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Accelerated Extragradient Algorithm for Equilibrium and Fixed Point Problems for Countable Family of Certain Multi-valued Mappings

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ABSTRACT: In this paper, we introduce a viscosity-type extragradient algorithm for finding a common point of the solution of a pseudomonotone equilibrium problem and a fixed point problem of an infinite family of multi-valued quasi-nonexpansive mappings in a real Hilbert space. Using our algorithm, we state and prove a strong convergence result of our iteration sequences. An application to variational inequality problem was considered. Lastly, we give a numerical example of our main result. The result presented in this paper extends and complements some recent results in literature.

Key Words: Equilibrium problem, quasi-nonexpansive mappings, multi-valued mappings, fixed point problems, variational inequality problem.

Contents

1. Introduction

Let C be a nonempty, closed and convex subset of a real Hilbert space H. We denote by $CB(C)$ and $K(C)$ the family of nonempty closed bounded subsets and nonempty compact subsets of C respectively. The Hausdorff metric distance on $CB(C)$ is defined by

$$\mathcal{H}(A,B) = \max\Big\{ \sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\Big\} \text{ for } A,B \in CB(C),$$

where $d(x,C) = \min\{||x-y|| : y \in C\}$.
Let $T : C \to CB(C)$ be a multivalued mapping, then $P_T x = \{u \in Tx : ||x-u|| = d(x,Tx)\}$. A point $x \in C$ is called a fixed point of T if $x \in Tx$. However, if $Tx = \{x\}$, then x is called a strict point of T. We denote the set of fixed point of T by $Fix(T)$. A multivalued mapping T is said to be *L-Lipschitzian* if there exists $L > 0$ such that

$$\mathcal{H}(Tx,Ty) \le L||x-y||,\ x,y \in C. \tag{1.1}$$

In (1.1), if $L \in (0,1)$, then T is called a strict contraction while T is called nonexpansive if $L = 1$.
T is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\mathcal{H}(Tx,Ty) \le ||x-y||,\ \forall\ x\ \in C, y \in F(T),$$

Let $g : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem (EP) is to find

$$x^* \in C \text{ such that } g(x^*,y) \ge 0,\ \forall\ y \in C. \tag{1.2}$$

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We denote by Ω, the set of solution of problem (1.2).
The application of EP have been found in practical problems arising from physics, engineering, game theory, transportation, economics to mention a few and network, (see [1,2,3,4,5,6,9,11,23,24,31,30] and other references contained in). Many authors have considered approximating solutions of fixed point problem together with $EP(C,g)$. For instance, in 2013, Anh [7] introduced the following extragradient method for approximating solutions of pseudomonotone EP as follows:

$$\begin{cases} y_n = argmin_{y\in C}\{\frac{1}{2}||y-x_n||^2 + \lambda_n g(x_n,y)\}; \\ t_n = argmin_{t\in C}\{\lambda_n g(y_n,t) + \frac{1}{2}||t-x_n||^2\}; \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n)T(t_n); \end{cases} \tag{1.3}$$

where $\{\lambda_n\} \subset (0,1]$, $x_0 \in C$, $\alpha_n \in (0,1)$ such that $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\lim_{n\to\infty}\alpha_n = 0$, $f(x,y) = \langle F(x), y-x\rangle$, T a nonexpansive mapping and g satisfies a Lipschitz-type property. The author showed that (1.3) converges strongly to x^* in $Sol(F,C)$ and $Fix(T)$ (see [7] for more details of their iterative sequence).
In 2012, Vuong et al. [14] employed a hybrid projection algorithm for approximating a common element of fixed point set of a pseudo-contractions and solution set of an equilibrium problem involving pseudomonotone bifunction g in the following manner: $x_0 \in C$ and

$$\begin{cases} y_n = argmin_{y\in C}\{\lambda_n g(x_n,y) + \frac{1}{2}||y-x_n||^2\}; \\ z_n = argmin_{y\in C}\{\lambda_n g(y_n,y) + \frac{1}{2}||y-x_n||^2\}; \\ t_n = \alpha_n x_n + (1-\alpha_n)\beta_n z_n + (1-\beta_n)Sz_n; \\ C_n = \{z \in C : ||t_n - z|| \leq ||x_n - z||\}; \\ D_n = \{z\in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}; \\ x_{n+1} = P_{C_n\cap D_n}x_0; \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\} \subset (0,1)$ and g satisfies a Lipschitz-type property. They proved a strong convergence theorem.
A very good approach to fasten up the convergence rate of iterative algorithms is to combine the iterative scheme with the inertial term. The inertial type iteration process originate from the heavy ball method of the two order dynamical system which is a two-step method for minimizing a smooth convex function. Polyak [27] was the first author to propose the heavy ball method, Alvarez and Attouch [8] employed this to the setting of a general maximal monotone operator using the PPA, which is called the inertial PPA, and is of the form:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (I + r_n B)^{-1}y_n, n \geq 1. \end{cases} \tag{1.5}$$

They proved that if $\{r_n\}$ is non-decreasing and $\{\theta_n\} \subset [0,1)$ with

$$\sum_{n=1}^{\infty}\theta_n||x_n - x_{n-1}||^2 < \infty, \tag{1.6}$$

then the sequence generated by Algorithm (1.5) converges weakly to a zero of a maximal monotone operator B.
The introduction of the term $\theta_n(x_n - x_{n-1})$ in (1.5), represents a remarkable tool employed in improving the performance of algorithms which in turn provides some remarkeable convergence properties (see [27] and the references therein). Motivated by the works of Trans et al. [13], Vuong et. al. [14] and other related works in literature, we introduce a viscosity type algorithm together with an inertial term to approximate a common solution of an EP involving a pseudomonotone bifunction and a fixed point problem of an infinite family of quasi-nonexpansive multi-valued mappings in the framework of real Hilbert spaces. We prove a strong convergence result and give application of our main result to variatioanal inequality problem. The result presented in this paper extends and complements the result of [14] and other related results in literature.

2. Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\to$" and "$\rightharpoonup$" respectively.

Definition 2.1. *The bifunction g is called (i) monotone, if*

$$g(x,y)+g(y,x)\leq 0,\ \forall\ x,y\in C;$$

(ii) pseudomonotone, if

$$g(x,y)\geq 0\Rightarrow g(y,x)\leq 0,\ \forall\ x,y\in C;$$

(iii) Lipschitz-type continuous with constants $c_1>0$ and $c_2>0$ if

$$g(x,y)+g(y,z)\geq g(x,z)-c_1||x-y||^2-c_2||y-z||^2,\ \forall\ x,y\in C.$$

To solve an EP involving a pseudomonotone bifunction, the following assumptions are needed:
(L1) $g(x,x)=0$ for all $x\in C$ and g is pseudomonotone on C;
(L2) g is Lipschitz-type continuous on C; with constants c_1 and c_2.
(L3) for each $x\in C,\ y\to g(x,y)$ is convex and subdifferentiable;
(L4) $g(x,y)$ is jointly weakly continuous on $C\times C$.
Let C be a nonempty, closed and convex subset of a real Hilbert space H and x be an element in H. We know that for each $x\in H$, there is a unique $P_Cx\in C$ such that

$$||x-P_Cx||=\inf\{||x-y||:y\in C\}. \tag{2.1}$$

The operator P_C so defined is referred to as the nearest point mapping or the metric projection onto C. It is known that

$$\langle x-P_Cx,y-P_Cx\rangle\leq 0, \tag{2.2}$$

holds for all $x\in H$ and $y\in C$ (see [10,32]).
Given a proper and convex function $h:H\to\mathbb{R}\cup\{+\infty\}$, the subdifferential of h as $x\in H$ is defined as (the set of all subgradients of h a x);

$$\partial h(x)=\{w\in H:h(y)-h(x)\geq\langle w,y-x\rangle\ \forall y\in H\}.$$

The function h is said to be sub-differentiable at x if $\partial h(x)\neq\emptyset$.

Definition 2.2. *Let H be a real Hilbert space and $T:H\to CB(H)$ a multivalued mapping. Then, T is said to be demiclosed at the origin if for any sequence $\{x_n\}\subset H$ with $x_n\rightharpoonup x^*$, and $d(x_n,T(x_n))\to 0$, we have $x^*\in Tx^*$.*

Lemma 2.3. *[20] Let H be a real Hilbert space, then $\forall\ x,y\in H$ and $\alpha\in(0,1)$, we have*

(i) $2\langle x,y\rangle=||x||^2+||y||^2-||x-y||^2=||x+y||^2-||x||^2-||y||^2$,

(ii) $||\alpha x+(1-\alpha)y||^2=\alpha||x||^2+(1-\alpha)||y||^2-\alpha(1-\alpha)||x-y||^2$,

(iii) $\langle x,y\rangle=\frac{1}{2}||x||^2+\frac{1}{2}||y||^2-\frac{1}{2}||x-y||^2$.

Lemma 2.4. *[21] Let E be a uniformly convex real Banach space. For arbitrary $r>0$, let $B_r(0):=\{x\in E:||x||\leq r\}$. Then, for any given sequence $\{x_i\}_{i=1}^{\infty}\subset B_r(0)$ and for any given sequence $\{\lambda_i\}_{i=1}^{\infty}$, in (0,1) with $\sum_{i=1}^{\infty}\lambda_i=1$, there exists a continuously strictly increasing convex function*

$$g:[0,2r]\to\mathbb{R}\ \text{with}\ g(0)=0,$$

such that for any positive integers i,j with $i<j$, the following inequality holds

$$||\sum_{i=1}^{\infty}\lambda_i x_i||^2=\sum_{i=1}^{\infty}\lambda_i||x||^2-\lambda_i\lambda_j g(||x_i-x_j||).$$

Lemma 2.5. *[15] Assume that f satisfies (L1)-(L4) such that EP(f) is nonempty and $0 < \lambda < \min\left(\frac{1}{2c_1}, \frac{1}{2c_2}\right)$. If $x_0 \in C$ and y_0, z_0 are defined by*

$$\begin{cases} y_0 = argmin_{y\in C}\{\lambda f(x_0, y) + \frac{1}{2}||y - x_0||^2\}; \\ z_0 = argmin_{y\in C}\{\lambda f(y_0, y) + \frac{1}{2}||y - x_0||^2\}; \end{cases}$$

then,
(i) $\lambda[f(x_0, y) - f(x_0, y_0)] \geq \langle y_0 - x_0, y_0 - y\rangle, \ \forall \ y \in C;$
(ii) $||z_0 - p||^2 \leq ||x_0 - p||^2 - (1 - 2\lambda c_1)||x_0 - y_0||^2 - (1 - 2\lambda c_2)||y_0 - z_0||^2, \ \forall \ p \in EP(f).$

Lemma 2.6. *[17] Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \to K(C)$ be a quasi-nonexpansive multi-valued mapping. Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $p \in Tp$.*

