



Some Existence Results on Positive Solutions for an Iterative Second-order Boundary-value Problem with Integral Boundary Conditions

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ABSTRACT: The present paper is devoted to study a nonlinear second order differential equation with iterative source term and integral boundary conditions. The utilization of some suitable fixed point theorems ultimately led us to establish some sufficient conditions that guarantee the existence, uniqueness and continuous dependence of positive bounded solutions. The obtained results are illustrated by an example.

Key Words: Iterative differential equation, Fixed point theorem, Integral boundary conditions.

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1. Introduction

Iterative differential equations constitute a particular type of the so-called delay differential equations where the delays which depend on both the time t and the state variable x are defined implicitly by the iterates. To the best of our knowledge, the concept of iterative equations appeared for the first time in 1815 by *Babbage* [2] in his essay towards the calculus of function in which he was interested in finding a function equalling its n -th iterate. But the first step of the theory for iterative differential equations is said to have begun in the late twentieth century see [14].

Recently they have been attracting great interest. Many researchers have concentrated on studying first order iterative differential equations by different approaches such as fixed point theory, Picard’s successive approximation and the technique of nonexpansive operators. But the literature related to the equations of higher order is limited since the presence of the iterates increases the difficulty of studying them. This motivates us to investigate the following second order iterative differential equation:

$$x''(t) + f(x^{[0]}(t), x^{[1]}(t), x^{[2]}(t), \dots, x^{[n]}(t)) = 0, \quad 0 < t < b, \quad (1.1)$$

$$x(0) = 0, \quad \alpha \int_0^\eta x(s) ds = x(b) \quad \text{with } \eta \in (0, b), \quad \alpha \in \mathbb{R}^*, \quad (1.2)$$

where $x^{[0]}(t) = t, x^{[1]}(t) = x(t), \dots, x^{[n]}(t) = x^{[n-1]}(x(t))$ and $f : [0, b] \times \mathbb{R}^n \rightarrow [0, +\infty)$ is a continuous function with respect to its arguments. Equation (1.1) describes diffusion phenomena with a source or a reaction term. For instance, in thermal conduction, it can be interpreted as the one-dimensional heat conduction equation which models the steady-states of a heated bar of length b with a controller at $x = b$ that adds or removes heat according to a sensor, while the left endpoint is maintained at 0°C and f is the distributed temperature source function depending on delayed temperatures. We refer the interested reader to [9,10,13] and the references therein for more details.

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We would like to mention some recent results on this type of equations with integral boundary conditions which arises in many areas of applied mathematics such as physics, electrodynamics, infectious diseases, population dynamics and many others.

In [12], *G. Infante* used the theory of fixed point index for treating the following equation:

$$\begin{aligned} u''(t) + f(t, u(t)) &= 0, \quad 0 < t < 1, \\ u'(0) = 0, \quad u(1) &= \int_0^1 \gamma(s, u(s)) ds. \end{aligned}$$

In [8], by virtue of Krasnoselskii's fixed point theorem, *A. Boucherif* investigated the existence of positive solutions of the following problem:

$$\begin{aligned} y''(t) &= f(t, y(t)), \quad 0 < t < 1, \\ y(0) - ay'(0) &= \int_0^1 g_0(s) y(s) ds, \\ y(1) - by'(1) &= \int_0^1 g_1(s) y(s) ds, \end{aligned}$$

where $a, b \geq 0$, $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ and $g_0, g_1 : [0, 1] \rightarrow [0, +\infty)$ are continuous functions.

