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Bi-derivations and Quasi-multipliers on Module Extensions Banach Algebras

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ABSTRACT: This paper characterizes two bi-linear maps bi-derivations and quasi-multipliers on the module extension Banach algebra $A \oplus_1 X$, where A is a Banach algebra and X is a Banach A-module. Under some conditions, it is shown that if every bi-derivation on $A \oplus_1 A$ is inner, then the quotient group of bounded bi-derivations and inner bi-derivations, is equal to the space of quasi-multipliers of A. Moreover, it is proved that $\mathrm{QM}(A \oplus_1 A) = \mathrm{QM}(A) \oplus (\mathrm{QM}(A) + \mathrm{QM}(A)')$, where $\mathrm{QM}(A)' = \{m \in \mathrm{QM}(A) : m(0, a) = m(a, 0) = 0\}$.

Key Words: Banach algebra, bi-derivation, derivation, locally compact group, module extension Banach algebra, quasi-multiplier.

Contents

1. Introduction

Let A be a Banach algebra and X a Banach A-bimodule. Throughout of this paper, all maps are continuous. A linear map $d : A \longrightarrow X$ is called a derivation if $d(aa') = a \cdot d(a') + d(a) \cdot a'$, for all $a, a' \in A$. The derivation $d : A \longrightarrow X$ is said to be inner, if there exists $x \in X$ such that $d(a) = a \cdot x - x \cdot a$, for every $a \in A$. An interesting generalization of derivations is the notion of bi-derivations, for example it ha a close connection with the second cohomology of Banach algebras. A bi-derivation $D : A \times A \longrightarrow X$ is a bi-linear map that is a derivation respect to both components, i.e.,

$$D(ab, c) = a \cdot D(b, c) + D(a, c) \cdot b \quad \text{and} \quad D(a, bc) = b \cdot D(a, c) + D(a, b) \cdot c,$$

for all $a, b, c \in A$. We define the following algebraic centers as follows

$$Z(A) = \{a \in A : aa' = a'a, \text{ for all } a' \in A\},$$

$$Z_A(X) = \{a \in A : a \cdot x = x \cdot a, \text{ for all } x \in X\},$$

and

$$Z_X(A) = \{x \in X : a \cdot x = x \cdot a, \text{ for all } a \in A\}.$$

We say a bi-derivation $D : A \times A \longrightarrow X$ is

(i) inner, if there exists $x \in Z_X(A)$ such that $D(a, a') = x[a, a']$, for all $a, a' \in A$, where $[a, a'] = aa' - a'a$.

(ii) inner respect to the first (second) component, if there exists $x \in X$ ($y \in X$) such that $D(a, a') = [a, x]$ ($D(a, a') = [a', y]$), for all $a, a' \in A$.

(iii) componential inner, if it is inner respect to the both components.

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Brešer et al., in [6] showed that all bi-derivations defined on noncommutative prime rings are inner and for the case semiprime rings Brešer in [7] considered bi-derivations on this class of rings. There are many literatures for bi-derivations that are studied by many authors, for example, we refer to [4,10,12,17,30].

Another interesting bi-linear maps defined on Banach algebras are quasi-multipliers; we refer to [2, 21,22], for general information regarding quasi-multipliers. Quasi-multipliers are a generalization of multipliers, where a bi-linear mapping $m : A \times A \longrightarrow A$ is called a quasi-multiplier if

$$m(ab, cd) = am(b, c)d,$$

for all $a, b, c, d \in A$. The set of all quasi-multipliers on A is denoted by $\mathrm{QM}(A)$.

Let A be a Banach algebra and X be a Banach A-bimodule. By a module extension Banach algebra corresponding to A and X, we will mean the ℓ^1-direct sum of A and X i.e., $A \oplus_1 X$ with the following algebra product and norm:

$$(a_1, x_1)(a_2, x_2) = (a_1a_2, a_1 \cdot x_2 + x_1 \cdot a_2),$$
$$\|(a, x)\| = \|a\|_A + \|x\|_X,$$

for all $a_1, a_2 \in A$ and $x_1, x_2 \in X$. These algebras were studied initially by Zhang [29]. Some homological, cohomological results, results related to derivations on the second dual and module extension of dual Banach algebras are given in [3,13,23,24]. Triangular Banach algebras are considered extensively by Forrest and Marcoux as examples of module extension Banach algebras [14,15,16]. We refer to [5,11, 19,20,25,26,27], for more results related to homological and cohomological results of triangular Banach algebras. For Banach algebra $A \oplus_1 X$, it is easy to see that

$$Z(A \oplus_1 X) = (Z(A) \cap Z_A(X)) \times Z_X(A). \tag{1.1}$$

for all $a, b, c, d \in A$.

Let A be a Banach algebra, X and Y two Banach A-bimodules. A linear operator $T : X \longrightarrow Y$ is called an A-bimodule map if $T(\alpha \cdot x \cdot \beta) = \alpha \cdot T(x) \cdot \beta$, for all $\alpha, \beta \in A$ and $x \in X$. We denote the set of all A-bimodule maps from X into Y by $\mathrm{Hom}_A(X, Y)$. If $A = X = Y$, then $\mathrm{Hom}_A(A, A)$ is the multiplier algebra defined on A which is denoted by $M(A)$ and denote the set of all bounded bi-derivations from $A \times A$ into X, by $\mathrm{BD}(A, X)$ and denote two subsets consist of all inner and componential inner bi-derivations from $A \times A$ into X, by $\mathrm{IBD}(A, X)$ and $\mathrm{IBD}_c(A, X)$, respectively. We define two quotient groups $\mathrm{HBD}(A, X)$ and $\mathrm{HBD}_c(A, X)$ as follows:

$$\mathrm{HBD}(A, X) = \frac{\mathrm{BD}(A, X)}{\mathrm{IBD}(A, X)} \quad \text{and} \quad \mathrm{HBD}_c(A, X) = \frac{\mathrm{BD}(A, X)}{\mathrm{IBD}_c(A, X)}.$$

If $A = X$, then we write $\mathrm{BD}(A, A) = \mathrm{BD}(A)$, $\mathrm{HBD}(A, A) = \mathrm{HBD}(A)$ and $\mathrm{HBD}_c(A, A) = \mathrm{HBD}_c(A)$.

