau\rho}\langle z_t(\rho,t), z(\rho,t)\rangle d\rho \\ &= -2e^{2\tau}\int_0^1 e^{-2\tau\rho}\frac{\partial}{\partial\rho}\|z(\rho,t)\|^2 d\rho.\end{aligned}$$

Then, by integrating by parts and $z(0,t)=u_t(t)$, we get

$$I_3'(t)=-2\tau e^{2\tau}\int_0^1 e^{-2\tau\rho}\|z(\rho,t)\|^2ds+e^{2\tau}\|u_t\|^2-\|z(1,t)\|^2,$$

which is (3.16) by using the fact that $e^{-2\tau\rho}\geq e^{-2\tau}$, for any $\rho\in]0,1[$. □

Lemma 3.4. *Let u be the solution of (2.10). Then the functional*

$$I_4(t)=\int_0^t f(t-s)\left\|B^{\frac{1}{2}}u(s)\right\|^2ds, \tag{3.17}$$

where $f(t)=\int_t^{+\infty}h(s)ds$, satisfies,

$$I_4'(t)\leq-\frac{1}{2}(h\diamond B^{\frac{1}{2}}u)(t)+3(1-l)\left\|A^{\frac{1}{2}}u\right\|^2, \qquad \forall t\geq 0. \tag{3.18}$$

Proof: By differentiating (3.17), we get

$$I_4'(t)=f(0)\left\|B^{\frac{1}{2}}u\right\|^2+\int_0^t f'(t-s)\left\|B^{\frac{1}{2}}u(s)\right\|^2ds.$$

Then, by using Young's inequality and the fact $f'(t)=-h(t)$

$$\begin{aligned}I_4'(t) &= f(0)\left\|B^{\frac{1}{2}}u\right\|^2-\int_0^t h(t-s)\left\|B^{\frac{1}{2}}u(s)\right\|^2ds\\ &\leq h_0\left\|B^{\frac{1}{2}}u\right\|^2-\int_0^t h(t-s)\left\|B^{\frac{1}{2}}(u(t)-u(s))\right\|^2ds-2\left\langle B^{\frac{1}{2}}u,\int_0^t h(t-s)B^{\frac{1}{2}}(u(t)-u(s))ds\right\rangle.\end{aligned}$$

But

$$\begin{aligned}-2\left\langle B^{\frac{1}{2}}u,\int_0^t h(t-s)B^{\frac{1}{2}}(u(t)-u(s))ds\right\rangle &\leq \frac{\int_0^t h(s)ds}{2h_0}\int_0^t h(t-s)\left\|B^{\frac{1}{2}}(u(t)-u(s))\right\|^2ds\\ &\quad+2h_0\left\|B^{\frac{1}{2}}u\right\|^2.\end{aligned}$$

Moreover, as $\int_0^t h(s)ds\leq f(0)=h_0$ and by using (2.1), we get (3.18) where $ah_0=1-l$. □

4. Stability results

In this section, we shall state and prove explicit and general decay rate results of the energy function E. For this purose, we construct a Lyapunov functional L equivalent to E, with which we can show the desired result. Let

$$L(t)=ME(t)+\sum_{i=1}^{3}N_iI_i(t), \tag{4.1}$$

where M, N_1, N_2 and N_3 are positive constants.

Lemma 4.1. *Assume that (A1)-(A3) hold, there exist two positive constants c_1 and c_2 such that*

$$c_1E(t)\leq L(t)\leq c_2E(t). \tag{4.2}$$

Proof: Using Cauchy-Schwarz's and Young's inequalities, we have

$$\begin{aligned}|L(t)-ME(t)| &\leq N_1|\langle u_t,u\rangle|+N_2\left|\left\langle u_t(t),\int_0^t h(t-s)(u(t)-u(s))ds\right\rangle\right|+N_3\tau e^{2\tau}\int_0^1 e^{-2\tau\rho}\|z(\rho,t)\|^2ds\\ &\leq \frac{N_1+N_2}{2}\|u_t\|^2+\frac{aN_1}{2b}\left\|A^{\frac{1}{2}}u\right\|^2+\frac{h_0N_2}{2b}(h\diamond B^{\frac{1}{2}}u)(t)+N_3\tau e^{2\tau}\int_0^1\|z(\rho,t)\|^2ds\\ &\leq CE(t),\end{aligned}$$

Then, by choosing M so large, we have $L\sim E$. □

Lemma 4.2. *The Lyapunov functional L defined in (4.1) satisfies*

$$L'(t) \leq -3(1-l)\left\|A^{\frac{1}{2}}u\right\|^2 - \|u_t\|^2 + \frac{1}{4}(h \diamond B^{\frac{1}{2}}u)(t), \quad \forall t \geq t_1, \tag{4.3}$$

under a suitable choice of M, N_1, N_2 and N_3.

