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q-analogue of a Class of Harmonic Functions

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Abstract: The purpose of the present paper is to introduce a new subclass of harmonic univalent functions associated with a q-Ruscheweyh derivative operator. A necessary and sufficient convolution condition for the functions to be in this class is obtained. Using this necessary and sufficient coefficient condition, results based on the extreme points, convexity and compactness for this class are also obtained.

Key Words: q-Ruscheweyh derivative operator, univalent functions, harmonic functions, subordination.

Contents

1. Introduction

The theory of q-calculus has motivated the researchers due to its applications in the field of physical sciences, specially in quantum physics. Jackson [9,10] was the first to give some applications of q-calculus by introducing the q-analogues of derivative and integral. Jackson's q-derivative operator ∂_q on a function h analytic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is defined for $0 < q < 1$, by

$$\partial_q h(z) = \begin{cases} \frac{h(z)-h(qz)}{(1-q)z} & z \neq 0, \\ h'(0) & z = 0. \end{cases}$$

For a power function $h(z) = z^k, \quad k \in \mathbb{N} = \{1, 2, 3, \cdots\}$,

$$\partial_q h(z) = \partial_q(z^k) = [k]_q z^{k-1},$$

where $[k]_q$ is the q-integer number k defined by

$$[k]_q = \frac{1-q^k}{1-q} = 1 + q + q^2 + ...q^{k-1}. \tag{1.1}$$

For any non-negative integer k the q-number factorial is defined by

$$[k]_q! = [1]_q[2]_q[3]_q \dots [k]_q \qquad ([0]_q! = 1).$$

For more detailed study see [1]. Clearly, $\lim_{q\to 1^-} [k]_q = k$ and $\lim_{q\to 1^-} \partial_q h(z) = h'(z)$.

Let $\mathcal{A}$ denote the class of functions $h(z)$ that are analytic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the normalization $h(0) = h'(0) - 1 = 0$. Complex-valued harmonic functions of the form: $f = u + iv$, where u and v are real-valued harmonic functions in $\mathbb{D}$, can also be expressed as $f = h + \bar{g}$, where h and g are analytic in $\mathbb{D}$. The Jacobian of the function $f = h + \overline{g}$ is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$. According to the Lewy [15], every harmonic function $f = h + \overline{g}$ is locally univalent and sense preserving in $\mathbb{D}$ if and only if $J_f(z) > 0$ in $\mathbb{D}$ which is equivalent to the existence of an analytic function $\omega(z) = g'(z)/h'(z)$ in $\mathbb{D}$ such that

$$|\omega(z)| < 1 \quad \text{for all} \quad z \in \mathbb{D}.$$

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The function $\omega(z)$ is called the dilatation of f. By requiring harmonic function to be sense-preserving, we retain some basic properties exhibited by analytic functions, such as the open mapping property, the argument principal, and zeros being isolated (see for detail [5]).

A class of harmonic functions $f = h+\overline{g}$ with the normalized conditions $h(0) = 0 = g(0)$ and $h'(0) = 1$ is denoted by $\mathcal{H}$ and functions therein are of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k z^k}. \tag{1.2}$$

A sub class of functions $f = h + \overline{g} \in \mathcal{H}$ with the additional condition $g'(0) = 0$ is denoted by $\mathcal{H}^0$. The class of all univalent, sense preserving harmonic functions $f = h + \overline{g} \in \mathcal{H}\ \left(\mathcal{H}^0\right)$ is denoted by $S_{\mathcal{H}}\ \left(S^0_{\mathcal{H}}\right)$. Further, if $g(z) \equiv 0$, the class $S_{\mathcal{H}}$ reduces to the class S of univalent functions in $\mathcal{A}$.

Motivated by Dziok *et al.* [3], the function $f \in \mathcal{H}^0$ is subordinate to a function F, and write $f(z) \prec F(z)$, if there exists a analytic function ω, which maps $\mathbb{D}$ into oneself with $\omega(0) = 0$, such that $f(z) = F(\omega(z))$ (see [4,24,3]).