In this paper, in Section 2, we investigate bi-derivations on the module extensions of Banach algebras and characterize these bi-linear maps. In Section 3, we consider quasi-multipliers on the module extension Banach algebra $A \oplus_1 X$.

2. Bi-derivations on $A \oplus_1 X$

Now by the following result, we characterize bi-derivations on $A \oplus_1 X$.

Theorem 2.1. *Let $A \oplus_1 X$ be a module extension Banach algebra, then $\mathcal{D} \in \mathrm{BD}(A \oplus_1 X)$ if and only if*

$$\mathcal{D}((a, x), (a', x')) = (\mathcal{D}_A(a, a') + \mathcal{D}_{A,X}(x, x'), \mathcal{D}_X(x, x') + \mathcal{D}_{X,A}(a, a')), \tag{2.1}$$

such that

(i) $\mathcal{D}_A \in \mathrm{BD}(A)$,

(ii) $\mathcal{D}_{X,A} \in \mathrm{BD}(A, X)$,

(iii) $\mathcal{D}_{A,X}$ *is an A-bimodule map such that* $x_1 \cdot \mathcal{D}_{A,X}(x_2, x_3) = -\mathcal{D}_{A,X}(x_1, x_2) \cdot x_3$ *and* $x_2 \cdot \mathcal{D}_{A,X}(x_1, x_3) = -\mathcal{D}_{A,X}(x_1, x_2) \cdot x_3$*, for all $a \in A$ and $x_1, x_2, x_3 \in X$.*

(iv) $\mathcal{D}_X(a\cdot x_1, x_2) = a\cdot\mathcal{D}_X(x_1,x_2) + \mathcal{D}_A(a,0)\cdot x_1$ *and* $\mathcal{D}_X(x_1, x_2\cdot a) = \mathcal{D}_X(x_1,x_2)\cdot a + x_2\cdot\mathcal{D}_A(0,a)$, *for all* $a\in A$ *and* $x_1,x_2\in X$.

Moreover,

(iv) $\mathcal{D}$ *is inner if and only if* $\mathcal{D}_A$, $\mathcal{D}_{X,A}$ *are inner,* $\mathcal{D}_{A,X}=0$ *and* $\mathcal{D}_X=0$.

(v) $\mathcal{D}$ *is inner respect to the first (second) component if and only if* $\mathcal{D}_A$, $\mathcal{D}_{X,A}$ *are inner respect to the first (second) component,* $\mathcal{D}_{A,X}=0$ *and* $\mathcal{D}_X=0$.

(vi) $\mathcal{D}$ *is componential inner if and only if* $\mathcal{D}_A$, $\mathcal{D}_{X,A}$ *are componential inner,* $\mathcal{D}_{A,X}=0$ *and* $\mathcal{D}_X=0$.

Proof. Let $\mathcal{D}\in \mathrm{BD}(A\oplus_1 X)$. Define the canonical injective maps $\imath_A : A\times A\longrightarrow (A\oplus_1 X)\times(A\oplus_1 X)$, $\imath_X : X\times X\longrightarrow (A\oplus_1 X)\times(A\oplus_1 X)$ by $\imath_A(a,a')=((a,0),(a',0))$, $\imath_X(x,x')=((0,x),(0,x'))$, for all $a,a'\in A$, $x,x'\in X$ and projective maps $\pi_A:(A\oplus_1 X)\longrightarrow A$ and $\pi_X:(A\oplus_1 X)\longrightarrow X$. Let $\mathcal{D}_A:=\pi_A\circ\mathcal{D}\circ\imath_A : A\times A\longrightarrow A$, $\mathcal{D}_X:=\pi_X\circ\mathcal{D}\circ\imath_X : X\times X\longrightarrow X$, $\mathcal{D}_{A,X}:=\pi_A\circ\mathcal{D}\circ\imath_X : X\times X\longrightarrow A$ and $\mathcal{D}_{X,A}:=\pi_X\circ\mathcal{D}\circ\imath_A : A\times A\longrightarrow X$. Since, $\mathcal{D}$ is bi-linear, the above-defined maps are bi-linear. Then

$$\mathcal{D}((a,x),(a',x')) = (\mathcal{D}_A(a,a')+\mathcal{D}_{A,X}(x,x'), \mathcal{D}_X(x,x')+\mathcal{D}_{X,A}(a,a')), \tag{2.2}$$

for all $a,a'\in A$ and $x,x'\in X$. For any $a_1,a_2,a_3\in A$ and $x_1,x_2,x_3\in X$, (2.2) implies that