Proof: Combining (4.1), (2.15), (3.2), (3.11) and (3.16). Then, by using (3.3) and for $h_1 = \int_0^{t_1} h(s)ds > 0$, where t_1 was introduced in (2.8), we have, for all $t \geq t_1$,

$$\begin{aligned} L'(t) \quad \leq \quad & -\left[(h_1-\varepsilon)N_2 - \left(1+\frac{\mu_1}{4\delta_1}\right)N_1 - N_3e^{2\tau}\right]\|u_t\|^2 - \left[\left(\frac{l}{2} - \frac{a}{b}\delta_1(\mu_1+|\mu_2|)\right)N_1 - \varepsilon N_2\right]\left\|A^{\frac{1}{2}}u\right\|^2 \\ & +\mu_2\left\langle z(1,t), N_2\int_0^t h(t-s)(u(t)-u(s))ds\right\rangle + \frac{\alpha M}{2}(h\diamond B^{\frac{1}{2}}u)(t) - 2N_3\tau\int_0^1 \|z(\rho,t)\|^2 ds \\ & -\left[\frac{M}{2} - \frac{aC_\alpha}{2l}N_1 - \frac{c(C_\alpha+1)}{\varepsilon}N_2\right](k\diamond B^{\frac{1}{2}}u)(t) - \left(N_3 - \frac{|\mu_2|}{4\delta_1}N_1\right)\|z(1,t)\|^2. \end{aligned}$$

By using Cauchy-Schwarz's and Young's inequalities, (2.1) and (3.9), we get

$$\left\langle \mu_2 z(1,t), N_2\int_0^t h(t-s)(u(t)-u(s))ds\right\rangle \leq |\mu_2|\left(\frac{1}{2}\|z(1,t)\|^2 + \frac{C_\alpha}{2b}N_2^2(k\diamond B^{\frac{1}{2}}u)(t)\right).$$

Consequently, by taking $\varepsilon = \frac{l}{4N_2}$, we obtain

$$\begin{aligned} L'(t) \quad \leq \quad & -\left[h_1N_2 - \frac{l}{4} - \left(1+\frac{\mu_1}{4\delta_1}\right)N_1 - N_3e^{2\tau}\right]\|u_t\|^2 - \left[\left(\frac{l}{2} - \frac{a}{b}\delta_1(\mu_1+|\mu_2|)\right)N_1 - \frac{l}{4}\right]\left\|A^{\frac{1}{2}}u\right\|^2 \\ & -\left[\frac{M}{2} - \frac{aC_\alpha}{2l}N_1 - \frac{c(C_\alpha+1)}{\varepsilon}N_2 - \frac{|\mu_2|C_\alpha}{2b}N_2^2\right](k\diamond B^{\frac{1}{2}}u)(t) + \frac{\alpha M}{2}(h\diamond B^{\frac{1}{2}}u)(t) \\ & -\left[N_3 - |\mu_2|\left(\frac{N_1}{4\delta_1}+\frac{1}{2}\right)\right]\|z(1,t)\|^2 - 2N_3\tau\int_0^1\|z(\rho,t)\|^2ds. \end{aligned}$$

At this point, let take $\delta_1 = \frac{bl}{4a(\mu_1+|\mu_2|)}$ and choose N_1 large enough so that

$$\frac{l}{4}N_1 - \frac{l}{4} > 4(1-l).$$

Then, let pick N_3 and N_2 big enough so that

$$N_3 - |\mu_2|\left(\frac{N_1}{4\delta_1}+\frac{1}{2}\right) > 0,$$

$$h_1N_2 - \frac{l}{4} - \left(1+\frac{\mu_1}{4\delta_1}\right)N_1 - N_3e^{2\tau} > 1.$$

Now, as $\frac{\alpha h^2(s)}{\alpha h(s)-h'(s)} < h(s)$ and by using the Lebesgue dominated convergence theorem, we have

$$\alpha C_\alpha = \int_0^{+\infty}\frac{\alpha h^2(s)}{\alpha h(s)-h'(s)}ds \to 0 \quad \text{as} \quad \alpha\to 0.$$

Consequently, there is $0<\alpha_0<1$ such that if $\alpha<\alpha_0$, then

$$\alpha C_\alpha < \frac{1}{8\left[\frac{a}{2l}N_1 + \left(\frac{4c}{l}+\frac{|\mu_2|}{2b}\right)N_2^2\right]}.$$