The convolution of two analytic functions $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ is defined by $(f * g)(z) = \sum_{n=1}^{\infty} a_n b_n z^n$. The convolution $\widetilde{*}$ of two harmonic functions $f = h + \overline{g}$ and $F = H + \overline{G}$ is defined by $(f\widetilde{*}F)(z) = (g * G)(z) + \overline{(h * H)(z)}$.
The q- generalized Pochammer symbol for $t \in \mathbb{R}$ and $n \in \mathbb{N}$ is defined as

$$([t]_q)_n = \begin{cases} [t]_q[t+1]_q[t+2]_q \dots [t+(n-1)]_q, & \text{if } n \geq 1 \\ 1, & \text{if } n = 0. \end{cases}$$

for $t > 0$. Let q-gamma function is defined as

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t) \qquad \textit{and} \qquad \Gamma_q(1) = 1.$$

A function $f \in S_{\mathcal{H}}$ is said to be starlike and convex, respectively, of order α and in the classes, respectively, $S^*_H(\alpha)$ and $S^c_H(\alpha)$ (investigated by Jahangiri [11]) if and only if

$$\Re e \left\{ \frac{\mathcal{D} f(z)}{f(z)} \right\} > \alpha$$

where

$$\mathcal{D} f(z) = z h'(z) - \overline{z g'(z)}.$$

In 2014, Kanas and Raducanu [14] also (see [16]) define the q-analogue of Ruscheweyh operator

$$D^n_q h(z) = h(z) * \phi_{q,n+1}(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma_q(k+n)}{[k-1]_q! \Gamma_q(n+1)} a_k z^k, \quad (n > -1),$$

where

$$\phi_{q,n+1}(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma_q(k+n)}{[k-1]_q! \Gamma_q(n+1)} z^k,$$

For convenience, we use

$$\psi_{k-1} = \frac{\Gamma_q(k+n)}{[k-1]_q! \Gamma_q(n+1)}.$$

The q-Ruscheweyh operator D^n_q of order $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ for an analytic function $h(z)$ is defined by

$$D^0_q h(z) = h(z),\ D^1_q h(z) = D_q h(z) = z \partial_q h(z).$$

and for $n \in \mathbb{N}$,

$$D^n_q h(z) = D_q(D^{n-1}_q h(z)), \tag{1.3}$$

$$D_q^n h(z) = \frac{z\partial_q^n(z^{n-1}h(z))}{[n]_q!},$$

Observe that

$$D_q h(z) = h(z) * D_q\left(\frac{z}{1-z}\right), \tag{1.4}$$

and

$$\begin{aligned} D_q\left(\frac{z}{1-z}\right) &= z + \sum_{k=2}^{\infty}[k]_q z^k, \\ &= \frac{z}{(1-z)(1-qz)}, \end{aligned} \tag{1.5}$$

where $[k]_q$ is the q-integer number defined by (1.1). The operator D_q^n reduces to the well known Ruscheweyh operator D^n [23] as $q \to 1^-$.

Further, the q-Ruscheweyh operator $\mathcal{D}_q^n$ of order $n \in \mathbb{N}_0$ for the harmonic function $f = h + \overline{g}$ is defined by ([13])

$$\mathcal{D}_q^n f(z) = D_q^n h(z) + (-1)^n \overline{D_q^n g(z)}.$$

As $q \to 1^-$, the operator $\mathcal{D}_q^n$ reduces to the operator $\mathcal{D}^n$ which is the Ruscheweyh operator for a harmonic function $f = h + \bar{g}$ ([17]).

Involving the q-Ruscheweyh operator $\mathcal{D}_q^n$, we define a subclass $S_H^0(n, q, A, B)$ of harmonic functions $f \in \mathcal{H}^0$ that satisfy the condition

$$\frac{\mathcal{D}_q\left(\mathcal{D}_q^n f(z)\right)}{\mathcal{D}_q^n f(z)} \prec \frac{1+Az}{1+Bz} \quad (-1 \leq A < B \leq 1; z \in \mathbb{D}). \tag{1.6}$$

We denote by $\mathcal{T}S_H^0(n, q, A, B)$ a subclass of harmonic functions $f = h + \bar{g} \in S_H^0(n, q, A, B)$ where n, h and g are of the form:

$$h(z) = z - \sum_{k=2}^{\infty}|a_k|z^k \quad \text{ and } \quad g(z) = (-1)^n \sum_{k=2}^{\infty}|b_k|z^k \quad (z \in \mathbb{D}). \tag{1.7}$$

When $q \to 1^-$ the class $S_H^0(n, q, A, B)$ reduces to the class $S_H^n(A, B)$ which was studied by Dziok [2] (see also [3]). Further, we denote the class $S_H^0(n, q, (1+q)\alpha - 1, q)$ $(0 \leq \alpha < 1)$ by $\mathcal{H}_q^n(\alpha)$ and hence, the classes $\mathcal{H}_q^0(\alpha)$ and $\mathcal{H}_q^1(\alpha)$ are the q-analogue of harmonic starlike and harmonic convex functions of order α, respectively. As $q \to 1^-$, the classes $\mathcal{H}_q^0(\alpha) =: S_H^*(\alpha)$ and $\mathcal{H}_q^1(\alpha) =: S_H^c(\alpha)$ are the well known classes of the functions $f \in S_H^0$ which are starlike and convex functions of order α, respectively, in $\mathbb{D}$ and are investigated by Jahangiri [11].