$$\begin{aligned}(a_1,x_1)\cdot\mathcal{D}((a_2,x_2),(a_3,x_3)) &= (a_1,x_1)\cdot(\mathcal{D}_A(a_2,a_3)+\mathcal{D}_{A,X}(x_2,x_3), \mathcal{D}_X(x_2,x_3)+\mathcal{D}_{X,A}(a_2,a_3))\\ &= (a_1\mathcal{D}_A(a_2,a_3)+a_1\mathcal{D}_{A,X}(x_2,x_3), a_1\cdot\mathcal{D}_X(x_2,x_3)+a_1\cdot\mathcal{D}_{X,A}(a_2,a_3)\\ &\quad +x_1\cdot\mathcal{D}_A(a_2,a_3)+x_1\cdot\mathcal{D}_{A,X}(x_2,x_3)),\end{aligned} \tag{2.3}$$

$$\begin{aligned}\mathcal{D}((a_1,x_1)(a_2,x_2),(a_3,x_3)) &= \mathcal{D}((a_1a_2, a_1\cdot x_2+x_1\cdot a_2),(a_3,x_3))\\ &= (\mathcal{D}_A(a_1a_2,a_3)+\mathcal{D}_{A,X}(a_1\cdot x_2+x_1\cdot a_2,x_3), \mathcal{D}_{X,A}(a_1a_2,a_3)\\ &\quad +\mathcal{D}_X(a_1\cdot x_2+x_1\cdot a_2,x_3)),\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\mathcal{D}((a_1,x_1),(a_3,x_3))\cdot(a_2,x_2) &= (\mathcal{D}_A(a_1,a_3)+\mathcal{D}_{A,X}(x_1,x_3), \mathcal{D}_X(x_1,x_3)+\mathcal{D}_{X,A}(a_1,a_3))\cdot(a_2,x_2)\\ &= (\mathcal{D}_A(a_1,a_3)a_2+\mathcal{D}_{A,X}(x_1,x_3)a_2, \mathcal{D}_X(x_1,x_3)\cdot a_2+\mathcal{D}_{X,A}(a_1,a_3)\cdot a_2\\ &\quad +\mathcal{D}_A(a_1,a_3)\cdot x_2+\mathcal{D}_{A,X}(x_1,x_3)\cdot x_2).\end{aligned} \tag{2.5}$$

Since $\mathcal{D}$ is a bi-derivation,

$$\mathcal{D}((a_1,x_1)(a_2,x_2),(a_3,x_3)) = (a_1,x_1)\cdot\mathcal{D}((a_2,x_2),(a_3,x_3))+\mathcal{D}((a_1,x_1),(a_3,x_3))\cdot(a_2,x_2). \tag{2.6}$$

Putting $x_1=x_2=x_3=0$, implies that

$$\mathcal{D}_A(a_1a_2,a_3) = a_1\mathcal{D}_A(a_2,a_3)+\mathcal{D}_A(a_1,a_3)a_2,$$

and

$$\mathcal{D}_{X,A}(a_1a_2,a_3) = a_1\cdot\mathcal{D}_{X,A}(a_2,a_3)+\mathcal{D}_{X,A}(a_1,a_3)\cdot a_2.$$

Thus, $\mathcal{D}_A$ and $\mathcal{D}_{X,A}$ are derivations respect to the first component. If we put $a_1=a_3=0$, then

$$\mathcal{D}_{A,X}(x_1\cdot a_2,x_3) = \mathcal{D}_{A,X}(x_1,x_3)a_2, \tag{2.7}$$

and if $a_2=a_3=0$, we have

$$\mathcal{D}_{A,X}(a_1\cdot x_2\cdot,x_3) = a_1\mathcal{D}_{A,X}(x_2,x_3). \tag{2.8}$$

Thus, by (2.7) and (2.8), $\mathcal{D}_{A,X}$ is an A-bimodule respect to the first component. Letting $a_1=a_2=a_3=0$ and $x_1=x_2=0$, imply that

$$\mathcal{D}_X(0,x_3)=0, \qquad (x_3\in X). \tag{2.9}$$

By assuming $a_1=a_2=a_3=0$ and by (2.9), we have

$$x_1\cdot\mathcal{D}_{A,X}(x_2,x_3)=-\mathcal{D}_{A,X}(x_1,x_2)\cdot x_3.$$

Taking $a_2=a_3=0$ and $x_1=0$, imply that

$$\mathcal{D}_X(a_1\cdot x_2,x_3)=a_1\cdot\mathcal{D}_X(x_2,x_3)+\mathcal{D}_A(a_1,0)\cdot x_2.$$

An argument similar to that in the above, for any $a_1,a_2,a_3\in A$ and $x_1,x_2,x_3\in X$, by (2.2), we have

$$\begin{aligned}(a_2,x_2)\cdot\mathcal{D}((a_1,x_1),(a_3,x_3)) &= (a_2,x_2)\cdot(\mathcal{D}_A(a_1,a_3)+\mathcal{D}_{A,X}(x_1,x_3),\mathcal{D}_X(x_1,x_3)+\mathcal{D}_{X,A}(a_1,a_3))\\ &= (a_2\mathcal{D}_A(a_1,a_3)+a_2\mathcal{D}_{A,X}(x_1,x_3),a_2\cdot\mathcal{D}_X(x_1,x_3)+a_2\cdot\mathcal{D}_{X,A}(a_1,a_3)\\ &\quad +x_2\cdot\mathcal{D}_A(a_1,a_3)+x_2\cdot\mathcal{D}_{A,X}(x_1,x_3)),\end{aligned} \tag{2.10}$$