Then, we choose M large enough such that (4.2) is satisfied and

$$\frac{M}{2} - \frac{4c}{l}N_2^2 > 0,$$

then, for M fixed, we choose α so that

$$\alpha = \frac{1}{2M} < \alpha_0,$$

which gives

$$\frac{M}{2} - \frac{4c}{l}N_2^2 - C_\alpha \left[\frac{a}{2l}N_1 + \left(\frac{4c}{l} + \frac{|\mu_2|}{2b}\right) N_2^2\right] > 0.$$

Therefore, we arrive at

$$L'(t) \le -4(1-l)\left\|A^{\frac{1}{2}}u\right\|^2 - \|u_t\|^2 + \frac{1}{4}(h \diamond B^{\frac{1}{2}}u)(t) - 2N_3\tau \int_0^1 \|z(\rho,t)\|^2 ds.$$

which yields (4.3). □

The stability results is ensuring by the following theorem.

Theorem 4.1. *Assume that (A1)-(A3) hold. Then there exist a positive constants k_1, k_2, k_3 and k_4 such that the solution of (1.1) satisfies, for all $t \ge t_1$,*

$$E(t) \le k_1 e^{-k_2 \int_{t_1}^t \zeta(s)ds}, \quad \text{if } G \text{ is linear} \tag{4.4}$$

$$E(t) \le k_4 G_1^{-1}\left(k_3 \int_{t_1}^t \zeta(s)ds\right), \quad \text{if } G \text{ is nonlinear}, \tag{4.5}$$

where $G_1(t) = \int_t^r \frac{ds}{sG'(s)}$, which is strictly decreasing and convex on $(0,r]$, with $\lim_{t\to 0} G_1(t) = +\infty$.

Proof: By using (2.9) and (2.15), we conclude that, for any $t \ge t_1$,

$$\int_0^{t_1} h(s)\left\|B^{\frac{1}{2}}(u(t)-u(t-s))\right\|^2 ds \le \frac{-h(0)}{\delta_1}\int_0^{t_1} h'(s)\left\|B^{\frac{1}{2}}(u(t)-u(t-s))\right\|^2 ds \le -cE'(t). \tag{4.6}$$

Inserting this estimate in (4.3) and introducing the following function F which is equivalent to E by

$$F(t) = L(t) + cE(t).$$

On the other hand, we have for some constant $m > 0$ and for all $t \ge t_1$,

$$\begin{aligned} L'(t) &\le -3(1-l)\left\|A^{\frac{1}{2}}u\right\|^2 - \|u_t\|^2 + \frac{1}{4}(h \diamond B^{\frac{1}{2}}u)(t) \\ &\le -mE(t) + c(h \diamond B^{\frac{1}{2}}u)(t) \\ &\le -mE(t) - cE'(t) + c\int_{t_1}^t h(s)\left\|B^{\frac{1}{2}}(u(t)-u(t-s))\right\|^2 ds, \end{aligned}$$

which gives

$$F'(t) \le -mE(t) + c\int_{t_1}^t h(s)\left\|B^{\frac{1}{2}}(u(t)-u(t-s))\right\|^2 ds. \tag{4.7}$$

Let consider the following two cases.

Case 1: G is linear.

By multiplying (4.7) by ζ and using (A2) and (2.15), we get

$$\begin{aligned} \zeta(t)F'(t) &\le -m\zeta(t)E(t) + c\zeta(t)\int_{t_1}^t h(s)\left\|B^{\frac{1}{2}}(u(t)-u(t-s))\right\|^2 ds \\ &\le -m\zeta(t)E(t) + c\int_{t_1}^t \zeta(t)h(s)\left\|B^{\frac{1}{2}}(u(t)-u(t-s))\right\|^2 ds \\ &\le -m\zeta(t)E(t) - c\int_{t_1}^t h'(s)\left\|B^{\frac{1}{2}}(u(t)-u(t-s))\right\|^2 ds \\ &\le -m\zeta(t)E(t) - cE'(t), \end{aligned}$$

by using the fact ζ is nonincreasing, we deduce

$$(\zeta F + cE)'(t) \le -m\zeta(t)E(t), \quad \forall t \ge t_1.$$

Consequently, by integrating this last over (t_1, t) and using the fact that $\zeta F + cE \sim E$, we obtain

$$E(t) \le k_1 e^{-k_2 \int_{t_1}^{t} \zeta(s)ds} \quad \forall t \ge t_1,$$

where k_1 and k_2 be a positive constants.

Case 2: G is nonlinear.

