



A Use of Pair (\mathcal{F}, h) Upper Class on Some Fixed Point Results in Probabilistic Menger Space

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ABSTRACT: In this paper, we define the concept of $(\mathcal{F}, h) - \alpha - \beta$ -contractive mappings in probabilistic Menger space and prove some fixed point theorems for such mappings. Some examples are given to support the obtained results.

Key Words: Fixed point, contractive mapping, probabilistic metric space.

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1. Introduction and preliminaries

probabilistic metric space were introduced by Menger in 1942, by using the notion of distribution functions in place of non-negative real numbers [5]. Sehgal and Bharucha-Reid proved the probabilistic version of the classical Banach contraction principle for B-contraction mappings in 1972 [8]. After this initial work, the fixed point theory in probabilistic metric spaces has been developed in many works such as [7,12,13,14,15,16,17,18]. The concepts of $\alpha - \psi$ -type contractive and α -admissible mappings were introduced by Gopal et. al. [3], who also established some fixed point theorems for these mappings in complete Menger spaces. After that, Shams and Jafari generalized this concept to (α, β, ψ) -contractive and $\alpha - \beta$ -admissible mappings and proved some fixed point theorems for such maps [11].

In this paper, we give a generalization of concept of contractive mapping in [11] and introduce the notion of $(\mathcal{F}, h) - \alpha - \beta$ -contractive mapping. Also we compare it with previous results in Menger space and prove some fixed point theorems for these contractive mappings. Our results generalize and improve the previous results in [3] and [11].

We first bring notion, definitions and known results, which are related to our work. For more details, we refer the reader to [4].

Definition 1.1. A distribution function is a function $F : (-\infty, \infty) \rightarrow [0, 1]$, that is non-decreasing and left continuous on \mathbb{R} . Moreover, $\inf_{t \in \mathbb{R}} F(t) = 0$ and $\sup_{t \in \mathbb{R}} F(t) = 1$. The set of all the distribution functions is denoted by D , and the set of those distribution functions such that $F(0) = 0$ is denoted by D^+ . We will denote the specific Heaviside distribution function by:

$$H(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0. \end{cases}$$

Definition 1.2. A binary operation $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t -norm if the following conditions hold:

- (a) T is commutative and associative,
- (b) T is continuous,
- (c) $T(a, 1) = a$ for all $a \in [0, 1]$,

(d) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$, for $a, b, c, d \in [0, 1]$.

The following are three basic continuous t-norms.

- (i) The minimum t-norm, say T_M , defined by $T_M(a, b) = \min\{a, b\}$.
 - (ii) The product t-norm, say T_P , defined by $T_P(a, b) = a.b$.
 - (iii) The Lukasiewicz t-norm, say T_L , defined by $T_L(a, b) = \max\{a + b - 1, 0\}$.
- These t-norms are related in the following way: $T_L \leq T_P \leq T_M$.

Definition 1.3. A Menger space is a triple (X, F, T) , where X is a nonempty set, T is a continuous t-norm, and F is a mapping from $X \times X$ into D^+ such that the following conditions hold:

- (PM1) $F_{x,y}(t) = H(t)$ if and only if $x = y$,
- (PM2) $F_{x,y}(t) = F_{y,x}(t)$
- (PM3) $F_{x,y}(t + s) \geq T(F_{x,z}(t), F_{z,y}(s))$ for all $x, y, z \in X$ and $s, t \geq 0$

Definition 1.4. Let (X, F, T) be a Menger space. Then

- (i) A sequence $\{x_n\}$ in X is said to converge to x if, for every $\epsilon > 0$ and $0 < \lambda < 1$, there exists a positive integer N such that $F_{x_n, x}(\epsilon) > 1 - \lambda$, whenever $n \geq N$.
- (ii) A sequence $\{x_n\}$ in X is called Cauchy sequence if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $F_{x_n, x_m}(\epsilon) > 1 - \lambda$ whenever $n, m \geq N$.
- (iii) A Menger space is said to be complete if and only if every Cauchy sequence in X converges to a point in X .

According to [6], the (ϵ, λ) -topology in Menger space (X, F, T) is introduced by the family of neighborhoods N_x of a point $x \in X$ given by

$$N_x = N_x(\epsilon, \lambda) : \epsilon > 0, \lambda \in (0, 1),$$

where

$$N_x(\epsilon, \lambda) = \{y \in X : F_{x,y}(\epsilon) > 1 - \lambda\}.$$

The (ϵ, λ) -topology is a Hausdorff topology. In this topology, a function f is continuous in $x_0 \in X$ if and only if $f(x_n) \rightarrow f(x_0)$, for every sequence $x_n \rightarrow x_0$.

Definition 1.5. [2] A function $\phi : [0, \infty) \rightarrow [0, \infty)$ is said to be a Φ -function if it satisfies the following conditions:

- (i) $\phi(t) = 0$ if and only if $t = 0$,
- (ii) $\phi(t)$ is strictly monotone increasing and $\phi(t) \rightarrow \infty$ as $t \rightarrow \infty$,
- (iii) ϕ is left continuous in $(0, \infty)$,
- (iv) ϕ is continuous at 0.

In the sequel, the class of all Φ -functions will be denoted by Φ .

Definition 1.6. [11] Let (X, F, T) be a Menger PM-space and $f : X \rightarrow X$ be a given mapping and $\alpha, \beta : X \times X \times (0, \infty) \rightarrow [0, \infty)$, be two functions, we say that f is $\alpha - \beta$ -admissible if

- (i) For all $x, y \in X$ and for all $t > 0$, $\alpha(x, y, t) \geq 1 \Rightarrow \alpha(fx, fy, t) \geq 1$,
- (ii) For all $x, y \in X$ and for all $t > 0$, $\beta(x, y, t) \leq 1 \Rightarrow \beta(fx, fy, t) \leq 1$.