In [4], *M. Benchohra* et al. used nonlinear alternative Leray Schauder type and Banach contraction principle to establish the existence of solutions of the following second-order boundary value problem:

$$\begin{aligned} -y''(t) &= f(t, y(t)), \quad \text{a.e } 0 < t < 1, \\ y(0) = 0, \quad y(1) &= \int_0^1 g(s) y(s) ds, \end{aligned}$$

where $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a given function and $g : [0, 1] \rightarrow \mathbb{R}$ is an integrable function. In [11], by using Schauder's fixed point theorem, Juan Galvis, Edixon M. Rojas, Alexander V. Sinityn proved the existence of positive solutions of the problem

$$\begin{aligned} -u''(t) + a(t) f(u(t)) &= 0, \quad 0 < t < \gamma, \\ u(0) = 0, \quad u(\gamma) &= \alpha \int_0^\eta u(s) ds \quad \text{with } \eta \in (0, \gamma). \end{aligned}$$

This work is a continuation of the above mentioned works, some other works on positivity and boundary value problems [1,3] and our recent papers on iterative problems (see [5,6]). Our main contribution to this important area is to show that the fixed point theory can be applied successfully to iterative problems with integral boundary conditions.

We briefly outline the structure of the paper. In the beginning of it, we provide some basic concepts which are useful in the sequel. In the third section we prove our main results by means of Banach and Schauder's fixed point theorems. Finally, an illustrative example is given in the last section.

2. Preliminaries

Before attempting to prove our desired results, we start by defining a subset of $\mathcal{C}([0, b], \mathbb{R})$ that it will be able to contain the iterates and the solutions if they exist. To this end, we define a subset \mathcal{CB}_{Int} of $\mathcal{C}([0, b], \mathbb{R})$ as follows:

$$\mathcal{CB}_{Int} = \left\{ x \in \mathcal{C}([0, b], \mathbb{R}) : x(0) = 0, \alpha \int_0^\eta x(s) ds = x(b), \alpha \in \mathbb{R}^*, \eta \in (0, b) \right\}.$$

It's obvious that \mathcal{CB}_{Int} , equipped with the supremum norm

$$\|x\| = \sup_{t \in [0, b]} |x(t)|,$$

is a Banach space as a closed linear subspace of the Banach space $\mathcal{C}([0, b], \mathbb{R})$.
For $0 \leq L \leq b$ and $M \geq 0$, let

$$\mathcal{CB}_{Int}(L, M) = \{x \in \mathcal{CB}_{Int}, 0 \leq x \leq L, |x(t_2) - x(t_1)| \leq M|t_2 - t_1|, \forall t_1, t_2 \in [0, b]\},$$

then $\mathcal{CB}_{Int}(L, M)$ is a closed convex and bounded subset of \mathcal{CB}_{Int} .

Throughout this paper we assume that the function $f(t, x_1, x_2, \dots, x_n)$ is globally Lipschitz in x_1, \dots, x_n . i.e., there exist n positive constants c_1, c_2, \dots, c_n such that

$$|f(t, x_1, \dots, x_n) - f(t, y_1, \dots, y_n)| \leq \sum_{i=1}^n c_i \|x_i - y_i\|, \quad (2.1)$$

and we introduce the following constants

$$\rho = \sup_{s \in [0, b]} |f(s, 0, 0, \dots, 0)|, \quad \zeta = \rho + L \sum_{i=1}^n c_i \sum_{j=0}^{i-1} M^j.$$

Lemma 2.1. [11] *Let $2T \neq \alpha\eta^2$, then for $y \in \mathcal{C}([0, T], [0, +\infty))$, the problem*

$$x''(t) + y(t) = 0, \quad (2.2)$$

$$x(0) = 0, \quad \alpha \int_0^\eta x(t) dt = x(T), \quad \eta \in (0, T), \quad \alpha \neq 0, \quad (2.3)$$

has a unique solution given by

$$x(t) = \frac{2t}{2T - \alpha\eta^2} \int_0^T (T-s)y(s) ds - \frac{\alpha t}{2T - \alpha\eta^2} \int_0^\eta (\eta-s)^2 y(s) ds - \int_0^t (t-s)y(s) ds.$$

Lemma 2.2. [11] *Let $0 < \alpha \leq \frac{2}{\eta^2}$. If $y \in \mathcal{C}[0, T]$ and $y(t) \geq 0$ on $(0, T)$, then the unique solution of the problem (2.2)-(2.3) satisfies $u(t) \geq 0$ for $t \in [0, T]$.*