Firstly, we introduce the following function

$$L_1(t) = L(t) + I_4(t),$$

which is nonnegative by using Lemmas (3.4) and (4.2). Moreover, it satisfies

$$L_1'(t) \le -(1-l)\left\|A^{\frac{1}{2}}u\right\|^2 - \|u_t\|^2 - \frac{1}{4}(h \diamond B^{\frac{1}{2}}u)(t) \le -\beta E(t).$$

Hence,

$$\beta \int_{t_1}^{t} E(s)ds \le L_1(t_1) - L_1(t) \le L_1(t_1),$$

this gives

$$\int_0^{+\infty} E(s)ds < +\infty. \tag{4.8}$$

Let now define the function I by

$$I(t) = p\int_{t_1}^{t} \left\|B^{\frac{1}{2}}(u(t) - u(t-s))\right\|^2 ds, \quad \forall t \ge t_1,$$

where p be a positive constant, so $I(t) > 0$, for all $t \ge t_1$, otherwise (4.7) leads to an exponential decay. Furthermore, by a particular choice of p so that

$$I(t) < 1. \tag{4.9}$$

We also define the function λ by

$$\lambda(t) = -\int_{t_1}^{t} h'(s)\left\|B^{\frac{1}{2}}(u(t) - u(t-s))\right\|^2 ds, \quad \forall t \ge t_1,$$

for t_1 small enough and by (2.15), we observe that

$$\lambda(t) \le -cE'(t). \tag{4.10}$$

Since G is strictly convex on $(0, r]$ and $G(0) = 0$, then

$$G(\theta x) \le \theta\, G(x), \quad \text{for some} \quad \theta \in [0,1] \quad \text{and} \quad x \in (0, r].$$

By using the assumption (A2), (4.9) and Jensen's inequality, we get

$$\begin{aligned}
\lambda(t) &= \frac{1}{p\ I(t)} \int_{t_1}^{t} I(t)\,(-h'(s))\,p\ \left\| B^{\frac{1}{2}}(u(t)-u(t-s)) \right\|^2 ds \\
&\geq \frac{1}{p\ I(t)} \int_{t_1}^{t} I(t)\zeta(s)G(h(s))p\ \left\| B^{\frac{1}{2}}(u(t)-u(t-s)) \right\|^2 ds \\
&\geq \frac{\zeta(t)}{p\ I(t)} \int_{t_1}^{t} G(I(t)h(s))p\ \left\| B^{\frac{1}{2}}(u(t)-u(t-s)) \right\|^2 ds \\
&\geq \frac{\zeta(t)}{p} G\left(\frac{1}{I(t)} \int_{t_1}^{t} I(t)h(s)p\ \left\| B^{\frac{1}{2}}(u(t)-u(t-s)) \right\|^2 ds \right) \\
&= \frac{\zeta(t)}{p} G\left(p\ \int_{t_1}^{t} h(s) \left\| B^{\frac{1}{2}}(u(t)-u(t-s)) \right\|^2 ds \right) \\
&= \frac{\zeta(t)}{p} \overline{G}\left(p\ \int_{t_1}^{t} h(s) \left\| B^{\frac{1}{2}}(u(t)-u(t-s)) \right\|^2 ds \right)
\end{aligned}$$

where $\overline{G}$ is an extension of G, which is strictly increasing and strictly convex C^2 on $[0,+\infty)$, see Remark (2.1). The use of this fact and since ζ is a positive nonincreasing function, we obtain

$$\int_{t_1}^{t} h(s) \left\| B^{\frac{1}{2}}(u(t)-u(t-s)) \right\|^2 ds \leq \frac{1}{p} \left(\overline{G}\right)^{-1} \left(\frac{p\ \lambda(t)}{\zeta(t)} \right),$$

and (4.7) becomes

$$F'(t) \leq -m\ E(t) + c\ \left(\overline{G}\right)^{-1} \left(\frac{p\ \lambda(t)}{\zeta(t)} \right), \quad \forall t \geq t_1. \tag{4.11}$$

Let $0 < r_1 < r$, then we define the functional F_1 by

$$F_1(t) = \overline{G}'\left(r_1 \frac{E(t)}{E(0)} \right) F(t) + E(t),$$

by using (4.11) and the fact that $E' \leq 0$, $G' > 0$ and $G'' > 0$, we conclude that F_1 is equivalent to E and

$$\begin{aligned}
F_1'(t) &= r_1 \frac{E'(t)}{E(0)} \overline{G}''\left(r_1 \frac{E(t)}{E(0)} \right) F(t) + \overline{G}'\left(r_1 \frac{E(t)}{E(0)} \right) F'(t) + E'(t), \\
&\leq -m\ E(t)\overline{G}'\left(r_1 \frac{E(t)}{E(0)} \right) + c\ \overline{G}'\left(r_1 \frac{E(t)}{E(0)} \right) \left(\overline{G}\right)^{-1} \left(\frac{p\ \lambda(t)}{\zeta(t)} \right) + E'(t).
\end{aligned} \tag{4.12}$$