Lemma 2.7. *[19] Assume $\{a_n\}$ is a sequence of nonnegative real sequence such that*

$$a_{n+1} \leq (1 - \sigma_n)a_n + \sigma_n\delta_n + \gamma_n, \ n > 1,$$

where $\{\sigma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a real sequence such that
(i) $\sum_{n=1}^{\infty} \sigma_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$,
(iii) $\gamma_n \geq 0, \ (n \geq 1)$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

In this section, we introduce an iterative algorithm for approximating a solution of pseudomonotone equilibrium problem and the set of fixed point problem of an infinite family of quasi-nonexpansive multi-valued mappings.
Algorithm 1:
Initialization: Choose $x_1 \in H$ and $i \in 1, 2,$ the sequences $\{\alpha_n\}, \{\beta_{n,0}\}$ and $\{\beta_{n,i}\}$ in $(0, 1)$ such that

$$\begin{cases} (i) \ \lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=0}^{\infty} \alpha_n = \infty; \\ (ii) \ \liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0, \ \sum_{i=0}^{\infty} \beta_{n,i} = 1, \ \forall \ i; \\ (iii) \ 0 < \underline{\mu} \leq \mu_n \leq \overline{\mu} < \min\left(\frac{1}{2c_1}, \frac{1}{2c_2}\right); \\ (iv) \ \sum_{n=1}^{\infty} \theta_n||x_n - x_{n-1}|| < \infty; \\ (v) \ \lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0. \end{cases}$$

Set $n = 0$ and go to step 1,
Step 1: Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1});$$

Step 2: Compute

$$t_n = argmin_{u\in C}\{\mu_n g(w_n, u) + \frac{1}{2}||w_n - u||^2\};$$

$$z_n = argmin_{u\in C}\{\mu_n g(t_n, u) + \frac{1}{2}||w_n - u||^2\};$$

Step 3: Let h_n be defined by:

$$h_n = \beta_{n,0}z_n + \sum_{i=1}^{\infty} \beta_{n,i}q_n^i \text{ where } q_n^i \in T_i z_n;$$

Step 4: Compute

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)h_n.$$

Step 5: If $x_{n+1} = x_n$, then stop. Otherwise, set $n := n+1$ for $n \geq 0$ and go to step 1.

Theorem 3.1. *Let C be a nonempty, closed and convex of a real Hilbert space H and $f : C \to C$ be a contraction with constant $\rho \in (0,1)$. Let $g : C \times C \to \mathbb{R}$ be a bifunction satisfying (L1)-(L4) and $\{T_i\}_{i=1}^{\infty} : C \to K(C)$ be an infinite family of multi-valued quasi-nonexpansive mappings. Assume that $\Gamma := \cap_{i=1}^{\infty} Fix(T_i) \cap \Omega \neq \emptyset$, then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to $\overline{x} = P_\Gamma f(\overline{x})$, where $P_\Gamma f$ is the metric projection from $f(\overline{x})$ onto C.*

Let $p \in \Gamma$, then we have from Lemma 2.5 and Algorithm 1 that

$$||z_n - p||^2 \leq ||w_n - p||^2 - (1-2\mu_n c_1)||w_n - t_n||^2 - (1-2\mu_n c_2)||t_n - z_n|| \tag{3.1}$$

which implies

$$||z_n - p||^2 \leq ||w_n - p||^2,$$

using this and Step 1, we have

$$\begin{aligned} ||z_n - p|| &\leq ||w_n - p|| \\ &= ||x_n + \theta_n(x_n - x_{n-1}) - p|| \\ &\leq ||x_n - p|| + \theta_n||x_n - x_{n-1}||. \end{aligned} \tag{3.2}$$

From step 3 of Algorithm 1 and (3.1), we have

$$\begin{aligned} ||h_n - p|| &= ||\beta_{n,0}z_n + \sum_{i=1}^{\infty}\beta_{n,i}q_n^i - p|| \\ &\leq \beta_{n,0}||z_n - p|| + \sum_{i=1}^{\infty}\beta_{n,i}||q_n^i - p|| \\ &\leq \beta_{n,0}||z_n - p|| + \sum_{i=1}^{\infty}\beta_{n,i}d(q_n^i, T_i p) \\ &\leq \beta_{n,0}||z_n - p|| + \sum_{i=1}^{\infty}\beta_{n,i}\mathcal{H}(T_i z_n, T_i p) \\ &\leq \beta_{n,0}||z_n - p|| + \sum_{i=1}^{\infty}\beta_{n,i}||z_n - p|| \\ &= ||z_n - p|| \\ &\leq ||x_n - p|| + \theta_n||x_n - x_{n-1}||. \end{aligned} \tag{3.3}$$

Now, we have from Algorithm 1 and (3.3) that

$$\begin{aligned} ||x_{n+1} - p|| &\leq \alpha_n||f(x_n) - p|| + (1-\alpha_n)||h_n - p|| \\ &\leq \alpha_n\big[||f(x_n) - f(p)|| + ||f(p) - p||\big] + (1-\alpha_n)||x_n - p|| + (1-\alpha_n)\theta_n||x_n - x_{n-1}|| \\ &= \alpha_n\rho||x_n - p|| + \alpha_n||f(p) - p|| + (1-\alpha_n)||x_n - p|| + (1-\alpha_n)\theta_n||x_n - x_{n-1}|| \\ &\leq (1-\alpha_n(1-\rho))||x_n - p|| + \alpha_n(1-\rho)\frac{||f(p) - p||}{1-\rho} + (1-\alpha_n)\theta_n||x_n - x_{n-1}|| \\ &\leq \Bigg\{\max\Big\{\max||x_{n-1} - p||, \frac{||f(p) - p||}{1-\rho}\Big\} + \theta_{n-1}||x_{n-1} - x_{n-2}||, \frac{||f(p) - p||}{1-\rho}\Bigg\} + \theta_n||x_n - x_{n-1}|| \\ &= \max\Bigg\{||x_n - p||, \frac{||f(p) - p||}{1-\rho}\Bigg\} + \theta_{n-1}||x_{n-1} - x_{n-2}|| + \theta_n||x_n - x_{n-1}||. \end{aligned}$$