Definition 1.7. [11] Let (X, F, T) be a Menger space and let $f : X \rightarrow X$ be a given mapping. We say that f is a generalized $\alpha - \beta$ -contractive mapping if there exist two functions $\alpha, \beta : X \times X \times (0, \infty) \rightarrow (0, \infty)$ such that

$$\begin{aligned} \beta(x, y, t) F_{fx, fy}(\varphi(t)) &\geq \alpha(x, y, t) \min\{F_{x,y}(\varphi(\frac{t}{c})), F_{x,fx}(\varphi(\frac{t}{c})), \\ &F_{y,fy}(\varphi(\frac{t}{c})), F_{x,fy}(2\varphi(\frac{t}{c})), F_{y,fx}(2\varphi(\frac{t}{c}))\} \end{aligned} \quad (1.1)$$

for all $x, y \in X$ and for all $t > 0$, where $\varphi \in \Phi$ and $c \in (0, 1)$.

Definition 1.8. [9,10] We say that the function $h: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is a function of subclass of type I, if $x \geq 1 \implies h(1, y) \leq h(x, y)$ for all $y \in \mathbb{R}^+$.

Example 1.9. [9,10] Define $h: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ by:

$$(a) \quad h(x, y) = (y + l)^x, l > 1;$$

$$(b) \quad h(x, y) = (x + l)^y, l > 1;$$

$$(c) \quad h(x, y) = x^n y, n \in \mathbb{N};$$

$$(d) \quad h(x, y) = y;$$

$$(e) \quad h(x, y) = \frac{1}{n+1} \left(\sum_{i=0}^n x^i \right) y, n \in \mathbb{N};$$

$$(f) \quad h(x, y) = \left[\frac{1}{n+1} \left(\sum_{i=0}^n x^i \right) + l \right]^y, l > 1, n \in \mathbb{N}$$

for all $x, y \in \mathbb{R}^+$. Then h is a function of subclass of type I.

Definition 1.10. [9,10] Let $h, \mathcal{F}: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$, then we say that the pair (\mathcal{F}, h) is an upper class of type I, if h is a function of subclass of type I and:

$$(i) \quad 0 \leq s \leq 1 \implies \mathcal{F}(s, t) \leq \mathcal{F}(1, t),$$

$$(ii) \quad h(1, y) \leq \mathcal{F}(1, t) \implies y \leq t \text{ for all } t, y \in \mathbb{R}^+.$$

Example 1.11. [9,10] Define $h, \mathcal{F}: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ by:

$$(a) \quad h(x, y) = (y + l)^x, l > 1 \text{ and } \mathcal{F}(s, t) = st + l;$$

$$(b) \quad h(x, y) = (x + l)^y, l > 1 \text{ and } \mathcal{F}(s, t) = (1 + l)^{st};$$

$$(c) \quad h(x, y) = x^m y, m \in \mathbb{N} \text{ and } \mathcal{F}(s, t) = st;$$

$$(d) \quad h(x, y) = y \text{ and } \mathcal{F}(s, t) = t;$$

$$(d) \quad h(x, y) = \frac{1}{n+1} \left(\sum_{i=0}^n x^i \right) y, n \in \mathbb{N} \text{ and } \mathcal{F}(s, t) = st;$$

$$(e) \quad h(x, y) = \left[\frac{1}{n+1} \left(\sum_{i=0}^n x^i \right) + l \right]^y, l > 1, n \in \mathbb{N} \text{ and } \mathcal{F}(s, t) = (1 + l)^{st}$$

for all $x, y, s, t \in \mathbb{R}^+$. Then the pair (\mathcal{F}, h) is an upper class of type I.

Definition 1.12. [9,10] We say that the function $h: \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is a function of subclass of type II, if $x, y \geq 1 \implies h(1, 1, z) \leq h(x, y, z)$ for all $z \in \mathbb{R}^+$.

Example 1.13. [9,10] Define $h: \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ by:

$$(a) \quad h(x, y, z) = (z + l)^{xy}, l > 1;$$

$$(b) \quad h(x, y, z) = (xy + l)^z, l > 1;$$

$$(c) \quad h(x, y, z) = z;$$

$$(d) \quad h(x, y, z) = x^m y^n z^p, m, n, p \in \mathbb{N};$$

$$(e) \quad h(x, y, z) = \frac{x^m + x^n y^p + y^q}{3} z^k, m, n, p, q, k \in \mathbb{N},$$

for all $x, y, z \in \mathbb{R}^+$. Then h is a function of subclass of type II.

Definition 1.14. [9,10] Let $h: \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ and $\mathcal{F}: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$, then we say that the pair (\mathcal{F}, h) is an upper class of type II, if h is a subclass of type II and:

- (i) $0 \leq s \leq 1 \implies \mathcal{F}(s, t) \leq \mathcal{F}(1, t)$,
(ii) $h(1, 1, z) \leq \mathcal{F}(s, t) \implies z \leq st$, for all $s, t, z \in \mathbb{R}^+$.