Let $\overline{G}^*$ be the convex conjugate of G in the sense of Young (see [4] pp. 61-64), which is given by

$$\overline{G}^*(s) = s \left(\overline{G}'\right)^{-1}(s) - \overline{G}\left[\left(\overline{G}'\right)^{-1}(s) \right] \tag{4.13}$$

and it satisfies the following Young's inequality

$$AB \leq \overline{G}^*(A) + \overline{G}(B). \tag{4.14}$$

By taking

$$A = \overline{G}'\left(r_1 \frac{E(t)}{E(0)} \right) \quad \text{and} \quad B = \left(\overline{G}\right)^{-1} \left(\frac{p\ \lambda(t)}{\zeta(t)} \right),$$

and using (4.14), we obtain

$$\begin{aligned}
\overline{G}'\left(r_1 \frac{E(t)}{E(0)} \right) \left(\overline{G}\right)^{-1} \left(\frac{p\ \lambda(t)}{\zeta(t)} \right) &\leq \overline{G}^*\left(\overline{G}'\left(r_1 \frac{E(t)}{E(0)} \right) \right) + \frac{p\ \lambda(t)}{\zeta(t)} \\
&\leq r_1 \frac{E(t)}{E(0)} \overline{G}'\left(r_1 \frac{E(t)}{E(0)} \right) - \overline{G}\left(r_1 \frac{E(t)}{E(0)} \right) + \frac{p\ \lambda(t)}{\zeta(t)},
\end{aligned}$$

then, using the fact that $\overline{G}$ is nonnegative, we get

$$\overline{G}'\left(r_1\,\frac{E(t)}{E(0)}\right)\left(\overline{G}\right)^{-1}\left(\frac{p\,\lambda(t)}{\zeta(t)}\right)\leq r_1\,\frac{E(t)}{E(0)}\overline{G}'\left(r_1\,\frac{E(t)}{E(0)}\right)+\frac{p\,\lambda(t)}{\zeta(t)}. \tag{4.15}$$

Inserting (4.15) in (4.12), we arrive at

$$F_1'(t)\leq -m\,E(t)\overline{G}'\left(r_1\,\frac{E(t)}{E(0)}\right)+c\,r_1\,\frac{E(t)}{E(0)}\overline{G}'\left(r_1\,\frac{E(t)}{E(0)}\right)+c\,\frac{p\,\lambda(t)}{\zeta(t)}+E'(t),$$

then, multiplying by $\zeta(t)$ and using (4.10) and the fact that, as $r_1\frac{E(t)}{E(0)}<r$, $\overline{G}'\left(r_1\frac{E(t)}{E(0)}\right)=G'\left(r_1\frac{E(t)}{E(0)}\right)$ to obtain

$$\zeta(t)F_1'(t)\leq -m\,\zeta(t)E(t)G'\left(r_1\,\frac{E(t)}{E(0)}\right)+c\,r_1\,\zeta(t)\frac{E(t)}{E(0)}G'\left(r_1\,\frac{E(t)}{E(0)}\right)-c\,E'(t).$$

On the other hand, the functional $F_2=\zeta F_1+cE$ is equivalent to E which means for some γ_1 and γ_2, we have

$$\gamma_1 F_2(t)\leq E(t)\leq \gamma_2 F_2(t), \tag{4.16}$$

and under a suitable choice of r_1 and for a positive constant k, we find

$$F_2'(t)\leq -k\,\zeta(t)\frac{E(t)}{E(0)}G'\left(r_1\,\frac{E(t)}{E(0)}\right)=-k\,\zeta(t)G_2\left(\frac{E(t)}{E(0)}\right), \tag{4.17}$$

where $G_2(t)=tG'(r_1t)$. Since $G_2'(t)=G'(r_1t)+r_1tG''(r_1t)$, and using the strict convexity of G on $(0,r]$, we find that $G_2,\ G_2'>0$ on $(0,1]$. Finally, with

$$R(t)=\gamma_1\frac{F_2(t)}{E(0)},$$

then, by using (4.16) and (4.17), we have $R\sim E$ and for some positive constant k_3, (4.17) gives

$$R'(t)\leq -k_3\,\zeta(t)G_2(R(t)),\quad \forall t\geq t_1.$$

A simple integration over (t_1,t), we find

$$\int_{t_1}^{t}\frac{-R'(s)}{G_2\left(R(s)\right)}ds\geq k_3\int_{t_1}^{t}\zeta(s)ds.$$