Let $M = \sum_{i=1}^{n} \theta_i ||x_i - x_{i-1}||$, we obtain that

$$||x_{n+1} - p|| \leq \max\{||x_1 - p||, \frac{||f(p) - p||}{1-\rho}\} + M.$$

Therefore, $\{x_n\}$ is bounded. Consequently, $\{t_n\}, \{z_n\}$ and $\{h_n\}$ are all bounded.
From (i) and Lemma 2.3(iii), we have that

$$\begin{aligned}
||w_n - p||^2 &= ||x_n + \theta_n(x_n - x_{n-1}) - p||^2 \\
&= ||x_n - p||^2 + 2\theta_n\langle x_n - p, x_n - x_{n-1}\rangle + \theta_n^2||x_n - x_{n-1}||^2 \\
&= ||x_n - p||^2 + \theta_n(||x_n - p||^2 + ||x_n - x_{n-1}||^2 - ||x_{n-1} - p||^2) + \theta_n^2||x_n - x_{n-1}||^2 \\
&\leq ||x_n - p||^2 + \theta_n(||x_n - p||^2 - ||x_{n-1} - p||^2) + 2\theta_n||x_n - x_{n-1}||^2. \qquad (3.4)
\end{aligned}$$

We conclude from (3.1) and (3.4), that

$$\begin{aligned}
||z_n - p||^2 \leq ||x_n - p||^2 &+ \theta_n(||x_n - p||^2 - ||x_{n-1} - p||^2) + 2\theta_n||x_n - x_{n-1}||^2 \\
&- (1 - 2\mu_n c_1)||w_n - t_n||^2 - (1 - 2\mu_n c_2)||t_n - z_n||^2. \qquad (3.5)
\end{aligned}$$

Using (3.5) and Lemma 2.4, we have that

$$\begin{aligned}
||h_n - p||^2 &= ||\beta_{n,0}z_n + \sum_{i=1}^{\infty}\beta_{n,i}q_n^i - p||^2 \\
&\leq \beta_{n,0}||z_n - p||^2 + \sum_{i=1}^{\infty}\beta_{n,i}d(q_n^i, T_i p)^2 - \beta_{n,0}\beta_{n,i}g(||z_n - q_n^i||) \\
&\leq \beta_{n,0}||z_n - p||^2 + \sum_{i=1}^{\infty}\beta_{n,i}\mathcal{H}(T_i z_n, T_i p)^2 - \beta_{n,0}\beta_{n,i}g(||z_n - q_n^i||) \\
&\leq \beta_{n,0}||z_n - p||^2 + \sum_{i=1}^{\infty}\beta_{n,i}||z_n - p||^2 - \beta_{n,0}\beta_{n,i}g(||z_n - q_n^i||) \\
&= ||z_n - p||^2 - \beta_{n,0}\beta_{n,i}g(||z_n - q_n^i||) \\
&\leq ||x_n - p||^2 + \theta_n(||x_n - p||^2 - ||x_{n-1} - p||^2) + 2\theta_n||x_n - x_{n-1}||^2 \\
&- (1 - 2\mu_n c_1)||w_n - t_n||^2 - (1 - 2\mu_n c_2)||t_n - z_n||^2 - \beta_{n,0}\beta_{n,i}g(||z_n - q_n^i||) \qquad (3.6)
\end{aligned}$$

From Algorithm 1 and (3.6), we have that

$$\begin{aligned}
||x_{n+1} - p||^2 \leq \alpha_n||f(x_n) - p||^2 &+ (1 - \alpha_n)||x_n - p||^2 + \theta_n(||x_n - p||^2 - ||x_{n-1} - p||^2) \\
&+ 2\theta_n||x_n - x_{n-1}||^2 - (1 - \alpha_n)(1 - 2\mu_n c_1)||w_n - t_n||^2 - (1 - \alpha_n)(1 - 2\mu_n c_2)||t_n - z_n||^2 \\
&- \beta_{n,0}\beta_{n,i}g(||z_n - q_n^i||). \qquad (3.7)
\end{aligned}$$

CASE A: Suppose there exists a natural number N such that $||x_{n+1} - p|| \leq ||x_n - p||$ for all $n \geq \mathbb{N}$. In this case, $||x_n - p||$ is convergent. Since $\{x_n\}$ is bounded, then

$$\lim_{n\to\infty} ||x_n - p|| = \lim_{n\to\infty} ||x_{n+1} - p||.$$

From (3.7), conditions (i), (iii) and (iv) of Algorithm 1, we have that

$$\lim_{n\to\infty} ||w_n - t_n|| = 0, \qquad (3.8)$$

and

$$\lim_{n\to\infty} ||t_n - z_n|| = 0. \qquad (3.9)$$

Furthermore, by applying Lemma (2.4) to (3.7), we obtain

$$\lim_{n\to\infty} ||z_n - q_n^i|| = 0 = \lim_{n\to\infty} d(z_n, T_i z_n). \tag{3.10}$$

From Algorithm 1, we have that

$$||h_n - z_n|| \leq \sum_{i=1}^{\infty} \beta_{n,i} ||q_n^i - z_n||. \tag{3.11}$$