Example 1.15. [9,10] Define $h: \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ and $\mathcal{F}: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ by:

- (a) $h(x, y, z) = (z + l)^{xy}$, $l > 1$, $\mathcal{F}(s, t) = st + l$;
(b) $h(x, y, z) = (xy + l)^z$, $l > 1$, $\mathcal{F}(s, t) = (1 + l)^{st}$;
(c) $h(x, y, z) = z$, $\mathcal{F}(s, t) = st$;
(d) $h(x, y, z) = x^m y^n z^p$, $m, n, p \in \mathbb{N}$, $\mathcal{F}(s, t) = s^p t^p$;
(e) $h(x, y, z) = \frac{x^m + x^n y^p + y^q}{3} z^k$, $m, n, p, q, k \in \mathbb{N}$, $\mathcal{F}(s, t) = s^k t^k$,

for all $x, y, z, s, t \in \mathbb{R}^+$. Then the pair (\mathcal{F}, h) is an upper class of type II.

2. Fixed point theorems for generalized (\mathcal{F}, h) - $\alpha - \beta$ -contractive mappings

In this section we introduce the notions of generalized (\mathcal{F}, h) - $\alpha - \beta$ -contractive mapping in probabilistic Menger spaces.

Lemma 2.1. [1] Let (X, F, T) be a complete Menger space and $\varphi: [0, \infty) \rightarrow [0, \infty)$ be a Φ -function. Then the following statement holds:

If for $x, y \in X, c \in (0, 1)$, we have $F_{x,y}(\varphi(t)) \geq F_{x,y}(\varphi(\frac{t}{c}))$ for all $t > 0$, then $x = y$.

Definition 2.2. [3] Let (X, F, T) be a Menger space and $f: X \rightarrow X$ be a given mapping. We say that f is a generalized β -type contractive mapping if there exists a function $\beta: X \times X \times (0, \infty) \rightarrow (0, \infty)$ such that

$$\begin{aligned} \beta(x, y, t) F_{f_x, f_y}(\varphi(t)) &\geq \min\{F_{x,y}(\varphi(\frac{t}{c})), F_{x,fx}(\varphi(\frac{t}{c})), \\ &F_{y,fy}(\varphi(\frac{t}{c})), F_{x,fy}(2\varphi(\frac{t}{c})), F_{y,fx}(2\varphi(\frac{t}{c}))\} \end{aligned} \quad (2.1)$$

for all $x, y \in X$ and for all $t > 0$, where $\varphi \in \Phi$ and $c \in (0, 1)$.

Theorem 2.3. [1] Let (X, F, T) be a complete Menger space with continuous t -norm T which satisfies $T(a, a) \geq a$ for each $a \in [0, 1]$. Let $c \in (0, 1)$ be fixed. If for a Φ -function φ and a self-mapping f on X , we have

$$\begin{aligned} F_{f_x, f_y}(\varphi(t)) &\geq \min\{F_{x,y}(\varphi(\frac{t}{c})), F_{x,fx}(\varphi(\frac{t}{c})), F_{y,fy}(\varphi(\frac{t}{c})), \\ &F_{x,fy}(2\varphi(\frac{t}{c})), F_{y,fx}(2\varphi(\frac{t}{c}))\}, \end{aligned} \quad (2.2)$$

for all $x, y \in X$ and for all $t > 0$, then f has a unique fixed point in X .

Now, we introduce the following definition:

Definition 2.4. Let (X, F, T) be a Menger space and $f: X \rightarrow X$ be a given mapping. We say that f is a generalized (\mathcal{F}, h) - $\alpha - \beta$ -contractive mapping if there exist two functions $\alpha, \beta: X \times X \times (0, \infty) \rightarrow (0, \infty)$ such that

$$\begin{aligned} \mathcal{F}(\beta(x, y, t), F_{f_x, f_y}(\varphi(t))) &\geq h(\alpha(x, y, t), \min\{F_{x,y}(\varphi(\frac{t}{c})), F_{x,fx}(\varphi(\frac{t}{c})), \\ &F_{y,fy}(\varphi(\frac{t}{c})), F_{x,fy}(2\varphi(\frac{t}{c})), F_{y,fx}(2\varphi(\frac{t}{c}))\}) \end{aligned} \quad (2.3)$$

for all $x, y \in X$ and for all $t > 0$, where pair (\mathcal{F}, h) is an upper class of type I, $\varphi \in \Phi$ and $c \in (0, 1)$.

Remark 2.5. If $\alpha(x, y, t) = \beta(x, y, t) = 1$ for all $x, y \in X$ and for all $t > 0$, then the condition (2.3) reduces to condition (2.2), but the converse is not true always, (see Example 2.12).

Remark 2.6. If $\mathcal{F}(s, t) = st$ and $h(x, y) = xy$, then the condition (2.3) reduces to condition (1.1).

Remark 2.7. If $\mathcal{F}(s, t) = st$, $h(x, y) = xy$ and $\alpha(x, y, t) = 1$, then the condition (2.3) reduces to condition (2.1), but the converse is not true always.

Theorem 2.8. Let (X, F, T) be a complete Menger space with continuous t -norm T which satisfies $T(a, a) \geq a$ with $a \in [0, 1]$, let $f : X \rightarrow X$ be a generalized (\mathcal{F}, h) - $\alpha - \beta$ - contractive mapping satisfying the following conditions:

- (i) f is $\alpha - \beta$ -admissible,
- (ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0, t) \geq 1$ and $\beta(x_0, fx_0, t) \leq 1$ for all $t > 0$,
- (iii) if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}, t) \leq 1$, $\alpha(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$ and for all $t > 0$, and $x_n \rightarrow x$ as $n \rightarrow \infty$, then $\beta(x_n, x, t) \leq 1$ and $\alpha(x_n, x, t) \geq 1$ for all $n \in \mathbb{N}$ and for all $t > 0$.