Since $r_1R(t_1)<r$, we obtain

$$G_1(r_1R(t))=\int_{r_1R(t)}^{r_1R(t_1)}\frac{ds}{s\,G'(s)}\geq k_3\int_{t_1}^{t}\zeta(s)ds.$$

Using the fact that G_1 is strictly decreasing function on $(0,r]$ and $\lim_{t\to 0}G_1(t)=+\infty$. Then

$$R(t)\leq\frac{1}{r_1}G_1^{-1}\left(k_3\int_{t_1}^{t}\zeta(s)ds\right),$$

consequently, by using the fact that R is equivalent to E, the stability estimate (4.5) is established. This completes the proof. □

Remark 4.1. *The decay rate of E given by (4.1) is optimal in the sense that it's consistent with the decay rate of h given by (2.4) where (4.4) and (4.5) provide the best decay rates expected under the very general assumption on h.*

Example 4.1. *Assume that (A2) holds with $G(s) = s^p$, where $1 \leq p < 2$. Then, the decay rate of E is given by*

$$E(t) \leq \begin{cases} \tilde{c}e^{-\tilde{c}_1 \int_0^t \zeta(s)ds}, & if \quad p = 1 \\ \tilde{c}_2 \left(1 + \int_0^t \zeta(s)ds\right)^{\frac{-1}{p-1}}, & if \quad 1 < p < 2. \end{cases} \tag{4.18}$$

In this example, we can show that h not be necessarily of exponential or polynomial decay but under general assumption on the relaxation function h which gives a much larger class of functions h, the uniform stability of the system (1.1) is established with an explicit formula of the decay rates of the energy.

For more examples of relaxation functions and the decay rates of the energy, see [29,6,8].

5. Applications

We can seek our result in many problems. In this section, we present only three applications. Let Ω be a bounded and regular domain of $\mathbb{R}^n$, with $n \geq 1$.

5.1. More general model

Our first application is the abstract system (1.1) with more general form

$$\begin{cases} u_{tt}(t) + Au(t) - \int_0^t h(t-s)Bu(s)ds + C_1C_1^*u_t(t) + C_2C_2^*u_t(t-\tau) = F(u(t)), & t \in (0,+\infty), \\ C_2^*u_t(t-\tau) = f_0(t-\tau) & t \in (0,\tau), \\ u(0) = u_0, \quad u_t(0) = u_1, & \end{cases} \tag{5.1}$$

where $C_i : W_i \to H$ be bounded linear operators and W_i be real Hilbert spaces with norm $\|.\|_{W_i}$. Moreover, we assume that

$$\exists\, 0 < \mu < 1, \quad \|C_2^*u\|_{W_2} \leq \mu \|C_1^*u\|_{W_1}, \quad \forall u \in H. \tag{5.2}$$

5.2. Wave equations

We consider the following equation

$$\begin{cases} u_{tt}(t) + Au(t) + \int_0^t h(t-s)\Delta u(s)ds + \mu_1 u_t(t) + \mu_2 u_t(t-\tau) = |u(t)|^\gamma u(t), & t \in (0,+\infty), \\ u(x,t) = 0, & x \in \partial\Omega, \\ u(0) = u_0, \quad u_t(0) = u_1, & x \in \Omega, \\ u_t(t-\tau) = f_0(t-\tau) & t \in (0,\tau), \end{cases} \tag{5.3}$$

with initial data $(u_0, u_1, f_0) \in [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega) \times H^1(-\tau, 0; L^2(\Omega))$ and γ be a positive number. Our results hold with $H = L^2(\Omega)$ and the operators A, B are given by

$$A : D(A) \longrightarrow H \; : u \mapsto -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right),$$

$$B : D(B) \longrightarrow H \; : u \mapsto -\Delta u,$$

where $D(A) = D(B)) = H^2(\Omega) \cap H_0^1(\Omega)$. $a_{ij} \in C^1(\overline{\Omega})$, is symmetric and

$$\exists a_0 > 0, \quad \sum_{i,j=1}^n a_{ij}(x)\zeta_j\zeta_i \geq a_0|\zeta|^2, \quad x \in \overline{\Omega}, \; \zeta = (\zeta_1, \cdots, \zeta_n) \in \mathbb{R}^n.$$