$$\begin{aligned}\mathcal{D}((a_1,x_1),(a_2,x_2)(a_3,x_3)) &= \mathcal{D}((a_1,x_1),(a_2a_3,a_2\cdot x_3+x_2\cdot a_3))\\ &= (\mathcal{D}_A(a_1,a_2a_3)+\mathcal{D}_{A,X}(x_1,a_2\cdot x_3+x_2\cdot a_3),\mathcal{D}_{X,A}(a_1,a_2a_3)\\ &\quad +\mathcal{D}_X(x_1,a_2\cdot x_3+x_2\cdot a_3)),\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\mathcal{D}((a_1,x_1),(a_2,x_2))\cdot(a_3,x_3) &= (\mathcal{D}_A(a_1,a_2)+\mathcal{D}_{A,X}(x_1,x_2),\mathcal{D}_X(x_1,x_2)+\mathcal{D}_{X,A}(a_1,a_2))\cdot(a_3,x_3)\\ &= (\mathcal{D}_A(a_1,a_2)a_3+\mathcal{D}_{A,X}(x_1,x_2)a_3,\mathcal{D}_X(x_1,x_2)\cdot a_3+\mathcal{D}_{X,A}(a_1,a_2)\cdot a_3\\ &\quad +\mathcal{D}_A(a_1,a_2)\cdot x_3+\mathcal{D}_{A,X}(x_1,x_2)\cdot x_3).\end{aligned} \tag{2.12}$$

Since $\mathcal{D}$ is a bi-derivation,

$$\mathcal{D}((a_1,x_1),(a_2,x_2)(a_3,x_3))=(a_2,x_2)\cdot\mathcal{D}((a_1,x_1),(a_3,x_3))+\mathcal{D}((a_1,x_1),(a_2,x_2))\cdot(a_3,x_3). \tag{2.13}$$

Putting $x_1=x_2=x_3=0$, implies that

$$\mathcal{D}_A(a_1,a_2a_3)=a_2\mathcal{D}_A(a_1,a_3)+\mathcal{D}_A(a_1,a_2)a_3,$$

and

$$\mathcal{D}_{X,A}(a_1,a_2a_3)=a_2\cdot\mathcal{D}_{X,A}(a_1,a_3)+\mathcal{D}_{X,A}(a_1,a_2)\cdot a_3.$$

Thus, $\mathcal{D}_A$ and $\mathcal{D}_{X,A}$ are derivations respect to the second component. These imply that $\mathcal{D}_A\in\mathrm{BD}(A)$ and $\mathcal{D}_{X,A}\in\mathrm{BD}(A,X)$. If we put $a_1=a_2=0$, then

$$\mathcal{D}_{A,X}(x_1,x_2\cdot a_3)=\mathcal{D}_{A,X}(x_1,x_2)a_3, \tag{2.14}$$

and if $a_1=a_3=0$, we have

$$\mathcal{D}_{A,X}(x_1,a_2\cdot x_3)=a_2\mathcal{D}_{A,X}(x_1,x_3). \tag{2.15}$$

Thus, by (2.14) and (2.15), $\mathcal{D}_{A,X}$ is an A-bimodule respect to the second component. Hence, $\mathcal{D}_{A,X}$ is an A-bimodule. Let $a_1=a_2=a_3=0$ and $x_2=x_3=0$, then

$$\mathcal{D}_X(0,x_1)=0, \qquad (x_1\in X). \tag{2.16}$$

This implies that, if we set $a_1=a_2=a_3=0$, then

$$x_2\cdot\mathcal{D}_{A,X}(x_1,x_3)=-\mathcal{D}_{A,X}(x_1,x_2)\cdot x_3.$$

By letting $a_1=a_2=0$, we have

$$\mathcal{D}_X(x_1,x_2\cdot a_3)=\mathcal{D}_X(x_1,x_3)\cdot a_+x_2\cdot\mathcal{D}_A(0,a_3).$$

This completes the proof and the converse is trivial. Now, suppose that $\mathcal{D}$ is inner, then there exists $(b,y) \in Z(\mathcal{D}_X)$ such that

$$\begin{aligned}\mathcal{D}((a,x),(a',x')) &= (b,y)[(a,x),(a',x')]\\ &= (aa'b - ba'a, y\cdot aa' - y\cdot a'a + ba\cdot x' - b\cdot x'\cdot a + b\cdot x\cdot a' - ba'\cdot x)\\ &= (\mathcal{D}_A(a,a') + \mathcal{D}_{A,X}(x,x'), \mathcal{D}_X(x,x') + \mathcal{D}_{X,A}(a,a')),\end{aligned} \tag{2.17}$$

for all $(a,x),(a',x') \in A \oplus_1 X$. If $x = x' = 0$, then $\mathcal{D}_A(a,a') = aa'b - ba'a = b[a,a']$ and $\mathcal{D}_{X,A}(a,a') = y\cdot aa' - y\cdot a'a = y[a,a']$. If $a = a' = 0$, then

$$\begin{aligned}(0,0) &= \mathcal{D}(0,0) = \mathcal{D}((0,x),(0,x'))\\ &= (0,\mathcal{D}_X(x,x')),\end{aligned}$$

for all $x, x'X$. This implies that $\mathcal{D}_X = 0$ and $\mathcal{D}_{A,X} = 0$. Thus (iv) holds.

(v) Let $\mathcal{D}$ be inner respect to the first component. Thus, there exists $(b,y) \in \mathcal{D}_X$ such that

$$\begin{aligned}\mathcal{D}((a,x),(a',x')) &= (a,x)(b,y) - (b,y)(a,x)\\ &= (ab - ba, a\cdot y - y\cdot a + x\cdot b - b\cdot x)\\ &= (\mathcal{D}_A(a,a') + \mathcal{D}_{A,X}(x,x'), \mathcal{D}_X(x,x') + \mathcal{D}_{X,A}(a,a')),\end{aligned}$$

for all $(a,x),(a',x') \in A \oplus_1 X$. Letting $x = x' = 0$ implies that $\mathcal{D}_A$ and $\mathcal{D}_{X,A}$ are inner respect to the first component. If $a = a' = 0$, then $\mathcal{D}_X = 0$ and $\mathcal{D}_{A,X} = 0$. Similarly, we can investigate the above obtained results for the second component.