Then f has a fixed point.

Proof. Since T is continuous and $T(a, a) \geq a$, for all $a \in [0, 1]$, then we have

$$T(a, a) \geq T(\min\{a, b\}, \min\{a, b\}) \geq \min\{a, b\},$$

for all $b \in [0, 1]$, and we can write $F_{x,y}(2t) \geq \min\{F_{x,z}(t), F_{z,y}(t)\}$, for all $x, y, z \in X$. Now, Let $x_0 \in X$ be such that (ii) holds and define a sequence $\{x_n\}$ in X such that $x_{n+1} = fx_n$, for all $n \in \mathbb{N}$. First, we suppose $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, otherwise f has trivially a fixed point. Now, since f is $\alpha - \beta$ -admissible, we have

$$\beta(x_0, fx_0, t) = \beta(x_0, x_1, t) \leq 1 \implies \beta(x_1, x_2, t) = \beta(fx_0, fx_1, t) \leq 1$$

and

$$\alpha(x_0, fx_0, t) = \alpha(x_0, x_1, t) \geq 1 \implies \alpha(x_1, x_2, t) = \alpha(fx_0, fx_1, t) \geq 1.$$

Consequently, by induction, we get $\beta(x_n, x_{n+1}, t) \leq 1$, and $\alpha(x_n, x_{n+1}, t) \geq 1$ for all $t > 0$. From the properties of Definition 1.5, we can find $r > 0$ such that $t > \varphi(r)$. Therefore we have:

$$\begin{aligned} \mathcal{F}(1, F_{fx_{n-1}, fx_n}(\varphi(r))) &\geq \mathcal{F}(\beta(x_{n-1}, x_n, r), F_{fx_{n-1}, fx_n}(\varphi(r))) \\ &\geq h(\alpha(x_{n-1}, x_n, r), \min\{F_{x_{n-1}, x_n}(\varphi(r/c)), F_{x_{n-1}, x_n}(\varphi(r/c)), \\ &\quad F_{x_n, x_{n+1}}(\varphi(r/c)), F_{x_{n-1}, x_{n+1}}(2\varphi(r/c)), F_{x_n, x_n}(2\varphi(r/c))\}) \\ &\geq h(1, \min\{F_{x_{n-1}, x_n}(\varphi(r/c)), F_{x_{n-1}, x_n}(\varphi(r/c)), \\ &\quad F_{x_n, x_{n+1}}(\varphi(r/c)), F_{x_{n-1}, x_{n+1}}(2\varphi(r/c)), F_{x_n, x_n}(2\varphi(r/c))\}). \end{aligned}$$

So we have

$$\begin{aligned} F_{x_n, x_{n+1}}(t) &\geq F_{fx_{n-1}, fx_n}(\varphi(r)) \geq \min\{F_{x_{n-1}, x_n}(\varphi(r/c)), F_{x_{n-1}, x_n}(\varphi(r/c)), \\ &\quad F_{x_n, x_{n+1}}(\varphi(r/c)), F_{x_{n-1}, x_{n+1}}(2\varphi(r/c)), F_{x_n, x_n}(2\varphi(r/c))\} \\ &= \min\{F_{x_{n-1}, x_n}(\varphi(r/c)), F_{x_n, x_{n+1}}(\varphi(r/c)), F_{x_{n-1}, x_{n+1}}(2\varphi(r/c))\} \\ &\geq \min\{F_{x_{n-1}, x_n}(\varphi(r/c)), F_{x_n, x_{n+1}}(\varphi(r/c)), \min\{F_{x_{n-1}, x_n}(\varphi(r/c)), \\ &\quad F_{x_n, x_{n+1}}(\varphi(r/c))\}\} \\ &= \min\{F_{x_{n-1}, x_n}(\varphi(r/c)), F_{x_n, x_{n+1}}(\varphi(r/c))\}. \end{aligned}$$

We shall prove that

$$F_{x_n, x_{n+1}}(\varphi(r)) \geq F_{x_{n-1}, x_n}(\varphi(\frac{r}{c})). \quad (2.4)$$

If we assume that $F_{x_n, x_{n+1}}(\varphi(\frac{r}{c}))$ is the minimum, that from lemma 2.1, we get that $x_n = x_{n+1}$, which leads to contradiction with the assumption that $x_{n+1} \neq x_n$ and so $F_{x_{n-1}, x_n}(\varphi(\frac{r}{c}))$ is the minimum and therefore (2.4) holds true. Since φ is strictly increasing, we have

$$F_{x_n, x_{n+1}}(t) \geq F_{x_n, x_{n+1}}(\varphi(r)) \geq F_{x_{n-1}, x_n}(\varphi(\frac{r}{c})) \geq \dots \geq F_{x_0, x_1}(\varphi(\frac{r}{c^n})),$$

that is, $F_{x_n, x_{n+1}}(t) \geq F_{x_0, x_1}(\varphi(\frac{r}{c^n}))$ for arbitrary $n \in \mathbb{N}$. Next, Let $m, n \in \mathbb{N}$ with $m > n$, then by (PM3) we have

$$\begin{aligned} F_{x_n, x_m}((m-n)t) &\geq \min\{F_{x_n, x_{n+1}}(t), \dots, F_{x_{m-1}, x_m}(t)\} \\ &\geq \min\{F_{x_0, x_1}(\varphi(\frac{r}{c^n})), \dots, F_{x_0, x_1}(\varphi(\frac{r}{c^{m-1}}))\}. \end{aligned}$$