Hence, using (3.10), we obtain that

$$\lim_{n\to\infty} ||h_n - z_n|| = 0. \tag{3.12}$$

Also, from Algorithm 1 and condition (iv), we have that

$$||w_n - x_n|| \leq \theta_n ||x_n - x_{n-1}|| \to 0, \; n \to \infty. \tag{3.13}$$

Again from Algorithm 1 and condition (i), we get

$$||x_{n+1} - h_n|| \leq \alpha_n ||f(x_n) - h_n|| \to 0, \; n \to \infty. \tag{3.14}$$

From (3.8) and (3.9), we have

$$||w_n - z_n|| \leq ||w_n - t_n|| + ||t_n - z_n|| \to 0, \; n \to \infty. \tag{3.15}$$

It follows from (3.12) and (3.15), that

$$\lim_{n\to\infty} ||h_n - w_n|| = 0. \tag{3.16}$$

From (3.13) and (3.16), we have that

$$\lim_{n\to\infty} ||h_n - x_n|| = 0. \tag{3.17}$$

Therefore, by (3.14) and (3.17), we obtain

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.18}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $\overline{x} \in H$. From (3.10) and Lemma (2.6), we obtain that $\overline{x} \in \cap_{i=1}^{\infty} F(T_i)$. We now show that $\overline{x} \in \Omega$. Using (3.8) and (3.13), we easily observe that the subsequences $\{w_{n_j}\}$ and $\{t_{n_j}\}$ which converges weakly to $\overline{x}$. Hence, from Lemma 2.5, we have that

$$\begin{aligned} \mu_{n_j}[g(w_{n_j}, u) - g(w_{n_j}, t_{n_j})] &\geq \langle t_{n_j} - w_{n_j}, t_{n_j} - u \rangle \; \forall \; u \in C \\ &\geq -||t_{n_j} - w_{n_j}|| \; ||t_{n_j} - u||, \; \forall \; u \in C; \end{aligned}$$

this implies that

$$g(w_{n_j}, u) - g(w_{n_j}, t_{n_j}) + \frac{1}{\mu_{n_j}} ||t_{n_j} - w_{n_j}|| \; ||t_{n_j} - u|| > 0, \; \forall \; u \in C;$$

Letting $j \to \infty$, by the hypothesis on $\{\mu_n\}$, (3.8) and condition L2 on g, we obtain that

$$g(\overline{x}, u) \geq 0, \; \forall \; u \in C.$$

This implies that $\overline{x} \in EP(g)$, that is $\overline{x} \in \Omega$. We therefore conclude that $\overline{x} \in \Gamma$.

We now prove that $\{x_n\}$ converges strongly to $\overline{x}$. Firstly, we show that $\limsup_{n\to\infty}\langle f(\overline{x})-\overline{x}, x_{n+1}-\overline{x}\rangle \leq 0$. Indeed, let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that $x_n \rightharpoonup x^*$ and

$$\limsup_{n\to\infty}\langle f(\overline{x})-\overline{x}, x_{n+1}-\overline{x}\rangle = \lim_{j\to\infty}\langle f(\overline{x})-\overline{x}, x_{n_j+1}-\overline{x}\rangle.$$

Since $||x_{n+1}-x_n|| \to 0$ as $n\to\infty$, we have by (2.2), that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(\overline{x})-\overline{x}, x_{n+1}-\overline{x}\rangle &= \lim_{j\to\infty}\langle f(\overline{x})-\overline{x}, x_{n_j+1}-\overline{x}\rangle\\ &\leq \langle f(\overline{x})-\overline{x}, x^*-\overline{x}\rangle\\ &\leq 0. \end{aligned}\tag{3.19}$$

From Algorithm 1 and (3.6), we have that

$$\begin{aligned}||x_{n+1}-\overline{x}||^2 &= ||\alpha_n f(x_n)+(1-\alpha_n)h_n-\overline{x}||^2\\
&\leq (1-\alpha_n)^2||h_n-\overline{x}||^2 + 2\alpha_n\langle f(x_n)-\overline{x}, x_{n+1}-\overline{x}\rangle\\
&\leq (1-\alpha_n)^2\left[||x_n-\overline{x}||^2+\theta_n(||x_n-\overline{x}||^2-||x_{n-1}-\overline{x}||^2)+2\theta_n||x_n-x_{n-1}||^2\right]\\
&+2\alpha_n\langle f(x_n)-f(\overline{x}),\ x_{n+1}-\overline{x}\rangle + 2\alpha_n\langle f(\overline{x})-\overline{x},\ x_{n+1}-\overline{x}\rangle\\
&\leq (1-\alpha_n)^2||x_n-\overline{x}||^2+(1-\alpha_n)^2\theta_n(||x_n-\overline{x}||^2-||x_{n-1}-\overline{x}||^2)\\
&+2(1-\alpha_n)^2\theta_n||x_n-x_{n-1}||^2+2\alpha_n\rho||x_n-\overline{x}||\ ||x_{n+1}-\overline{x}||+2\alpha_n\langle f(\overline{x})-\overline{x},\ x_{n+1}-\overline{x}\rangle\\
&\leq ((1-\alpha_n)^2+\alpha_n\rho)||x_n-\overline{x}||^2+\alpha_n\rho||x_{n+1}-\overline{x}||^2\\
&+(1-\alpha_n)^2\theta_n(||x_n-\overline{x}||^2-||x_{n-1}-\overline{x}||^2)+2(1-\alpha_n)^2\theta_n||x_n-x_{n-1}||^2\\
&+2\alpha_n\langle f(\overline{x})-\overline{x},\ x_{n+1}-\overline{x}\rangle\\
&\leq ((1-\alpha_n)^2+\alpha_n\rho)||x_n-\overline{x}||^2+\alpha_n\rho||x_{n+1}-\overline{x}||^2+\theta_n[(1-\alpha_n)^2||x_n-p||+||x_{n-1}-p||\\
&+2(1-\alpha_n)^2||x_n-x_{n-1}||]||x_n-x_{n_1}||+2\alpha_n\langle f(\overline{x})-\overline{x}, x_{n+1}-\overline{x}\rangle\\
&\leq (1-\alpha_n(2-\rho))||x_n-\overline{x}||^2+\alpha_n^2||x_n-p||^2+\alpha_n\rho||x_{n+1}-\overline{x}||^2+\theta_n[(1-\alpha_n)^2||x_n-p||\\
&+||x_{n-1}-p||+2(1-\alpha_n)^2||x_n-x_{n-1}||]||x_n-x_{n-1}||+2\alpha_n\langle f(\overline{x})-\overline{x}, x_{n+1}-\overline{x}\rangle. \end{aligned}\tag{3.20}$$