For (vi) apply (v). □

Corollary 2.2. *Let $A \oplus_1 X$ be a module extension Banach algebra such that* $\mathrm{HBD}(A) = 0$ *and* $\mathrm{HBD}(A,X) = 0$. *Then* $\mathrm{HBD}(A \oplus_1 X) = 0$

Similarly, we have:

Corollary 2.3. *Let $A \oplus_1 X$ be a module extension Banach algebra such that* $\mathrm{HBD}_c(A) = 0$ *and* $\mathrm{HBD}_c(A,X) = 0$. *Then* $\mathrm{HBD}_c(A \oplus_1 X) = 0$

Corollary 2.4. *Let $A \oplus_1 A$ be a module extension Banach algebra such that* $\mathrm{HBD}_c(A) = 0$. *Then* $\mathrm{HBD}_c(A \oplus_1 A) = 0$

Example 2.5. *Let A be a super amenable Banach algebra i.e., every derivation from A into any Banach A-bimodule X is inner (see [28]). Then by Corollary 2.4, we have* $\mathrm{HBD}_c(A \oplus_1 A) = 0$.

Proposition 2.6. *Let $A \oplus_1 X$ be a module extension Banach algebra and $T \in B^2(X,X)$ be an A-bimodule map. Then $\mathcal{D} : (A \oplus_1 X) \times (A \oplus_1 X) \longrightarrow A \oplus_1 X$ defined by $\mathcal{D}((a,x),(a',x')) = (0, T(x,x'))$, for all $(a,x),(a',x') \in A \oplus_1 X$, is a bi-derivation. Moreover, $\mathcal{D}$ is inner if and only if $T = 0$.*

Proof. Straightforward. □

Lemma 2.7. *Let $A \oplus_1 X$ be a module extension Banach algebra and $\mathcal{D}_A : A \times A \longrightarrow A$ be an inner bi-derivation. Then there is an inner bi-derivation $\mathcal{D}$ on $A \oplus_1 X$ related to $\mathcal{D}_A$.*

Proof. If $\mathcal{D}_A$ is an inner bi-derivation, then there exists $c \in Z(A)$ such that $\mathcal{D}_A(a,a') = c[a,a']$, for all $a, a' \in A$. Define $\mathcal{D} : (A \oplus_1 X) \times (A \oplus_1 X) \longrightarrow (A \oplus_1 X)$ by

$$\mathcal{D}((a,x),(a',x')) = (\mathcal{D}_A(a,a'), c[a,x'] + c[a',x]), \tag{2.18}$$

for all $(a,x),(a',x') \in A \oplus_1 X$. Clearly, $\mathcal{D}$ is bounded and bi-linear. We show that there exists $(b,y) \in Z(A \oplus_1 X)$ such that $\mathcal{D}((a,x),(a',x')) = (b,y)[(a,x),(a',x')]$, for all $(a,x),(a',x') \in A \oplus_1 X$. We set $(b,y) = (c,0)$. Then it is easy to see that $\mathcal{D}((a,x),(a',x')) = (c,0)[(a,x),(a',x')]$, for all $(a,x),(a',x') \in A \oplus_1 X$. □

We denote the set of all A-bimodule bi-linear maps from a Banach A-bimodule $Y\times Y$ into an other Banach A-bimodule Z by $\mathbb{HOM}_A(Y\times Y, Z)$. We now give an interesting result related to the bi-derivations on module extension algebras.

Theorem 2.8. *Let $A\oplus_1 X$ be a module extension Banach algebra,* $\mathrm{HBD}(A)=0$ *and let the only A-bimodule map $\mathcal{P}\in B^2(X,A)$ satisfies $x_1\cdot\mathcal{P}(x_2,x_3)+\mathcal{P}(x_1,x_2)\cdot x_3=0$, for all $x_1,x_2,x_3\in X$ be 0, then*

$$\mathrm{HBD}(A\oplus_1 X)\cong \mathrm{HBD}(A,X)\oplus\mathbb{HOM}_A(X\times X,X) \tag{2.19}$$

as vector spaces.

Proof. Since $\mathrm{HBD}(A)=0$, for any $\mathcal{D}_A\in \mathrm{BD}(A)$ and $a\in A$, $X\cdot\mathcal{D}_A(0,a)=\mathcal{D}_A(a,0)\cdot X=0$. Thus, $\mathcal{D}_X$ is an A-bimodule. Define $\Phi:\mathrm{BD}(A,X)\oplus\mathbb{HOM}_A(X\times X,X)\longrightarrow \mathrm{HBD}(A\oplus_1 X)$ by $\Phi(R,S)=[\mathcal{D}'_{R,S}]$, where $[\mathcal{D}'_{R,S}]$ is the equivalence class of $\mathcal{D}'_{R,S}$ in $\mathrm{HBD}(A\oplus_1 X)$ and $\mathcal{D}'_{R,S}((a,x),(a,x'))=(0,R(x,x')+S(a,a'))$, for all $(a,x),(a,x')\in A\oplus_1 X$. Clearly, Φ is linear. We show that Φ is surjective. Let $\mathcal{D}\in \mathrm{BD}(A,X)$, then by Theorem 2.1, $\mathcal{D}$ is as the following form:

$$\mathcal{D}((a,x),(a',x'))=(\mathcal{D}_A(a,a'),\mathcal{D}_X(x,x')+\mathcal{D}_{X,A}(a,a')),$$