Lemma 2.3. [16] *For any $\varphi, \psi \in \mathcal{CB}_{Int}(L, M)$,*

$$\|\varphi^{[m]} - \psi^{[m]}\| \leq \sum_{j=0}^{m-1} M^j \|\varphi - \psi\|, \quad m = 1, 2, \dots$$

Theorem 2.1. [15](Schauder) *Let \mathbb{M} be a non-empty compact convex subset of a Banach space $(\mathbb{X}, \|\cdot\|)$ and let $\mathcal{A} : \mathbb{M} \rightarrow \mathbb{M}$ be a continuous mapping. Then \mathcal{A} has a fixed point in \mathbb{M} .*

3. Main results

3.1. Existence results

The aim of this section is to convert our boundary-value problem (1.1)-(1.2) to a fixed point problem where the proof of our main result relies on Schauder's fixed point Theorem.

By virtue of Lemma 2.1, we define an operator $\mathcal{A} : \mathcal{CB}_{Int}(L, M) \rightarrow \mathcal{CB}_{Int}$ as follows:

$$\begin{aligned} (\mathcal{A}\varphi)(t) &= \frac{2t}{2b - \alpha\eta^2} \int_0^b (b-s)f(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)) ds \\ &\quad - \frac{\alpha t}{2b - \alpha\eta^2} \int_0^\eta (\eta-s)^2 f(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)) ds \\ &\quad - \int_0^t (t-s)f(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)) ds. \end{aligned} \quad (3.1)$$

It follows that, φ is a solution of the boundary-value problem (1.1)-(1.2) if and only if φ is a fixed point of the operator \mathcal{A} . Using the Arzelà-Ascoli theorem, we can show that the closed subset $\mathcal{CB}_{Int}(L, M)$ of \mathcal{CB}_{Int} is compact since it is an uniformly bounded and equicontinuous part of the space $\mathcal{C}([0, b], \mathbb{R})$. So, to prove the existence of solutions for (1.1)-(1.2), we must prove that \mathcal{A} is well defined, continuous and $\mathcal{A}(\mathcal{CB}_{Int}(L, M)) \subset \mathcal{CB}_{Int}(L, M)$.

Lemma 3.1. *Let $2b \neq \alpha\eta^2$, then operator $\mathcal{A} : \mathcal{CB}_{Int}(L, M) \longrightarrow \mathcal{CB}_{Int}$ given by (3.1) is well defined.*

Proof: To show that \mathcal{A} is well defined it suffices to show that $(\mathcal{A}\varphi)(0) = 0$ and $\alpha \int_0^\eta (\mathcal{A}\varphi) dt = (\mathcal{A}\varphi)(b)$

for all $\varphi \in \mathcal{CB}_{Int}(L, M)$. Clearly $(\mathcal{A}\varphi)(0) = 0$.

Let $\varphi \in \mathcal{CB}_{Int}(L, M)$, we have

$$\begin{aligned} (\mathcal{A}\varphi)(b) &= \frac{2b}{2b - \alpha\eta^2} \int_0^b (b-s)f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) ds \\ &\quad - \frac{\alpha b}{2b - \alpha\eta^2} \int_0^\eta (\eta-s)^2 f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) ds \\ &\quad - \int_0^b (b-s)f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) ds \\ &= \frac{\alpha\eta^2}{2b - \alpha\eta^2} \int_0^b (b-s)f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) ds \\ &\quad - \frac{\alpha b}{2b - \alpha\eta^2} \int_0^\eta (\eta-s)^2 f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) ds, \end{aligned}$$

and

$$\begin{aligned} \alpha \int_0^\eta (\mathcal{A}\varphi)(t) dt &= \alpha \int_0^\eta \left(\frac{2t}{2b - \alpha\eta^2} \int_0^b (b-s)f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) ds \right) dt \\ &\quad + \alpha \int_0^\eta \left(-\frac{\alpha t}{2b - \alpha\eta^2} \int_0^\eta (\eta-s)^2 f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) ds \right) dt \\ &\quad + \alpha \int_0^\eta \left(-\int_0^t (t-s)f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) ds \right) dt, \end{aligned}$$