From (3.20), we obtain

$$\begin{aligned}||x_{n+1}-p||^2 \leq& \left(1-\frac{\alpha_n(2-\rho)}{1-\alpha_n\rho}\right)||x_n-p||^2+\frac{\alpha_n(2-\rho)}{1-\alpha_n\rho}\left(\frac{2\langle f(\overline{x})-\overline{x}, x_{n+1}-\overline{x}\rangle+\alpha_n M_1}{(2-\rho)}\right)\\ &+\frac{\theta_n||x_n-x_{n-1}||M_2}{(1-\alpha_n\rho)}\end{aligned}\tag{3.21}$$

where $M_1=\sup_{n\geq 1}||x_n-p||^2$ and $M_2=\sup_{n\geq1}((1-\alpha_n)^2(||x_n-p||+||x_{n-1}-p||)+2(1-\alpha_n)^2||x_n-x_{n-1}||)$. Using (3.19), conditions (i),(iv) and Lemma 2.7, we obtain that $\lim_{n\to\infty}||x_n-\overline{x}||^2=0$, that is, $x_n\to\overline{x}$.
CASE B: Assume that $\{||x_n-\overline{x}||\}$ is not a monotone decreasing sequence. Then, we define an integer sequence $\{\sigma(n)\}$ for all $n\geq n_0$ (for some n_0 large enough) by

$$\sigma(n) := \max\{k\in\mathbb{N};\ k\leq n : ||x_k-\overline{x}|| < ||x_{k+1}-\overline{x}||\}.$$

clearly, σ is a nondecreasing sequence such that $\sigma(n)\to\infty$ as $n\to\infty$ and for all $n\geq n_0$. From (3.7) and conditions (i)-(iv) of Algorithm 1, we have that

$$\lim_{k\to\infty}||w_{n_k}-t_{n_k}||=0,$$

$$\lim_{k\to\infty}||t_{n_k}-z_{n_k}||=0,$$

and

$$\lim_{k\to\infty} d(z_{n_k}, T_i u_{n_k}) = 0.$$

Following the same argument as in CASE A, we have that there exists a subsequence $\{x_{\sigma(n)}\}$ which converges weakly to $\overline{x} \in \Gamma$. Now, for all $n \geq n_0$, we have from (3.21) that

$$\begin{aligned} 0 &< ||x_{\sigma(n)+1} - \overline{x}||^2 - ||x_{\sigma(n)} - \overline{x}||^2 \\ &< \left(1 - \frac{\alpha_{\sigma(n)}(2-\rho)}{1-\alpha_{\sigma(n)}\rho}\right)||x_{\sigma(n)} - p||^2 + \frac{\alpha_{\sigma(n)}(2-\rho)}{1-\alpha_{\sigma(n)}\rho}\left(\frac{2\langle f(\overline{x}) - \overline{x}, x_{\sigma(n)+1} - \overline{x}\rangle + \alpha_{\sigma(n)} M_1}{(2-\rho)}\right) \\ &+ \frac{\theta_{\sigma(n)}||x_{\sigma(n)} - x_{\sigma(n)-1}||M_2}{(1-\alpha_{\sigma(n)}\rho)} - ||x_{\sigma(n)} - \overline{x}||^2, \end{aligned}$$

which implies that

$$\begin{aligned} \left(\frac{\alpha_{\sigma(n)}(2-\rho)}{1-\alpha_{\sigma(n)}\rho}\right)||x_{\sigma(n)} - p||^2 &< \frac{\alpha_{\sigma(n)}(2-\rho)}{1-\alpha_{\sigma(n)}\rho}\left(\frac{2\langle f(\overline{x}) - \overline{x}, x_{\sigma(n)+1} - \overline{x}\rangle + \alpha_{\sigma(n)} M_1}{(2-\rho)}\right) \\ &+ \frac{\theta_{\sigma(n)}||x_{\sigma(n)} - x_{\sigma(n)-1}||M_2}{(1-\alpha_{\sigma(n)}\rho)}. \end{aligned}$$

Therefore,

$$||x_{\sigma(n)} - p||^2 \leq 2\langle f(\overline{x}) - \overline{x}, x_{\sigma(n)+1} - \overline{x}\rangle + \alpha_{\sigma(n)} M_1 + \frac{\theta_{\sigma(n)}||x_{\sigma_n} - x_{\sigma_n - 1}||M_2}{\alpha_{\sigma(n)}(2\rho)}.$$