for all $(a,x),(a,x')\in A\oplus_1 X$, note that according to the our assumption $\mathcal{D}_{A,X}=0$. Since $\mathrm{HBD}(A)=0$, there exists $c\in Z(A)$ such that $\mathcal{D}_A(a,a')=c[a,a']$, for all $a,a'\in A$. Define $T:(A\oplus_1 X)\times(A\oplus_1 X)\longrightarrow X$ by $T((a,x),(a',x'))=c[a,x']+c[a',x]$ and

$$\begin{aligned}\mathcal{D}_{R,S}((a,x),(a,x')) &= \mathcal{D}'_{\mathcal{D}_X,\mathcal{D}_{X,A}-T}((a,x),(a,x'))\\ &= (0,\mathcal{D}_X(x,x')+\mathcal{D}_{X,A}(a,a')-c[a,x']-c[a',x]),\end{aligned} \tag{2.20}$$

for all $(a,x),(a,x')\in A\oplus_1 X$. Then

$$\mathcal{D}((a,x),(a',x'))-\mathcal{D}'_{\mathcal{D}_X,\mathcal{D}_{X,A}-T}((a,x),(a,x'))=(\mathcal{D}_A(a,a'),c[a,x']+c[a',x]),$$

for all $(a,x),(a',x')\in A\oplus_1 X$. Then by the proof of Lemma 2.7(i), we have $\mathcal{D}-\mathcal{D}'_{\mathcal{D}_X,\mathcal{D}_{X,A}-T}$ is an inner bi-derivation. Thus, $\Phi(R,S)=[\mathcal{D}_{R,S}]=[\mathcal{D}]$. Finally, by Proposition 2.6, we have

$$\begin{aligned}\ker\Phi &= \left\{(\mathcal{D}_{X,A},\mathcal{D}_X)\in \mathrm{BD}(A,X)\oplus\mathbb{HOM}_A(X\times X,X):\mathcal{D}_{\mathcal{D}_{X,A},\mathcal{D}_X}\text{ is central inner}\right\}\\ &= \{(\mathcal{D}_{X,A},\mathcal{D}_X)\in \mathrm{BD}(A,X)\oplus\mathbb{HOM}_A(X\times X,X):\mathcal{D}_{X,A}\in \mathrm{IBD}(A,X)\text{ and }\mathcal{D}_X=0\}\\ &= \mathrm{IBD}(A,X).\end{aligned}$$

This implies that (2.19) holds. □

Note that in the above Theorem if $X=A$, then $\mathbb{HOM}_A(A\times A,A)=\mathrm{QM}(A)$ and so, by assumptions in Theorem 2.8, we have

$$\mathrm{HBD}(A\oplus_1 X)\cong \mathrm{QM}(A). \tag{2.21}$$

Example 2.9. *Let M_n be an algebra consists of all $n\times n$ matrices over $\mathbb{C}$. Let $\mathcal{P}\in B^2(M_n,M_n)$ be an M_n-bimodule map such that $A\cdot\mathcal{P}(B,C)=-\mathcal{P}(A,B)\cdot C$, for all $A,B,C\in M_n$. Note that there are $A,B\in M_n$ such that $\mathcal{P}(A,B)\neq-\mathcal{P}(B,A)$. Suppose that $\mathcal{P}(A,B)=(\alpha_{ij})_{n\times n}$ and $\mathcal{P}(B,A)=(\beta_{ij})_{n\times n}$. Let $a\in A$ and set $A=C=(a_{ij})_{n\times n}\in M_n$ such that $a_{ii}=a$ and $a_{ij}=0$, for all $i\neq j$, where $1\le i,j\le n$. Then*

$$\begin{aligned}\left(a\beta_{ij}\right)_{n\times n} &= A\cdot\mathcal{P}(B,A)=-\mathcal{P}(A,B)\cdot A\\ &= -\left(\alpha_{ij}a\right)_{n\times n}.\end{aligned}$$

This implies that $\alpha_{ij}=-\beta_{ij}$, for all $1\le i,j\le n$, a contradiction. Thus, $\mathcal{P}=0$. By [8, Propositions 1.3.51 and 1.3.52], M_n is a simple algebra and consequently is a prime Banach algebra. From [6, Theorem 3.3], we have $\mathrm{HBD}(M_n)=0$. Then (2.21) implies that $\mathrm{HBD}(M_n\oplus_1 M_n)\cong\mathrm{QM}(M_n)$.

3. Quasi-multipliers

As we mentioned in the first section, quasi multipliers are a generalization of multipliers. In [9], Daws introduced a module version of multiplies that is another generalization of multipliers. He called a linear map T from a Banach algebra A into a Banach A-bimodule X, a left multiplier of X; if $T(ab) = T(a) \cdot b$, for all $a, b \in A$. Similarly, T is a right multiplier of X; if $T(ab) = a \cdot T(b)$, for all $a, b \in A$. In this section, we say that $m : A \times A \longrightarrow X$ is a quasi-multiplier of X or $m \in \mathrm{QM}(A, X)$, if $m(ab, cd) = a \cdot m(b, c) \cdot d$, for all $a, b, c, d \in A$. Our aim in this section is characterizing of quasi-multipliers on the module extensions $A \oplus_1 X$.

Theorem 3.1. *Let $A \oplus_1 X$ be a module extension Banach algebra, then $m \in \mathrm{QM}(A \oplus_1 X)$ if and only if*

$$m((a, x), (a', x')) = (m_A(a, a') + m_{A,X}(x, x'), m_X(x, x') + m_{X,A}(a, a')), \tag{3.1}$$

such that

(i) $m_A \in \mathrm{QM}(A)$,

(ii) $m_{X,A} \in \mathrm{QM}(A, X)$,

(iii) $m_{A,X}$ *is an A-bimodule map such that* $x_1 \cdot m_{A,X}(x_2, x_3) = m_{A,X}(x_1, x_2) \cdot x_3 = 0$.