Research work in connection with function theory and q-calculus was first introduced by Ismail *et al.* [8]. Recently, q-calculus is involved in the theory of analytic functions in the work [6,7,16]. But research on q-calculus in connection with harmonic functions is fairly new and not much published (one may find papers [20], [21], [12], [13], [18], [22], [19]).

In this paper, a class $S_H^0(n, q, A, B)$ of harmonic functions $f \in \mathcal{H}^0$, associated with q-Ruscheweyh operator is defined as above (1.6). A necessary and sufficient convolution condition for the functions $f \in \mathcal{H}^0$ to be in this class is proved as Theorem 2.1 below. A sufficient coefficient condition for the functions $f \in \mathcal{H}^0$ to be sense preserving and univalent and in the same class is obtained as Theorem 2.4. It is proved that this coefficient condition is necessary for the functions in its sub class $\mathcal{T}S_H^0(n, q, A, B)$ as Theorem 2.5. Using this necessary and sufficient coefficient condition, in the subsequent work, and extreme points for the functions in the class $\mathcal{T}S_H^0(n, q, A, B)$ are obtained. This research work will motivate future research to work in the area of q-calculus operators together with harmonic functions.

2. Main Results

Theorem 2.1. *Let $f \in \mathcal{H}^0$. Then the function $f \in S_H^0(n,q,A,B)$ if and only if*

$$\mathcal{D}_q^n f(z) \widetilde{*} \Phi(z;\zeta) \neq 0 \qquad (\zeta \in \mathbb{C}, |\zeta| = 1, z \in \mathbb{D}\backslash\{0\}),$$

where

$$\Phi(z;\zeta) = \frac{(B-A)\,\zeta z + (1+A\zeta)\,qz^2}{(1-z)\,(1-qz)} - \frac{2\bar{z} + (B+A)\,\zeta\bar{z} - (1+A\zeta)\,q\bar{z}^2}{(1-\bar{z})\,(1-q\bar{z})}. \tag{2.1}$$

Proof. Let $f \in \mathcal{H}^0$ be of the form (1.2). Then $f \in S_H^0(n,q,A,B)$ if and only if (1.6) holds or equivalently

$$\frac{\mathcal{D}_q\left(\mathcal{D}_q^n f(z)\right)}{\mathcal{D}_q^n f(z)} \neq \frac{1+A\zeta}{1+B\zeta} \quad (\zeta \in \mathbb{C}, |\zeta| = 1, z \in \mathbb{D}\backslash\{0\}),$$

or,

$$(1+B\zeta)\mathcal{D}_q\left(\mathcal{D}_q^n f(z)\right) - (1+A\zeta)\,\mathcal{D}_q^n f(z) \neq 0. \tag{2.2}$$

On using (1.4) and (1.5), the condition (2.2) may also be given by

$$\mathcal{D}_q^n h(z) * \left[(1+B\zeta)\frac{z}{(1-z)(1-qz)} - (1+A\zeta)\,\frac{z}{1-z}\right] - (-1)^n\overline{\mathcal{D}_q^n g(z)} * \left[(1+B\zeta)\frac{\bar{z}}{(1-\bar{z})(1-q\bar{z})} + (1+A\zeta)\,\frac{\bar{z}}{1-\bar{z}}\right] \neq 0$$

or

$$\mathcal{D}_q^n f(z) \widetilde{*} \Phi(z;\zeta) \neq 0.$$

where the function $\Phi(z;\zeta)$ is given by (2.1). ☐

Remark 2.2. *The result of Theorem 2.1 coincides with the result of Dziok [2] for $(k=2)$, If we consider $q \to 1^-$ in Theorem 2.1, we get following result involving the Ruscheweyh operator $\mathcal{D}^n$:*