Since φ is strictly increasing and $\varphi(t) \rightarrow \infty$ as $t \rightarrow \infty$, then for any fixed $\epsilon \in (0, 1)$, so there exists $n_0 \in \mathbb{N}$ such that $F_{x_0, x_1}(\varphi(\frac{r}{c^n})) > 1 - \epsilon$, whenever $n \geq n_0$. This implies that, for every $m > n \geq n_0$, we get $F_{x_n, x_m}((m-n)t) \geq 1 - \epsilon$. Since $t > 0$ and $\epsilon \in (0, 1)$ is arbitrary, we deduce that $\{x_n\}$ is a Cauchy sequence in the complete Menger space (X, F, T) . Then, $x_n \rightarrow u$ as $n \rightarrow \infty$ for some $u \in X$. We will show that u is a fixed point of f . By (PM3), we have

$$\begin{aligned} F_{fu, u}(t) &\geq T(F_{fu, x_n}(\varphi(r)), F_{x_n, u}(t - \varphi(r))) \\ &\geq \min\{F_{fu, x_n}(\varphi(r)), F_{x_n, u}(t - \varphi(r))\}. \end{aligned}$$

Notice that, if $x_n = fu$ for infinitely many values of n , then $u = fu$ and hence the proof finishes. Assume that $x_n \neq fu$ for all $n \in \mathbb{N}$. Thus, since $\lim_{n \rightarrow \infty} x_n = u$, for any arbitrary $\epsilon \in (0, 1)$ and n large enough, we get $F_{x_n, u}(t - \varphi(r)) > 1 - \epsilon$ and hence, we have $F_{u, fu}(t) \geq \min\{F_{fu, x_n}(\varphi(r)), 1 - \epsilon\}$. Since $\epsilon > 0$ is arbitrary, we can write $F_{fu, u}(t) \geq F_{fu, x_n}(\varphi(r))$. Next, we get

$$\begin{aligned} \mathcal{F}(1, F_{fu, x_n}(\varphi(r))) &= \mathcal{F}(1, F_{fu, fx_{n-1}}(\varphi(r))) \\ &\geq \mathcal{F}(\beta(u, x_{n-1}, r), F_{fu, fx_{n-1}}(\varphi(r))) \\ &\geq h(\alpha(u, x_{n-1}, r), \min\{F_{u, x_{n-1}}(\varphi(\frac{r}{c})), F_{x_{n-1}, x_n}(\varphi(\frac{r}{c})), \\ &\quad F_{fu, u}(\varphi(\frac{r}{c})), F_{fu, x_{n-1}}(2\varphi(\frac{r}{c})), F_{u, x_n}(2\varphi(\frac{r}{c}))\}) \\ &\geq h(1, \min\{F_{u, x_{n-1}}(\varphi(\frac{r}{c})), F_{x_{n-1}, x_n}(\varphi(\frac{r}{c})), F_{fu, u}(\varphi(\frac{r}{c})), \\ &\quad F_{fu, x_{n-1}}(2\varphi(\frac{r}{c})), F_{u, x_n}(2\varphi(\frac{r}{c}))\}). \end{aligned}$$

Hence we have

$$\begin{aligned} F_{fu, u}(t) &\geq F_{fu, x_n}(\varphi(r)) = F_{fu, fx_{n-1}}(\varphi(r)) \\ &\geq \min\{F_{u, x_{n-1}}(\varphi(\frac{r}{c})), F_{x_{n-1}, x_n}(\varphi(\frac{r}{c})), F_{fu, u}(\varphi(\frac{r}{c})), \\ &\quad F_{fu, x_{n-1}}(2\varphi(\frac{r}{c})), F_{u, x_n}(2\varphi(\frac{r}{c}))\} \\ &\geq \min\{F_{u, x_{n-1}}(\varphi(\frac{r}{c})), F_{fu, u}(\varphi(\frac{r}{c})), F_{x_{n-1}, x_n}(\varphi(\frac{r}{c}))\}. \end{aligned}$$

It follows that

$$\begin{aligned} F_{fu, u}(t) &\geq \liminf_{n \rightarrow \infty} F_{fu, x_n}(\varphi(r)) \\ &\geq \liminf_{n \rightarrow \infty} \min\{F_{u, x_{n-1}}(\varphi(\frac{r}{c})), F_{fu, u}(\varphi(\frac{r}{c})), F_{x_{n-1}, x_n}(\varphi(\frac{r}{c}))\} \\ &\geq \min\{1 - \epsilon, F_{fu, u}(\varphi(\frac{r}{c})), 1 - \epsilon\}. \end{aligned}$$

Finally, since $\epsilon \in (0, 1)$ is arbitrary, we have $F_{fu, u}(\varphi(r)) \geq F_{fu, u}(\varphi(\frac{r}{c}))$ and so, by Lemma 2.1, we deduce that $u = fu$. This completes the proof. \square

Corollary 2.9. [11] Let (X, F, T) be a complete Menger space with continuous t -norm T which satisfies $T(a, a) \geq a$ with $a \in [0, 1]$, let $f : X \rightarrow X$ satisfy the following conditions:

- (i) f is a generalized $\alpha - \beta$ - contractive mapping,
- (ii) f is $\alpha - \beta$ -admissible,
- (iii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0, t) \geq 1$ and $\beta(x_0, fx_0, t) \leq 1$ for all $t > 0$,
- (iv) if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}, t) \leq 1$, $\alpha(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$ and for all $t > 0$, and $x_n \rightarrow x$ as $n \rightarrow \infty$, then $\beta(x_n, x, t) \leq 1$ and $\alpha(x_n, x, t) \geq 1$ for all $n \in \mathbb{N}$ and for all $t > 0$.