(iv) m_X *is an A-bimodule map such that* $m_X(x, 0) = m_X(0, x) = 0$, *for every* $x \in X$.

Proof. Let $m \in \mathrm{QM}(A \oplus_1 X)$. Suppose that the mappings $\imath_A$, $\imath_X$, $\pi_A : (A \oplus_1 X) \longrightarrow A$ and $\pi_X : (A \oplus_1 X) \longrightarrow X$ are the same as the proof of Theorem 2.1. Let $m_A := \pi_A \circ m \circ \imath_A : A \times A \longrightarrow A$, $m_X := \pi_X \circ m \circ \imath_X : X \times X \longrightarrow X$, $m_{A,X} := \pi_A \circ m \circ \imath_X : X \times X \longrightarrow A$ and $m_{X,A} := \pi_X \circ m \circ \imath_A : A \times A \longrightarrow X$. Since, m is bi-linear, the above-defined maps are bi-linear. Then

$$m((a, x), (a', x')) = (m_A(a, a') + m_{A,X}(x, x'), m_X(x, x') + m_{X,A}(a, a')), \tag{3.2}$$

for all $a, a' \in A$ and $x, x' \in X$. For any $a_1, a_2, a_3 \in A$ and $x_1, x_2, x_3 \in X$, (2.2) implies that

$$\begin{aligned}(a_1, x_1)m((a_2, x_2), (a_3, x_3)) &= (a_1, x_1)(m_A(a_2, a_3) + m_{A,X}(x_2, x_3), m_X(x_2, x_3) + m_{X,A}(a_2, a_3)) \\ &= (a_1 m_A(a_2, a_3) + a_1 m_{A,X}(x_2, x_3), a_1 \cdot m_X(x_2, x_3) + a_1 \cdot m_{X,A}(a_2, a_3) \\ &\quad + x_1 \cdot m_A(a_2, a_3) + x_1 \cdot m_{A,X}(x_2, x_3)),\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}m((a_1, x_1)(a_2, x_2), (a_3, x_3)) &= m((a_1 a_2, a_1 \cdot x_2 + x_1 \cdot a_2), (a_3, x_3)) \\ &= (m_A(a_1 a_2, a_3) + m_{A,X}(a_1 \cdot x_2 + x_1 \cdot a_2, x_3), m_{X,A}(a_1 a_2, a_3) \\ &\quad + m_X(a_1 \cdot x_2 + x_1 \cdot a_2, x_3)) \\ &= (m_A(a_1 a_2, a_3) + m_{A,X}(a_1 \cdot x_2, x_3) + m_{A,X}(x_1 \cdot a_2, x_3), m_{X,A}(a_1 a_2, a_3) \\ &\quad + m_X(a_1 \cdot x_2, x_3) + m_X(x_1 \cdot a_2, x_3))\end{aligned} \tag{3.4}$$

Putting $x_1 = x_2 = x_3 = 0$, implies that

$$m_A(a_1 a_2, a_3) = a_1 m_A(a_2, a_3), \tag{3.5}$$

and

$$m_{X,A}(a_1 a_2, a_3) = a_1 \cdot m_{X,A}(a_2, a_3). \tag{3.6}$$

Moreover, putting $a_1 = a_2 = a_3 = 0$ and $x_1 = 0$, imply that $m_X(0, x_3) = 0$ and for $a_1 = a_2 = a_3 = 0$,

$$x_1 \cdot m_{A,X}(x_2, x_3) = 0. \tag{3.7}$$

By letting $a_2 = a_3 = 0$ and (3.7), we have

$$m_X(a_1 \cdot x_2 \cdot, x_3) = a_1 \cdot m_X(x_2, x_3), \tag{3.8}$$

and

$$m_{A,X}(a_1 \cdot x_2, x_3) = a_1 \cdot m_{A,X}(x_2, x_3). \tag{3.9}$$

An argument similar to that in the above, for any $a_1, a_2, a_3 \in A$ and $x_1, x_2, x_3 \in X$, by (2.2), we have

$$\begin{aligned} m((a_1, x_1), (a_2, x_2)(a_3, x_3)) &= m((a_1, x_1), (a_2a_3, a_2 \cdot x_3 + x_2 \cdot a_3)) \\ &= (m_A(a_1, a_2a_3) + m_{A,X}(x_1, a_2 \cdot x_3 + x_2 \cdot a_3), m_{X,A}(a_1, a_2a_3) \\ &\quad + m_X(x_1, a_2 \cdot x_3 + x_2 \cdot a_3)), \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} m((a_1, x_1), (a_2, x_2)) \cdot (a_3, x_3) &= (m_A(a_1, a_2) + m_{A,X}(x_1, x_2), m_X(x_1, x_2) + m_{X,A}(a_1, a_2)) \cdot (a_3, x_3) \\ &= (m_A(a_1, a_2)a_3 + m_{A,X}(x_1, x_2)a_3, m_X(x_1, x_2) \cdot a_3 + m_{X,A}(a_1, a_2) \cdot a_3 \\ &\quad + m_A(a_1, a_2) \cdot x_3 + m_{A,X}(x_1, x_2) \cdot x_3). \end{aligned} \tag{3.11}$$