Corollary 2.3. *[2] Let $f \in \mathcal{H}^0$. Then the function $f \in S_H^0(n,A,B)$ if and only if*

$$\mathcal{D}^n f(z) \widetilde{*} \phi(z;\zeta) \neq 0 \qquad (\zeta \in \mathbb{C}, |\zeta| = 1, z \in \mathbb{D}\backslash\{0\}),$$

where

$$\phi(z;\zeta) = \frac{(B-A)\zeta z + (1+A\zeta)\,z^2}{(1-z)^2} - \overline{\frac{2z + (A+B)\,\overline{\zeta}z - \left(1+A\overline{\zeta}\right)z^2}{(1-z)^2}}.$$

Theorem 2.4. *Let $f = h + \bar{g} \in \mathcal{H}^0$ be of the form (1.2) and let $-1 \leq A < B \leq 1$. If*

$$\sum_{k=2}^{\infty} \left(C_k|a_k| + D_k|b_k|\right) \leq B - A, \tag{2.3}$$

where

$$C_k = \psi_{k-1}\{[k]_q(1+B) - (1+A)\}, \tag{2.4}$$

$$D_k = \psi_{k-1}\{[k]_q(1+B) + (1+A)\}, \tag{2.5}$$

and $[k]_q$ is the q-integer number k defined by (1.1), then

(i) the function f is locally univalent and sense-preserving as $q \to 1^-$ and univalent in $\mathbb{D}$,

(ii) the function $f \in S_H^0(n,q,A,B)$.

Proof. It is clear that the theorem is true for the function $f(z) \equiv z$. Let $f = h + \bar{g}$, where h and g of the form (1.2) and assume that there exist $k \in \{2, 3, \dots\}$ such that $a_k \neq 0$ or $b_k \neq 0$. Since, from (1.1), $[k]_q > 1$, we observe from (2.4) and (2.5) that $D_k \geq C_k > [k]_q(B - A) \quad (k = 2, 3, \dots)$, by which the condition (2.3) implies the condition

$$\sum_{k=2}^{\infty} [k]_q \left(|a_k| + |b_k|\right) < 1 \tag{2.6}$$

and hence, we get for any $q \ (0 < q < 1)$,

$$\begin{aligned}
|\partial_q h(z)| - |\partial_q g(z)| &\geq 1 - \sum_{k=2}^{\infty} [k]_q |a_k| |z^{k-1}| - \sum_{k=2}^{\infty} [k]_q |b_k| |z^{k-1}| \\
&> 1 - |z| \sum_{k=2}^{\infty} [k]_q \left(|a_k| + |b_k|\right) > 1 - |z| > 0.
\end{aligned}$$

in $\mathbb{D}$ which implies as $q \to 1^-$ that $|h'(z)| > |g'(z)|$ in $\mathbb{D}$ that is the function f is locally univalent and sense-preserving in $\mathbb{D}$. Moreover, if $z_1, z_2 \in \mathbb{D}$ and for some $q \, (0 < q < 1), z_1 \neq q z_2$. Then for that q,

$$\left| \frac{z_1^k - (q z_2)^k}{z_1 - (q z_2)} \right| = \left| \sum_{l=1}^{k} z_1^{l-1} (q z_2)^{k-l} \right| \leq \sum_{l=1}^{k} |z_1|^{l-1} q^{k-l} |z_2|^{k-l} < [k]_q \quad (k = 2, 3, \dots).$$

Hence, for that value of q, from (2.6), we have

$$\begin{aligned}
|f(z_1) - f(q z_2)| &\geq |h(z_1) - h(q z_2)| - |g(z_1) - g(q z_2)| \\
&\geq \left| z_1 - q z_2 - \sum_{k=2}^{\infty} a_k \left(z_1^k - (q z_2)^k\right) \right| - \left| \sum_{k=2}^{\infty} \overline{b_k \left(z_1^k - (q z_2)^k\right)} \right| \\
&\geq |z_1 - q z_2| \left(1 - \sum_{k=2}^{\infty} |a_k| \left| \frac{z_1^k - (q z_2)^k}{z_1 - q z_2} \right| - \sum_{k=2}^{\infty} |b_k| \left| \frac{z_1^k - (q z_2)^k}{z_1 - q z_2} \right| \right) \\
&> |z_1 - q z_2| \left(1 - \sum_{k=2}^{\infty} [k]_q |a_k| - \sum_{k=2}^{\infty} [k]_q |b_k| \right) > 0
\end{aligned}$$