which implies

$$\begin{aligned} \alpha \int_0^\eta (\mathcal{A}\varphi)(t) dt &= \frac{\alpha\eta^2}{2b - \alpha\eta^2} \int_0^b (b-s)f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) ds \\ &\quad - \frac{1}{2} \frac{\alpha^2\eta^2}{2b - \alpha\eta^2} \int_0^\eta (\eta-s)^2 f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) ds \\ &\quad - \frac{1}{2} \alpha \int_0^\eta (\eta-s)^2 f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) ds \\ &= \frac{\alpha\eta^2}{2b - \alpha\eta^2} \int_0^b (b-s)f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) ds \\ &\quad - \frac{\alpha b}{2b - \alpha\eta^2} \int_0^\eta (\eta-s)^2 f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) ds. \end{aligned}$$

So $\alpha \int_0^\eta (\mathcal{A}\varphi)(t) dt = (\mathcal{A}\varphi)(b)$. Consequently \mathcal{A} is well defined. \square

Lemma 3.2. *Suppose that condition (2.1) holds. Then the operator $\mathcal{A} : \mathcal{CB}_{Int}(L, M) \longrightarrow \mathcal{CB}_{Int}$ given by (3.1) is continuous.*

Proof: For $\varphi, \psi \in \mathcal{CB}_{Int}(L, M)$, we have

$$\begin{aligned}
|(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)| &\leq \frac{2t}{|2b - \alpha\eta^2|} \int_0^b (b-s) \left| f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) \right. \\
&\quad \left. - f\left(\psi^{[0]}(s), \psi^{[1]}(s), \psi^{[2]}(s), \dots, \psi^{[n]}(s)\right) \right| ds \\
&\quad + \frac{|\alpha|t}{|2b - \alpha\eta^2|} \int_0^\eta (\eta-s)^2 \left| f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) \right. \\
&\quad \left. - f\left(\psi^{[0]}(s), \psi^{[1]}(s), \psi^{[2]}(s), \dots, \psi^{[n]}(s)\right) \right| ds \\
&\quad + \int_0^t |(t-s)| \left| f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) \right. \\
&\quad \left. - f\left(\psi^{[0]}(s), \psi^{[1]}(s), \psi^{[2]}(s), \dots, \psi^{[n]}(s)\right) \right| ds.
\end{aligned}$$

According to condition (2.1), we obtain that

$$\begin{aligned}
|(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)| &\leq \frac{2t}{|2b - \alpha\eta^2|} \left(\int_0^b (b-s) ds \right) \left(\sum_{i=1}^n c_i \|\varphi^{[i]} - \psi^{[i]}\| \right) \\
&\quad + \frac{|\alpha|t}{|2b - \alpha\eta^2|} \left(\int_0^\eta (\eta-s)^2 ds \right) \left(\sum_{i=1}^n c_i \|\varphi^{[i]} - \psi^{[i]}\| \right) \\
&\quad + \left(\int_0^t |(t-s)| ds \right) \left(\sum_{i=1}^n c_i \|\varphi^{[i]} - \psi^{[i]}\| \right) \\
&= \frac{tb^2}{|2b - \alpha\eta^2|} \left(\sum_{i=1}^n c_i \|\varphi^{[i]} - \psi^{[i]}\| \right) \\
&\quad + \frac{1}{3} \frac{t\eta^3 |\alpha|}{|2b - \alpha\eta^2|} \left(\sum_{i=1}^n c_i \|\varphi^{[i]} - \psi^{[i]}\| \right) + \frac{1}{2} t^2 \left(\sum_{i=1}^n c_i \|\varphi^{[i]} - \psi^{[i]}\| \right) \\
&\leq \frac{b^3}{|2b - \alpha\eta^2|} \left(\sum_{i=1}^n c_i \|\varphi^{[i]} - \psi^{[i]}\| \right) \\
&\quad + \frac{1}{3} \frac{b\eta^3 |\alpha|}{|2b - \alpha\eta^2|} \left(\sum_{i=1}^n c_i \|\varphi^{[i]} - \psi^{[i]}\| \right) + \frac{1}{2} b^2 \left(\sum_{i=1}^n c_i \|\varphi^{[i]} - \psi^{[i]}\| \right) \\
&= \left(\frac{b^3}{|2b - \alpha\eta^2|} + \frac{1}{3} \frac{b\eta^3 |\alpha|}{|2b - \alpha\eta^2|} + \frac{1}{2} b^2 \right) \left(\sum_{i=1}^n c_i \|\varphi^{[i]} - \psi^{[i]}\| \right) \\
&= \left(\frac{1}{3} \frac{b(3b^2 + |\alpha|\eta^3)}{|2b - \alpha\eta^2|} + \frac{1}{2} b^2 \right) \left(\sum_{i=1}^n c_i \|\varphi^{[i]} - \psi^{[i]}\| \right).
\end{aligned}$$