Then f has a fixed point.

Corollary 2.10. Let (X, F, T) be a complete Menger space with continuous t -norm T which satisfies $T(a, a) \geq a$ with $a \in [0, 1]$, let there exists a function $\beta : X \times X \times (0, \infty) \rightarrow (0, \infty)$ such that $f : X \rightarrow X$ satisfy the following conditions:

- (i) f is a β -type contractive mapping,
- (ii) For any $x, y \in X$ and for all $t > 0$, $\beta(x, y, t) \leq 1 \Rightarrow \beta(fx, fy, t) \leq 1$,
- (iii) there exists $x_0 \in X$ such that $\beta(x_0, fx_0, t) \leq 1$ for all $t > 0$,
- (iv) if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N}$ and for all $t > 0$, and $x_n \rightarrow x$ as $n \rightarrow \infty$, then $\beta(x_n, x, t) \leq 1$ for all $n \in \mathbb{N}$ and for all $t > 0$.

Then f has a fixed point.

Two examples of the generalized contractions of Definition 2.4 possessing fixed points according to the above results follow below:

Example 2.11. Let $X = \mathbb{R}$, $T(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$ and $F_{x,y}(t) = \frac{t}{t+|x-y|}$ for all $x, y \in X$ and for all $t > 0$. Clearly (X, F, T) is a complete Menger space. Define the mapping $f : X \rightarrow X$ by

$$fx = \begin{cases} \frac{1}{3} & x \in [0, 1) \\ 1 & x = 1 \\ 2 & \text{otherwise} \end{cases}$$

and two functions $\alpha, \beta : X \times X \times (0, \infty) \rightarrow (0, \infty)$ by

$$\beta(x, y, t) = \begin{cases} 1 & x, y \in [0, 1) \\ 0 & x = 1 \text{ or } y = 1 \\ 1 & x = y = 1 \\ 3 & \text{otherwise} \end{cases}, \quad \alpha(x, y, t) = \begin{cases} 1 & x, y \in [0, 1) \\ \frac{1}{2} & x = 1 \text{ or } y = 1 \\ 1 & x = y = 1 \\ 0 & \text{otherwise,} \end{cases}$$

We define $\mathcal{F}, h : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ by $h(x, y) = (y+3)^x$ and $\mathcal{F}(s, t) = st+3$. Now, consider $\varphi : [0, \infty) \rightarrow [0, \infty)$ defined by $\varphi(t) = t$. Let $c = \frac{1}{2}$. We show that f satisfies the hypotheses of Theorem 2.8. At first we prove f is $\alpha - \beta$ -admissible. If $\beta(x, y, t) = 1$, this implies $x, y \in [0, 1]$, so $\beta(fx, fy, t) = 1$. If $\beta(x, y, t) = 0$, this implies $x = 1$ or $y = 1$. There are two cases:

- (1) If $x = 1$ and $y \in [0, 1]$, hence $\beta(fx, fy, t) = \beta(1, 1, t) = 1$.
- (2) If $x = 1$ and y is not in $[0, 1]$, so by the definitions of f and β , we have $\beta(fx, fy, t) = \beta(1, 3, t) = 0$. Similarly when $\alpha(x, y, t) \geq 1$, then $\alpha(fx, fy, t) \geq 1$. Hence f is $\alpha - \beta$ -admissible. On the other hand for $x_0 = 1$ we have $\alpha(1, f(1), t) = 1$ and $\beta(1, f(1), t) = 1$. Finally we show that f satisfies (2.3).

If $x, y \in [0, 1)$ or $x = y = 1$, then $\mathcal{F}(1, 1) = 4$ and hence the inequality (2.3) is true. If $x = 1$ or $y = 1$, we have $\mathcal{F}(0, 1) = 3$ and the inequality is true. In the other cases $\alpha = 0$ and the inequality is obviously true. Thus all the conditions of Theorem 2.8 hold and f has two fixed points, $x = 1$, $x = 2$ and $x = \frac{1}{3}$. On the other hand, f does not satisfy (1.1). Indeed for $x = 1$ and $y = \frac{1}{2}$, we get $\beta(x, y, t) = 0$ and $\alpha(x, y, t) = \frac{1}{2}$, then the left inequality of (1.1) is equal zero. Hence f does not satisfy condition (i) of Corollary 2.9.

Example 2.12. Let $X = [0, \infty)$, $T(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$ and $F_{x,y}(t) = \frac{t}{t+|x-y|}$ for all $x, y \in X$ and for all $t > 0$. Clearly (X, F, T) is a complete Menger space. Define the mapping $f : X \rightarrow X$ by

$$fx = \begin{cases} \frac{1}{2} & x \in [0, 1) \\ 1 & x = 1 \\ 2 & \text{otherwise} \end{cases}$$

and two functions $\alpha, \beta : X \times X \times (0, \infty) \rightarrow (0, \infty)$ by

$$\beta(x, y, t) = \begin{cases} 2 & x, y \in [0, 1) \\ 1 & x = y = 1 \\ 1 & \text{otherwise} \end{cases}, \quad \alpha(x, y, t) = \begin{cases} 1 & x, y \in [0, 1] \\ 0 & \text{otherwise,} \end{cases}$$