The function $F(u) = u|u|^\gamma$ satisfies the assumption (A3) with $0 < \gamma < \frac{2}{n-2}$

5.3. Coupled systems

We can also consider the following coupled systems with Dirichlet condition:

$$\begin{cases} w_{tt}(t) - \alpha \Delta w(t) + \displaystyle\int_0^t h(t-s) div(a_1(x)\nabla w(s))ds + \mu_1 w_t(t) + \mu_2 w_t(t-\tau) & \\ \qquad + dv(t) = f_1(w(t)), & t \in (0,+\infty), \\ v_{tt}(t) - \beta \Delta v(t) + \displaystyle\int_0^t h(t-s) div(a_2(x)\nabla v(s))ds + \mu_1 v_t(t) + \mu_2 v_t(t-\tau) & \\ \qquad + dw(t) = f_2(v(t)), & t \in (0,+\infty), \\ w(x,t) = v(x,t) = 0, & x \in \partial\Omega, \\ w(0) = w_0, \quad v(0) = v_0, & x \in \Omega, \\ w_t(0) = w_1, \quad v_t(0) = v_1, & x \in \Omega, \\ w_t(t-\tau) = l_0(t-\tau), \quad v_t(t-\tau) = m_0(t-\tau) & t \in (0,\tau), \end{cases} \tag{5.4}$$

where α and β are positive constants, $a_1,\ a_2 \in C^1(\Omega),\ a_1(x),\ a_2(x) > 0$. The above system is equivalent to (1.1) where $u = (w, v)$, $f_0 = (l_0, m_0)$ and $H = (L^2(\Omega))^2$ with

$$\langle (w_1, v_1), (w_2, v_2) \rangle = \int_\Omega w_1 w_2 + v_1 v_2 dx.$$

We take $D(A) = D(B)) = (H^2(\Omega) \cap H_0^1(\Omega))^2$ and the operators A, B are given by

$$Au = -(\alpha \Delta w, \beta \Delta v) + d(v, w),$$

$$Bu = -(div(a_1(x)\nabla w), div(a_2(x)\nabla w)).$$

The function $F_2(u(t)) = (f_1(w(t)), f_2(v(t)))$ satisfies (A3).

Conclusion

In conclusion, this work improves the previous results; we have considered a semilinear abstract second-order viscoelastic equation with time delay. For a much larger class of kernel functions, we have established explicit and general decay results of the energy solution by introducing a suitable Lyapunov functional and some properties of the convex functions. Moreover, we have given some applications in particular case of Hilbert space.

Acknowledgements

The authors are highly grateful to the anonymous referee for his/her valuable comments and suggestions for the improvement of the paper. This research work is supported by the General Direction of Scientific Research and Technological Development (DGRSDT), Algeria.

References

1. Alabau-Boussouira F, Cannarsa P and Sforza D. Decay estimates for second order evolution equations with memory. J. Funct. Anal. **254**(2008), 1342-1372.
2. F. Alabau-Boussouira, P. Cannarsa. A general method for proving sharp energy decay rates for memory-dissipative evolution equations. C. R. Math. **347**(2009), no. 15-16, 867-872.
3. K. Ammari, S. Nicaise and C. Pignotti. Feedback boundary stabilization of wave equations with interior delay. Syst. Control Lett. **59**(2010), 623-628.
4. V. Arnold. Mathematical methods of classical mechanics. New York: Springer, 1989.
5. A. Bàtkai, S. Piazzera. Semigroups for delay equations. Research Notes in Mathematics, 10. AK Peters, Ltd., Wellesley, MA, (2005).
6. F. Belhannache, M. M. Algharabli and S.A. Messaoudi. Asymptotic Stability for a Viscoelastic Equation with Nonlinear Damping and Very General Type of Relaxation Functions. J. Dyna. Control Syst. **26**(2020), no. 1, 45-67.
7. A. Benaissa, A. K. Benaissa and S. A. Messaoudi. Global existence and energy decay of solutions for the wave equation with a time varying delay term in the weakly nonlinear internal feedbacks. J. Math. Phys. **53**(2012), no. 12, 123514

8. Y. Boukhatem, B. Benabderrahmane. General decay for a viscoelastic equation of variable coefficients with a time-varying delay in the boundary feedback and acoustic boundary conditions. ACTA Math. Sci. **37**(2017), no. 5, 1453-1471.

9. Y. Boukhatem, B. Benabderrahmane. General Decay for a Viscoelastic Equation of Variable Coefficients in the Presence of Past History with Delay Term in the Boundary Feedback and Acoustic Boundary Conditions. ACTA Appl. Math. **154**(2018), no. 1, 131-152.