From Lemma 2.3, we can also obtain

$$|(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)| \leq \left(b \left(\frac{3b^2 + |\alpha|\eta^3}{3|2b - \alpha\eta^2|} + \frac{1}{2} b \right) \sum_{i=1}^n c_i \sum_{j=0}^{i-1} M^j \right) \|\varphi - \psi\|,$$

which proves the continuity of the operator \mathcal{A} . □

Lemma 3.3. *Suppose that condition (2.1) holds. If*

$$b\zeta \left(\frac{3b^2 + |\alpha|\eta^3}{3|2b - \alpha\eta^2|} + \frac{1}{2} b \right) \leq L, \tag{3.2}$$

and

$$\zeta \left(\frac{3b^2 + |\alpha|\eta^3}{3|2b - \alpha\eta^2|} + \zeta b \right) \leq M, \quad (3.3)$$

then $\mathcal{A}(\mathcal{CB}_{Int}(L, M)) \subset \mathcal{CB}_{Int}(L, M)$.

Proof: From Lemma 2.2, $(\mathcal{A}\varphi)(t) \geq 0$ for $t \in [0, b]$. It remains to show that $(\mathcal{A}\varphi)(t) \leq L$ and $|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| \leq M|t_2 - t_1|$, for all $\varphi \in \mathcal{CB}_{Int}(L, M)$ and $t_1, t_2 \in [0, b]$. For φ in $\mathcal{CB}_{Int}(L, M)$, we have

$$\begin{aligned} |(\mathcal{A}\varphi)(t)| &\leq \frac{2t}{|2b - \alpha\eta^2|} \int_0^b (b-s) \left| f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) \right| ds \\ &\quad + \frac{|\alpha|t}{|2b - \alpha\eta^2|} \int_0^\eta (\eta-s)^2 \left| f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) \right| ds \\ &\quad + \int_0^t |(t-s)| \left| f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) \right| ds. \end{aligned}$$

Since

$$\begin{aligned} \left| f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) \right| &= \left| f\left(s, \varphi(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) - f(s, 0, 0, \dots, 0) + f(s, 0, 0, \dots, 0) \right| \\ &\leq \left| f\left(s, \varphi(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) - f(s, 0, 0, \dots, 0) \right| \\ &\quad + |f(s, 0, 0, \dots, 0)| \\ &\leq \rho + \sum_{i=1}^n c_i \sum_{j=0}^{i-1} M^j \|\varphi\| \\ &\leq \rho + L \sum_{i=1}^n c_i \sum_{j=0}^{i-1} M^j = \zeta, \end{aligned}$$

then

$$\begin{aligned} |(\mathcal{A}\varphi)(t)| &\leq \frac{2t\zeta}{|2b - \alpha\eta^2|} \int_0^b (b-s) ds + \frac{|\alpha|t\zeta}{|2b - \alpha\eta^2|} \int_0^\eta (\eta-s)^2 ds + \zeta \int_0^t |(t-s)| ds \\ &= b\zeta \left(\frac{|\alpha|\eta^3 + 3b^2}{3|2b - \alpha\eta^2|} + \frac{1}{2}b \right) \leq L. \end{aligned}$$