We define $\mathcal{F}, h : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ by $h(x, y) = (x + l)^y$ and $\mathcal{F}(s, t) = (1 + l)^{st}$. Now, consider $\varphi : [0, \infty) \rightarrow [0, \infty)$ defined by $\varphi(t) = t$. Let $c = \frac{1}{2}$. We show that f satisfies the hypotheses of Theorem 2.8. At first we prove that f is $\alpha - \beta$ -admissible. If $\beta(x, y, t) \leq 1$, this implies $x, y \notin [0, 1)$, so by the definitions of f and β , we have $\beta(fx, fy, t) = 1$. Similarly when $\alpha(x, y, t) \geq 1$, then $\alpha(fx, fy, t) \geq 1$. Hence f is $\alpha - \beta$ -admissible. On the other hand for $x_0 = 1$ we have $\alpha(1, f(1), t) = 1$ and $\beta(1, f(1), t) = 1$. It is easy to show that f satisfies (2.3) and hence f has three fixed points, $x = 1$, $x = \frac{1}{2}$ and $x = 2$. On the other hand, f does not satisfy (2.2) and (1.1). Indeed for $x = 2$ and $y = 1$, we get $c \geq 1$, that is a contradiction.

We prove, with next theorem, uniqueness of the fixed point.

Theorem 2.13. With the same hypotheses of Theorem 2.8, if for all $u \in \text{Fix}(f)$ (The set of fixed points of f) and for all $t > 0$ there exists $z \in X$ such that $\beta(z, fz, t) \leq 1$ with $\beta(u, z, t) \leq 1$ and $\alpha(z, fz, t) \geq 1$ with $\alpha(u, z, t) \geq 1$, then f has a unique fixed point.

Proof. Let $u, v \in X$ be such that $fu = u$ and $fv = v$. From the hypotheses, there exists $z \in X$ such that

$$\beta(z, fz, t) \leq 1, \text{ with } \beta(u, z, t) \leq 1 \text{ and } \beta(v, z, t) \leq 1$$

and

$$\alpha(z, fz, t) \geq 1, \text{ with } \alpha(u, z, t) \geq 1 \text{ and } \alpha(v, z, t) \geq 1.$$

Since f is $\alpha - \beta$ -admissible, then we have $\beta(fz, f^2z, t) \leq 1$, $\alpha(fz, f^2z, t) \geq 1$. Also

$$\beta(u, fz, t) \leq 1 \text{ and } \beta(v, fz, t) \leq 1, \alpha(u, fz, t) \geq 1 \text{ and } \alpha(v, fz, t) \geq 1.$$

By induction, for all $t > 0$ we get

$$\beta(z_n, z_{n+1}, t) \leq 1, \beta(u, z_n, t) \leq 1, \beta(v, z_n, t) \leq 1$$

and

$$\alpha(z_n, z_{n+1}, t) \geq 1, \alpha(u, z_n, t) \geq 1, \alpha(v, z_n, t) \geq 1,$$

where $z_n = f^n z$.

From the properties of function φ , we can find $r > 0$ such that $t > \varphi(r)$ and therefore we have:

$$\begin{aligned} \mathcal{F}(1, F_{u, fz_n}(\varphi(r))) &= \mathcal{F}(1, F_{fu, fz_n}(\varphi(r))) \geq \mathcal{F}(\beta(u, z_n, r), F_{fu, fz_n}(\varphi(r))) \\ &\geq h(\alpha(u, z_n, r), \min\{F_{u, z_n}(\varphi(\frac{r}{c})), F_{u, fu}(\varphi(\frac{r}{c})), F_{z_n, z_{n+1}}(\varphi(\frac{r}{c})), \\ &\quad F_{u, z_{n+1}}(2\varphi(\frac{r}{c})), F_{z_n, fu}(2\varphi(\frac{r}{c}))\}) \\ &\geq h(1, \min\{F_{u, z_n}(\varphi(\frac{r}{c})), F_{u, fu}(\varphi(\frac{r}{c})), F_{z_n, z_{n+1}}(\varphi(\frac{r}{c})), \\ &\quad F_{u, z_{n+1}}(2\varphi(\frac{r}{c})), F_{z_n, fu}(2\varphi(\frac{r}{c}))\}), \end{aligned} \tag{2.5}$$

where $c \in (0, 1)$. Hence from (2.5) we get:

$$F_{u, z_{n+1}}(t) \geq F_{u, fz_n}(\varphi(r)) \min\{F_{u, z_n}(\varphi(\frac{r}{c})), F_{u, fu}(\varphi(\frac{r}{c})), F_{z_n, z_{n+1}}(\varphi(\frac{r}{c})), \\ F_{u, z_{n+1}}(2\varphi(\frac{r}{c})), F_{z_n, fu}(2\varphi(\frac{r}{c}))\},$$

which implies $F_{u, z_{n+1}}(t) \geq \min\{F_{u, z_n}(\varphi(\frac{r}{c})), F_{z_n, z_{n+1}}(\varphi(\frac{r}{c}))\}$. Now, we have two cases:

(i) We assume that $F_{z_n, z_{n+1}}(\varphi(\frac{r}{c}))$ is the minimum. Then, by applying (2.3), we can write

$$F_{u, z_{n+1}}(\varphi(r)) \geq F_{z_n, z_{n+1}}(\varphi(\frac{r}{c})) \geq \min\{F_{z_{n-1}, z_n}(\varphi(\frac{r}{c^2})), F_{z_n, z_{n+1}}(\varphi(\frac{r}{c^2}))\}.$$