Then, by putting $x_1 = x_2 = x_3 = 0$, we have

$$m_A(a_1, a_2a_3) = m_A(a_1, a_2)a_3, \tag{3.12}$$

and

$$m_{X,A}(a_1, a_2a_3) = m_{X,A}(a_1, a_2) \cdot a_3. \tag{3.13}$$

Thus, (3.5), (3.6), (3.12) and (3.13) imply that $m_A \in \mathrm{QM}(A)$ and $m_{X,A} \in \mathrm{QM}(A, X)$. Set $a_1 = a_2 = a_3 = 0$ and $x_3 = 0$, then $m_X(x_1, 0) = 0$. This implies that, for $a_1 = a_2 = a_3 = 0$,

$$m_{A,X}(x_1, x_2) \cdot x_3 = 0. \tag{3.14}$$

If we put $a_1 = a_2 = 0$, by (3.14),

$$m_X(x_1, x_2 \cdot a_3) = m_X(x_1, x_2) \cdot a_3, \tag{3.15}$$

and if $a_1 = a_3 = 0$, we have

$$m_{A,X}(x_1, x_2 \cdot a_3) = m_{A,X}(x_1, x_3)a_3. \tag{3.16}$$

Thus, by (3.15) and (3.16), $m_{A,X}$ and m_X are right A-module. Hence, $m_{A,X}$ and m_X are A-bimodule. □

Remark 3.2. *In the module extension Banach algebra $A \oplus_1 X$, if $A = X$ is one of the following Banach algebra: (1) without of order, i.e., for any $a, b \in A$, $ab = 0$ implies that $a = 0$ or $b = 0$, (2) unital, (3) a Banach algebra with a non-zero idempotent element or (4) a Banach algebra with a left (right) bounded approximate identity, then the map $m_{A,X}$ in Theorem 3.1 is zero. Thus, $m \in \mathrm{QM}(A \oplus_1 X)$ if and only if*

$$m((a, x), (a', x')) = (m_A(a, a'), m_X(x, x') + m_{X,A}(a, a')), \tag{3.17}$$

such that

(i) $m_A \in \mathrm{QM}(A)$,

(ii) $m_{X,A} \in \mathrm{QM}(A)$,

(iii) $m_X \in \mathrm{QM}(A)$ *such that* $m_X(x, 0) = m_X(0, x) = 0$, *for every* $x \in A$.

If we denote the set of all quasi-multipliers such as m_X by $\mathrm{QM}(A)'$, then we can write

$$\mathrm{QM}(A \oplus_1 A) = \mathrm{QM}(A) \oplus (\mathrm{QM}(A) + \mathrm{QM}(A)'). \tag{3.18}$$

Example 3.3. *Let G be a locally compact group, $L^1(G)$ and $M(G)$ be the group and the measure algebras on G, respectively. Then by [22] and Remark 3.2, we have*

$$\begin{aligned} \mathrm{QM}(L^1(G) \oplus_1 L^1(G)) &= \mathrm{QM}(L^1(G)) \oplus (\mathrm{QM}(L^1(G)) + \mathrm{QM}(L^1(G))') \\ &= M(G) \oplus (M(G) + \mathrm{QM}(L^1(G))'). \end{aligned}$$

Example 3.4. *Let G be a non-compact locally compact abelian group, $A(G)$ and $B(G)$ be the Fourier and the Fourier-Stieltjes algebras on G, respectively. We have $L^1(G) = A(\hat{G})$, where $\hat{G}$ is the dual of G and $M(\hat{G}) \cong B(G)$. Then by Example 3.3, we have*

$$\begin{aligned} \mathrm{QM}(A(G) \oplus_1 A(G)) &= \mathrm{QM}(L^1(\hat{G}) \oplus_1 L^1(\hat{G})) \\ &= M(\hat{G}) \oplus (M(\hat{G}) + \mathrm{QM}(L^1(\hat{G}))') \\ &= B(G) \oplus (B(G) + \mathrm{QM}(L^1(G))'). \end{aligned}$$

Let S be a locally compact semigroup and $M(S)$ be the space of all bounded complex regular Borel measures on S. A locally compact semigroup S is called a foundation semigroup if $\bigcup\{\mathrm{supp}(\mu) : \mu \in M_a(S)\}$ is dense in S, where $M_a(S)$ is a subspace of $M(S)$ contains all $\mu \in M(S)$ such that the maps $s \mapsto \delta_s * |\mu|$ and $s \mapsto |\mu| * \delta_s$ from S into $M(S)$ are continuous (δ_s denotes the Dirac measure at s). A complex-valued bounded function g on S is called $M_a(S)$-measurable, if it is μ-measurable, for all $\mu \in M_a(S)$. The space of such functions denotes by $L^\infty(S, M_a(S))$ and, for every $g \in L^\infty(S, M_a(S))$,

$$\|g\|_\infty = \sup\{\|g\|_{\infty,|\mu|} : \mu \in M_a(S)\},$$

where $\|g\|_{\infty,|\mu|}$ denotes the essential supremum norm with respect to $|\mu|$. The semigroup S is called compactly cancellative if for any two compact subsets C and D of S, the following two sets are compact subsets of S

$$CD^{-1} = \{x \in S : xd \in C \text{ for some } d \in D\},$$

$$DC^{-1} = \{x \in S : cx \in D \text{ for some } c \in D\}.$$

Example 3.5. *Let S be a compactly cancellative foundation semigroup with identity. Then by [1, Corollary 3.2], $\mathrm{QM}(M_a(S)) = M(S)$. Thus*

$$\mathrm{QM}(M_a(S) \oplus_1 M_a(S)) = M(S) \oplus (M(S) + \mathrm{QM}(M_a(S))').$$

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