By using (3.2), we get

$$(\mathcal{A}\varphi)(t) \leq |(\mathcal{A}\varphi)(t)| \leq L.$$

Let $t_1, t_2 \in [0, b]$ with $t_1 < t_2$, we have

$$\begin{aligned} |(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| &\leq \left| \frac{2t_2 - 2t_1}{|2b - \alpha\eta^2|} \int_0^b (b-s) \left| f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) \right| ds \right. \\ &\quad + \frac{|\alpha|t_2 - |\alpha|t_1}{|2b - \alpha\eta^2|} \int_0^\eta (\eta-s)^2 \left| f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) \right| ds \\ &\quad + \left| \int_0^{t_1} (t_2 - s) f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) ds \right. \\ &\quad + \int_{t_1}^{t_2} (t_2 - s) f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) ds \\ &\quad \left. - \int_0^{t_1} (t_1 - s) f\left(\varphi^{[0]}(s), \varphi^{[1]}(s), \varphi^{[2]}(s), \dots, \varphi^{[n]}(s)\right) ds \right|, \end{aligned}$$

so

$$\begin{aligned} |(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| &\leq \frac{\zeta b^2 |t_2 - t_1|}{|2b - \alpha\eta^2|} + \frac{1}{3} \frac{\zeta\eta^3 |\alpha| |t_2 - t_1|}{|2b - \alpha\eta^2|} + \zeta b |t_2 - t_1| + \frac{1}{2} \zeta b |t_2 - t_1| \\ &= \left(\frac{\zeta (3b^2 + \eta^3 |\alpha|)}{3|2b - \alpha\eta^2|} + \frac{3}{2} \zeta b \right) |t_2 - t_1|. \end{aligned}$$

Using (3.3), we find

$$|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| \leq M |t_2 - t_1|.$$

Since \mathcal{A} is well defined, i.e. $(\mathcal{A}\varphi)(t) \in \mathcal{CB}_{Int}$ for all $\varphi \in \mathcal{CB}_{Int}(L, M)$, we conclude that $\mathcal{A}(\mathcal{CB}_{Int}(L, M)) \subset \mathcal{CB}_{Int}(L, M)$. \square

Theorem 3.1. *Suppose that conditions (2.1) and (3.2)-(3.3) hold. Then the problem (1.1)-(1.2) has at least one positive bounded solution x in $\mathcal{CB}_{Int}(L, M)$.*

Proof: From Lemma 2.1, the problem (1.1)-(1.2) has a solution x on $\mathcal{CB}_{Int}(L, M)$ if and only if the operator \mathcal{A} defined by (3.1) has a fixed point.

From Lemmas 3.1, 3.2 and 3.3, all conditions of Schauder's fixed point theorem 2.1 are satisfied. Consequently, \mathcal{A} has at least one fixed point on $\mathcal{CB}_{Int}(L, M)$ and these fixed points are solutions of problem (1.1)-(1.2). \square

3.2. Uniqueness

In this section, we present our uniqueness result.

Theorem 3.2. *Under hypotheses of Theorem 3.1, assume further that*

$$b \left(\frac{3b^2 + |\alpha|\eta^3}{3|2b - \alpha\eta^2|} + \frac{1}{2}b \right) \sum_{i=1}^n c_i \sum_{j=0}^{j=i-1} M^j < 1, \quad (3.4)$$

then problem (1.1)-(1.2) has a unique solution in $\mathcal{CB}_{Int}(L, M)$.