10. Y. Boukhatem, B. Benabderrahmane, Existence and exponential decay of solutions for the variable coefficients wave equations. Analele Universitatii Oradea Fasc. Matematica, Tom XXIII (2), (2016), 93-108.

11. M.M. Cavalcanti,A.D. Cavalcanti, I. Lasiecka and X. Wang. Existence and sharp decay rate estimates for a von Karman system with long memory. NA: Real World Appl. **22**(2015), 289-306.

12. H. Chellaoua, Y. Boukhatem. Optimal decay for second-order abstract viscoelastic equation in Hilbert spaces with infinite memory and time delay. Math. Meth. Appl. Sci. **44**(2021), no. 2, 2071-2095.

13. H. Chellaoua, Y. Boukhatem. Stability results for second-order abstract viscoelastic equation in Hilbert spaces with time-varying delay. Z. Angew. Math. Phys. **72**(2021), no. 2, 1-18.

14. CM. Dafermos. Asymptotic stability in viscoelasticity. Arch. Rational Mech. An. **37**(1970), 297-308.

15. R. Datko. Two questions concerning the boundary control of certain elastic systems, J. Differ. Equa. **92**(1991), no. 1, 27-44.

16. M. Fabrizio, S. Polidoro, Asymptotic decay for some differential systems with fading memory. Appl. Anal. **81**(2002), no. 6, 1245-1264.

17. J. H. Hassan, S. A. Messaoudi. General decay rate for a class of weakly dissipative second-order systems with memory. Math. Meth. Appl. Sci. **42**(2019), no. 8, 2842-2853.

18. J. R. Kang. General decay for viscoelastic plate equation with p-Laplacian and time-varying delay. Bound. Value Probl. **2018**(2018), no. 1, 1-11.

19. M. Kirane, B. Said-Houari, Existence and asymptotic stability of a viscoelastic wave equation with a delay. Z. Angew. Math. Phys. **62**(2011), 1065-1082.

20. I. Lasiecka, S. A. Messaoudi and M. I. Mustafa. Note on intrinsic decay rates for abstract wave equations with memory. J Math Phys, **54**(2013), no. 3, 031504.

21. Z. Liu, C.Gao and Z. B. Fang. A general decay estimate for the nonlinear transmission problem of weak viscoelastic equations with time-varying delay. J. Math. Phys. **60**(2019), no. 10, 101507.

22. S. A. Messaoudi, General decay of solutions of a viscoelastic equation. J. Math. Anal. Appl. **341**(2008), no. 2, 1457-1467.

23. Messaoudi S. General stability in viscoelasticity, viscoelastic and viscoplastic materials, Mohamed El-Amin. IntechOpen (2016). https://doi.org/10.5772/64217.

24. J. E. Muñoz Rivera, A. Peres Salvatierra, Asymptotic behaviour of the energy in partially viscoelastic materials. Q. Appl. Math. **59**(2001), no. 3, 557-578.

25. J. E. Muñoz Rivera, M. G. Naso and F. M. Vegni, Asymptotic behavior of the energy for a class of weakly dissipative second-order systems with memory. J. Math. Anal. Appl. **286**(2003), no. 2, 692-704.

26. M. I. Mustafa. On the control of the wave equation by memory-type boundary condition. Discrete Conti. Dyn-A, **35**(2014), no. 3, 1179.

27. M. I. Mustafa. Memory-type plate system with nonlinear delay. Adv. Pure Appl. Math. **8**(2017), no. 4, 227-240.

28. M. I. Mustafa. Asymptotic stability for the second order evolution equation with memory. J. Dyna. Control Syst. **25**(2018), no. 2, 263-273.

29. M. I. Mustafa. Optimal decay rates for the abstract viscoelastic equation. J. Evol. Equ. (2019), 1-17.

30. M. I. Mustafa. Optimal decay rates for the viscoelastic wave equation. Math. Meth. Appl. Sci. (2017), 1-13.

31. S. Nicaise, C. Pignotti. Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. SIAM J. Control Optim. **45**(2006), 1561-1585.

32. S. Nicaise, J. Valein, Stabilization of second order evolution equations with unbounded feedback with delay. ESAIM Control Optim. Calc. Var. **16**(2010), no. 2, 420-456.

33. V. Pata, Exponential stability in linear viscoelasticity. Q. Appl. Math. **64**(2006), no. 3, 499-513.

34. T. J. Xiao, J. Liang. Coupled second order semilinear evolution equations indirectly damped via memory effects. J. Differ. Equa. **254**(2013), no. 5, 2128-2157.

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