Now, if $F_{z_n, z_{n+1}}(\varphi(\frac{r}{c}))$ is the minimum for some $n \in \mathbb{N}$, by Lemma 2.4, we deduce that $z_n = z_{n+1} = u$. Consequently, we deduce that $\beta(v, u, t) \leq 1$ and $\alpha(v, u, t) \geq 1$ and so by (2.3) we have

$$F_{u, v}(\varphi(t)) \geq \min\{F_{u, v}(\varphi(\frac{t}{c})), F_{v, v}(\varphi(\frac{t}{c})), F_{u, u}(\varphi(\frac{t}{c})), F_{v, u}(2\varphi(\frac{t}{c})), \\ F_{u, v}(2\varphi(\frac{t}{c}))\} = F_{v, u}(\varphi(\frac{t}{c})).$$

Again, by Lemma 2.4, we conclude that $u = v$. On the other hand, if $F_{z_{n-1}, z_n}(\varphi(\frac{r}{c^2}))$ is the minimum, then

$$F_{z_n, z_{n+1}}(\varphi(\frac{r}{c})) \geq F_{z_{n-1}, z_n}(\varphi(\frac{r}{c^2})) \geq \dots \geq F_{z_0, z_1}(\varphi(\frac{r}{c^{n+1}}))$$

and, letting $n \rightarrow \infty$, we get

$$F_{z_n, z_{n+1}}(\varphi(\frac{r}{c})) \rightarrow 1.$$

Therefore $F_{u, z_{n+1}}(t) \rightarrow 1$ as $n \rightarrow \infty$, which implies $z_{n+1} \rightarrow u$ as $n \rightarrow \infty$.

(ii) Suppose that $F_{u, z_n}(\varphi(\frac{r}{c}))$ is the minimum, then we have

$$F_{u, z_{n+1}}(\varphi(r)) \geq F_{u, z_n}(\varphi(\frac{r}{c})) \geq F_{u, z_{n-1}}(\varphi(\frac{r}{c^2})) \geq \dots \geq F_{u, z_0}(\varphi(\frac{r}{c^{n+1}})).$$

Letting $n \rightarrow \infty$, we obtain $F_{u, z_{n+1}}(\varphi(r)) \rightarrow 1$ as $n \rightarrow \infty$, i.e, $z_{n+1} \rightarrow u$ as $n \rightarrow \infty$.

A similar argument shows that $z_{n+1} \rightarrow v$, for $n \rightarrow \infty$. Now, the uniqueness of the limit, gives us $u = v$ and the proof is complete. \square

References

- [1] N. A. Babacev, *Nonlinear generalized contraction on Menger PM-spaces*, Appl. Anal. Discrete Math. 6, 257–264 (2012)
- [2] B. S. Choudhury, K. P. Das, *A new contraction principle in Menger spaces*, Acta Math. Sin. 24, 1379–1386 (2008) .
- [3] D. Gopal, M. Abbas, C. Vetro, *Some new fixed point theorems in Menger PM-spaces with application to Volterra type integral equation*, Appl Math and Comput. 232 , 955–967 (2014).
- [4] O. Hadžić, E. Pap, *Fixed Point Theory in Probabilistic Metric Spaces*, Kluwer, Dordrecht (2001).
- [5] K. Menger, *Statistical metric*, Proc. Natl. sci. 28, 535–538 (1942) .
- [6] B. Schweizer, A. Sklar, *Probabilistic Metric Space*, Elsevier, North-Holland, New York (1983).
- [7] J. Jachymski, *On probabilistic φ -contractions on Menger spaces*, Nonlinear Anal. 73, 2199–2203 (2010).
- [8] V. M. Sehgal, T. A. Bharucha-Reid, *Fixed point of contraction mappings in PM-spaces*, Math. Syst. Theory. 6, 97–102 (1972).
- [9] A. H. Ansari, *Note on α -admissible mappings and related fixed point theorems* , The 2nd Regional Conference on Mathematics And Applications, PNU, 373–376 (2014).
- [10] A. H. Ansari and S. Shukla, *Some fixed point theorems for ordered F - (\mathcal{F}, h) -contraction and subcontractions in 0-f-orbitally complete partial metric spaces*, J. Adv. Math. Stud. 9(1), 37–53 (2016).
- [11] M. Shams, Sh. Jafari, *Some fixed point results in probabilistic Menger space*, Bol. Soc. Paran. Mat. 35 (3) 9–24 (2017).

- [12] Y. Liu, Zh. Li, *Coincidence point theorems in probabilistic and fuzzy metric spaces*, Fuzzy Sets Syst. 158, 58–70 (2007).
- [13] D. Mihet, *A Banach contraction theorem in fuzzy metric spaces*, Fuzzy Sets Syst. 144, 431–439 (2004).
- [14] D. Mihet, *Fixed point theorems in probabilistic metric spaces*, Chaos Solitons Fractals 41, 1014–1019 (2009).
- [15] J. Jachymski, *On probabilistic φ -contractions on Menger spaces*, Nonlinear Anal. 73, 2199–2203 (2010).
- [16] T. Došenović, P. Kumam, D. Gopal, D. K. Patel, and A. Takači, *On fixed point theorems involving altering distances in Menger probabilistic metric spaces*, J Inequal Appl 1, 1–10 (2013).
- [17] M. Hasanuzzaman, and M. Imdad, *Relation theoretic metrical fixed point results for Suzuki type ZR-contraction with an application*, AIMS Mathematics, 5(3), 2071–2087 (2020).
- [18] N. Wairojjana, T. Došenović, D. Rakić, D. Gopal, and P. Kumam, *An altering distance function in fuzzy metric fixed point theorems*, Fixed Point Theory A, 1, 1–19 (2015).

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