Proof: Let φ, ψ be two distinct solutions of (1.1)-(1.2). Using the same technique as in the proof of 3.3 we can prove

$$\begin{aligned} |\varphi(t) - \psi(t)| &= |(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)| \\ &\leq \left(b \left(\frac{3b^2 + |\alpha|\eta^3}{3|2b - \alpha\eta^2|} + \frac{1}{2}b \right) \sum_{i=1}^n c_i \sum_{j=0}^{j=i-1} M^j \right) \|\varphi - \psi\|. \end{aligned}$$

Using (3.4) leads us to

$$\|(\mathcal{A}\varphi) - (\mathcal{A}\psi)\| \leq \Gamma \|\varphi - \psi\|,$$

where

$$\Gamma = b \left(\frac{3b^2 + |\alpha|\eta^3}{3|2b - \alpha\eta^2|} + \frac{1}{2}b \right) \sum_{i=1}^n c_i \sum_{j=0}^{j=i-1} M^j.$$

Since we have a contradiction, the fixed point must be unique. \square

3.3. Continuous dependence

In this section, we show that the unique solution proved in the last theorem depends continuously on the function f .

Theorem 3.3. *Suppose that the conditions of Theorem 3.2 hold. The unique solution of (1.1)-(1.2) depends continuously on the function f .*

Proof: Let $f_1, f_2 : [0, b] \times \mathbb{R}^n \rightarrow [0, +\infty)$ be two continuous functions with respect to their arguments. From Theorem 3.2, it follows that there exist two unique corresponding functions x_1 and x_2 in $\mathcal{CB}_{Int}(L, M)$ such that:

$$\begin{aligned} x_1(t) &= \frac{2t}{2b - \alpha\eta^2} \int_0^b (b-s) f_1 \left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), \dots, x_1^{[n]}(s) \right) ds \\ &\quad - \frac{\alpha t}{2b - \alpha\eta^2} \int_0^\eta (\eta-s)^2 f_1 \left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), \dots, x_1^{[n]}(s) \right) ds \\ &\quad - \int_0^t (t-s) f_1 \left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), \dots, x_1^{[n]}(s) \right) ds, \end{aligned}$$

and

$$\begin{aligned} x_2(t) &= \frac{2t}{2b - \alpha\eta^2} \int_0^b (b-s) f_2 \left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s), \dots, x_2^{[n]}(s) \right) ds \\ &\quad - \frac{\alpha t}{2b - \alpha\eta^2} \int_0^\eta (\eta-s)^2 f_2 \left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s), \dots, x_2^{[n]}(s) \right) ds \\ &\quad - \int_0^t (t-s) f_2 \left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s), \dots, x_2^{[n]}(s) \right) ds. \end{aligned}$$

We have

$$\begin{aligned} |x_2(t) - x_1(t)| &\leq \frac{2t}{|2b - \alpha\eta^2|} \int_0^b (b-s) \left| f_2 \left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s), \dots, x_2^{[n]}(s) \right) \right. \\ &\quad \left. - f_1 \left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), \dots, x_1^{[n]}(s) \right) \right| ds \\ &\quad + \frac{|\alpha|t}{2b - \alpha\eta^2} \int_0^\eta (\eta-s)^2 \left| f_2 \left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s), \dots, x_2^{[n]}(s) \right) \right. \\ &\quad \left. - f_1 \left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), \dots, x_1^{[n]}(s) \right) \right| ds \\ &\quad + \int_0^b |(t-s)| \left| f_2 \left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s), \dots, x_2^{[n]}(s) \right) \right. \\ &\quad \left. - f_1 \left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), \dots, x_1^{[n]}(s) \right) \right| ds \\ &= \frac{2t}{|2b - \alpha\eta^2|} \int_0^b (b-s) \left| f_2 \left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s), \dots, x_2^{[n]}(s) \right) \right. \\ &\quad \left. - f_2 \left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), \dots, x_1^{[n]}(s) \right) \right| ds. \end{aligned}$$