Since $\alpha_{\sigma(n)} \to 0$ as $n \to \infty$ and $\limsup_{n\to\infty}\langle f(\overline{x}) - \overline{x}, x_{\sigma(n)+1} - \overline{x}\rangle \leq 0$, we have by condition (v) that

$$\lim_{n\to\infty} ||x_{\sigma(n)} - p|| = 0. \tag{3.22}$$

Consequently, we obtain for all $n \geq n_0$, that

$$0 \leq ||x_n - \overline{x}|| \leq \max\{||x_{\sigma_n} - \overline{x}||^2, ||x_{\sigma_n+1} - \overline{x}||^2\} = ||x_{\sigma_n+1} - \overline{x}||^2.$$

Thus, $||x_n - p|| \to 0$ as $n \to \infty$. Therefore, the sequence $\{x_n\}$ converges strongly to $\overline{x}$.

Corollary 3.2. *Let C be a nonempty, closed and convex of a real Hilbert space H and $f : C \to C$ be a contraction with constant $\rho \in (0,1)$. Let $g : C \times C \to \mathbb{R}$ be a bifunction satisfying (L1)-(L4) and $T : C \to C$ be a nonexpansive mapping. Assume that $\Gamma := Fix(T) \cap \Omega \neq \emptyset$, then the sequence $\{x_n\}$ generated by the following algorithm converges strongly to $\overline{x} = P_\Gamma f(\overline{x})$, where $P_\Gamma f$ is the metric projection from $f(\overline{x})$ onto C. Initialization: Choose $x_1 \in H$ and $i \in 1, 2$, the sequences $\{\alpha_n\}, \{\beta_{n,0}\}$ and $\{\beta_{n,i}\}$ in $(0,1)$ such that*

$$\begin{cases} (i) \ \lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=0}^{\infty} \alpha_n = \infty; \\ (ii) \ \liminf_{n\to\infty} \beta_n(1-\beta_n) > 0, \\ (iii) \ 0 < \underline{\mu} \leq \mu_n \leq \overline{\mu} < \min\left(\frac{1}{2c_1}, \frac{1}{2c_2}\right); \\ (iv) \ \sum_{n=1}^{\infty} \theta_n ||x_n - x_{n-1}|| < \infty; \\ (v) \ \lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0. \end{cases}$$

Set n=0 and go to step 1,
Step 1: Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1});$$

Step 2: Compute

$$t_n = argmin_{u\in C}\{\mu_n g(w_n, u) + \frac{1}{2}||w_n - u||^2\};$$
$$z_n = argmin_{u\in C}\{\mu_n g(t_n, u) + \frac{1}{2}||w_n - u||^2\};$$

Step 3: Let h_n be defined by:

$$h_n = \beta_n z_n + (1-\beta_n)Tz_n;$$

Step 4: Compute

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)h_n.$$

Step 5: If $x_{n+1} = x_n$, then stop. Otherwise, set $n := n+1$ and go to step 1.

4. Application

In this section, we apply our result to solve the Variational Inequality Problem (VIP). Recall that the VIP consists of finding a point $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \ \forall \ y \in C. \tag{4.1}$$

with A a nonlinear mapping on a nonempty, closed and convex subset C of a real Hilbert space H. We denote the solution of $VIP(C, A)$ (4.1), by $Sol(C, A)$. A is said to be pseudomonotone on H if, for all $x, y \in H$, $\langle Ax, y - x\rangle \geq 0 \implies \langle Ay, y - x\rangle \geq 0$.
Suppose we define

$$g(x,y) := \begin{cases} \langle Ax, y - x\rangle, & \text{if } x, y \in C \\ +\infty, & \text{otherwise} \end{cases} \tag{4.2}$$

with $A : C \to H$, then the equilibrium (1.2) coincides with the VIP (4.1).
In this situation, Algorithm 1 provides a new method for solving a variational inequality problem and fixed points of a nonlinear mapping.
A convergence result for solving pseudomonotone variational inequality problem and fixed point of an infinite family of multi-valued quasi-nonexpansive mappings in a real Hilbert space is given below.

Theorem 4.1. *Let C be a nonempty, closed and convex of a real Hilbert space H and $f : C \to C$ be a contraction with constant $\rho \in (0,1)$. Let $A : C \to H$ be a pseudomonotone operator and $\{T_i\}_{i=1}^{\infty} : C \to K(C)$ be an infinite family of multi-valued quasi-nonexpansive mappings. Assume that $\Upsilon := \cap_{i=1}^{\infty} Fix(T_i) \cap Sol(C, A) \neq \emptyset$, then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to $\overline{x} = P_{\Upsilon} f(\overline{x})$, where $P_{\Upsilon} f$ is the metric projection from H onto C.*

For each pair $x, y \in C$, we define the bifunction g by (4.2). From the theorem's assumption, it is easily observed that the conditions of Theorem 3.1 are satisfied. Note that $\partial g(x, \cdot)(x) = Ax$. By Theorem 3.1, the sequence $\{x_n\}$ converges strongly to $\overline{x} \in \Gamma$. This implies that the sequence $\{x_n\}$ converges strongly to $\overline{x} \in \Upsilon$.