It follows from (2.1) and Lemma 2.3, that

$$\begin{aligned} &\left| f_2 \left(x_2^{[0]}(s), x_2^{[1]}(s), x_2^{[2]}(s), \dots, x_2^{[n]}(s) \right) - f_1 \left(x_1^{[0]}(s), x_1^{[1]}(s), x_1^{[2]}(s), \dots, x_1^{[n]}(s) \right) \right| \\ &\leq \|f_2 - f_1\| + \sum_{i=1}^n c_i \sum_{j=0}^{i-1} M^j \|x_2 - x_1\|. \end{aligned}$$

This implies that

$$\begin{aligned} \|x_2 - x_1\| &\leq b \left(\frac{3b^2 + |\alpha|\eta^3}{3|2b - \alpha\eta^2|} + \frac{1}{2}b \right) \|f_2 - f_1\| \\ &\quad + b \left(\frac{3b^2 + |\alpha|\eta^3}{3|2b - \alpha\eta^2|} + \frac{1}{2}b \right) \sum_{i=1}^n c_i \sum_{j=0}^{i-1} M^j \|x_2 - x_1\|. \end{aligned}$$

Therefore,

$$\|x_2 - x_1\| \leq \frac{b \left(\frac{3b^2 + |\alpha|\eta^3}{3|2b - \alpha\eta^2|} + \frac{1}{2}b \right)}{1 - b \left(\frac{3b^2 + |\alpha|\eta^3}{3|2b - \alpha\eta^2|} + \frac{1}{2}b \right) \sum_{i=1}^n c_i \sum_{j=0}^{i-1} M^j} \|f_2 - f_1\|.$$

This completes the proof. \square

4. Example

To illustrate the results established in Theorems 3.1, 3.2 and 3.3, we consider the following boundary-value problem:

$$x''(t) + f \left(\sin^2 t + \frac{1}{10} (\cos^2 t) x^{[1]}(t) + \frac{1}{25} (\sin^2 t) x^{[2]}(t) \right) = 0, \quad (4.1)$$

$$x(0) = 0, \quad \alpha \int_0^\eta x(t) dt = x(b) \quad \text{with } \eta \in (0, b), \quad (4.2)$$

where

$$f(t, x, y) = \sin^2 t + \frac{1}{10} x \cos^2 t + \frac{1}{25} y \sin^2 t.$$

We have

$$|f(t, y_1, y_2) - f(t, z_1, z_2)| \leq \frac{1}{10} |y_1 - z_1| + \frac{1}{25} |y_2 - z_2|,$$

therefore

$$|f(t, y_1, y_2) - f(t, z_1, z_2)| \leq \sum_{i=1}^2 c_i \|y_i - z_i\|,$$

where $c_1 = \frac{1}{10}$ and $c_2 = \frac{1}{25}$. Furthermore, if $b = \frac{\pi}{2}$ and $L = 7$, $M = 6$ in the definition of $\mathcal{CB}_{Int}(L, M)$, we have $f > 0$, $\rho = \sup_{s \in [0, b]} |f(s, 0, 0)| = 1$ and $\zeta = \frac{7}{4}$.

For $\alpha = \frac{1}{2}$ and $\eta = \frac{1}{3}$, we have $2b = \pi \neq \alpha\eta^2 = \frac{1}{18}$, $\alpha = \frac{1}{2} \leq \frac{2}{\eta^2} = 18$ and

$$b\zeta \left(\frac{3b^2 + |\alpha|\eta^3}{3|2b - \alpha\eta^2|} + \frac{1}{2}b \right) \simeq 4.3623 \leq L = 7,$$

$$\zeta \left(\frac{3b^2 + |\alpha|\eta^3}{3|2b - \alpha\eta^2|} + \frac{3}{2}b \right) \simeq 5.526 \leq M = 6,$$

$$\left(b \left(\frac{3b^2 + |\alpha|\eta^3}{3|2b - \alpha\eta^2|} + \frac{1}{2}b \right) \sum_{i=1}^n c_i \sum_{j=0}^{i-1} M^j \right) \simeq 0.94725 < 1.$$

From Theorems 3.1, 3.2 and 3.3, problem (4.1)-(4.2) has a unique positive bounded solution.

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