5. Numerical Example

We now display a numerical example to show the applicability of our main result.
Let $H = \mathbb{R}$ and define $g : C \times C \to \mathbb{R}$ by $g(x, y) = M(x)(y - x)$, where

$$M(x) = \begin{cases} 0, \ 0 \leq x \leq \frac{1}{80}; \\ sin(x - \frac{1}{80}), \ \frac{1}{80} \leq x \leq 1; \end{cases}$$

where g satisfies (L1)-(L4) with $c_1 = 1 = c_2$. Define $T_i : \mathbb{R} \to K(\mathbb{R})$ $(i = 1, 2, 3, ...)$ by

$$T_i x = \begin{cases} [0, \frac{x}{2i}] \ x \in [0, \infty); \\ [\frac{x}{2i}, 0] \ x \in (-\infty, 0]; \end{cases}$$

where $K(R)$ is the family of nonempty, closed and bounded subsets of $\mathbb{R}$. Clearly, T_i is a multi-valued quasi-nonexpansive mapping. Let $f : \mathbb{R} \to \mathbb{R}$ be given as $f(x) = \frac{x}{6}$ and take $\beta_{n,0} = \frac{1}{2}$, $\beta_{n,i} = \frac{1}{2^{i+1}}$, $q_n^i \in T_i z_n$ and $\alpha_n = \frac{1}{n+1}$. Then, conditions (i)-(v) of Theorem (3.1) are satisfied. Hence, Algorithm 1 becomes: For arbitrary $x_1 \in \mathbb{R}$:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \ n \geq 1, \\ t_n = w_n - \mu_n M(w_n); \\ z_n = t_n - \mu_n M(t_n); \\ h_n = \frac{1}{2} z_n + \sum_{i=1}^{\infty} \frac{1}{2^{i+1}} q_n^i; \\ x_{n+1} = \frac{x}{6(n+1)} + \frac{n}{n+1} h_n. \end{cases} \tag{5.1}$$

In what follows, we consider varying values of x_0, x_1 and μ in the different cases presented. We then plot a graph of errors against the number of iteration on a personal laptop Dell E6320 core i7 with MATLAB version 2019b. The figures are included to show the difference in convergence rate of the accelerated algorithm of Theorem 3.1 and an un-accelerated one.

Case 1:(a) $x_0 = 5/8$, $x_1 = -3/4$ and $\mu = 0.5$.

(b) $x_0 = 5/8$, $x_1 = -3/4$ and $\mu = 0.25$.

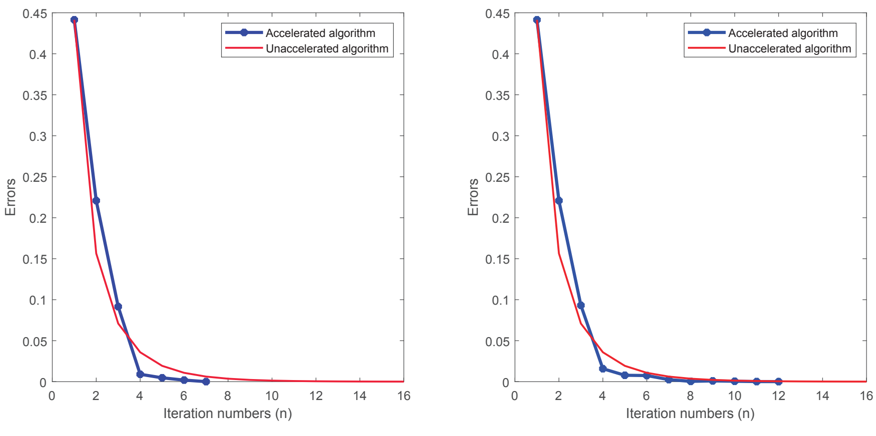

Figure 1: Left: Case (1a); Right: Case (1b).

Case 2:(a) $x_0 = 5/8$, $x_1 = -1/4$ and $\mu = 2$.

(b) $x_0 = 7/8$, $x_1 = -1/4$ and $\mu = 0.25$.

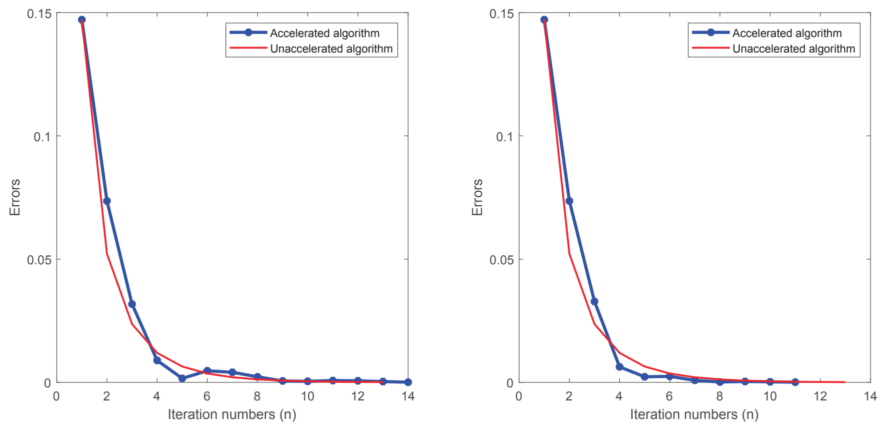

Figure 2: Left: Case (2a); Right: Case (2b).

6. Conclusion

We introduce a viscosity-type extragradient algorithm for finding a common point of the solution of a pseudomonotone equilibrium problem and a fixed point problem of an infinite family of multi-valued quasi-nonexpansive mappings in a real Hilbert space. The iterative scheme considered in this article has an advantage over the one considered in [14] in the sense that we do not use any projection of a point on the intersection of closed and convex sets which creates some difficulties in a practical calculation of the iterative sequence. The Halpern iteration considered in this article provides more flexibility in defining the algorithm parameters which is important for the numerical implementation perspective. We prove a strong convergence result for approximating the solution of the aforementioned problems and display a numerical example to show the applicability of our result.

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