which proves that f is univalent in $\mathbb{D}$. This proves the result (i). On the other hand $f \in S_H^0(n, q, A, B)$ if there exists a function w, $w(0) = 0$, $|w(z)| < 1$, $(z \in \mathbb{D})$ such that

$$\frac{\mathcal{D}_q \left(\mathcal{D}_q^n f(z)\right)}{\mathcal{D}_q^n f(z)} = \frac{1 + Az}{1 + Bz}.$$

or equivalently

$$\left| \frac{\mathcal{D}_q \left(\mathcal{D}_q^n f(z)\right) - \mathcal{D}_q^n f(z)}{B \left(\mathcal{D}_q \left(\mathcal{D}_q^n f(z)\right)\right) - A \mathcal{D}_q^n f(z)} \right| < 1 (z \in \mathbb{D}). \tag{2.7}$$

The above inequality (2.7) holds, since for $f = h + \bar{g}$, where h and g of the form (1.2) and for $|z| = r$

$(0 < r < 1)$, we obtain

$$\begin{aligned}
&\left|\mathcal{D}_q\left(\mathcal{D}_q^n f(z)\right) - \mathcal{D}_q^n f(z)\right| - \left|B\mathcal{D}_q\left(\mathcal{D}_q^n f(z)\right) - A\mathcal{D}_q^n f(z)\right| \\
&\qquad = \left|\sum_{k=2}^{\infty}(\psi_{k-1})([k]_q - 1)a_k z^k - (-1)^n \sum_{k=2}^{\infty}(\psi_{k-1})([k]_q + 1)\overline{b_k z^k}\right| \\
&\qquad\quad - \left|(B-A)z + \sum_{k=2}^{\infty}(\psi_{k-1})(B[k]_q - A)a_k z^k \right. \\
&\qquad\qquad \left. -(-1)^n \sum_{k=2}^{\infty}(\psi_{k-1})(B[k]_q + A)\overline{b_k z^k}\right| \\
&\qquad \le \sum_{k=2}^{\infty}(\psi_{k-1})([k]_q - 1)|a_k| r^k + \sum_{k=2}^{\infty}(\psi_{k-1})([k]_q + 1)|b_k| r^k \\
&\qquad\quad - (B-A)r + \sum_{k=2}^{\infty}(\psi_{k-1})(B([k]_q - A)|a_k| r^k \\
&\qquad\qquad + \sum_{k=2}^{\infty}(\psi_{k-1})(B[k]_q + A)|b_k| r^k \\
&\qquad < \sum_{k=2}^{\infty}(C_k|a_k| + D_k|b_k|)r^{k-1} - (B-A), \\
&\qquad \le \sum_{k=2}^{\infty}(C_k|a_k| + D_k|b_k|)r^{k-1} - (B-A) < 0.
\end{aligned}$$

if the condition (2.3) holds, where C_k and D_k are given, respectively, by (2.4) and (2.5). This proves the result (ii). This completes the proof of theorem. □

Theorem 2.5. *Let $f = h + \bar{g} \in \mathcal{H}^0$, where h and g are given by (1.7). Then $f \in \mathcal{T}S_H^0(n, q, A, B)$, if and only if the condition (2.3) holds.*

Proof. If part is proved in Theorem 2.4. To prove only if part let $f = h + \bar{g} \in \mathcal{T}S_H^0(n, q, A, B)$, where h and g are given by (1.7). Then by the class condition (1.6) we have from (2.7) that for any $z \in \mathbb{D}$,

$$\left|\frac{\sum_{k=2}^{\infty}(\psi_{k-1})([k]_q - 1)|a_k| z^k + \sum_{k=2}^{\infty}(\psi_{k-1})([k]_q + 1)|b_k|\overline{z}^k}{(B-A)z - \sum_{k=2}^{\infty}(\psi_{k-1})(B[k]_q - A)|a_k| z^k - \sum_{k=2}^{\infty}(\psi_{k-1})(B[k]_q + A)|b_k|\overline{z}^k}\right| < 1,$$

where for $z = r$ $(0 \le r < 1)$, we obtain

$$\frac{\sum_{k=2}^{\infty}(\psi_{k-1})([k]_q - 1)|a_k| r^{k-1} + \sum_{k=2}^{\infty}(\psi_{k-1})([k]_q + 1)|b_k| r^{k-1}}{(B-A) - \sum_{k=2}^{\infty}(\psi_{k-1})(B[k]_q - A)|a_k| r^{k-1} - \sum_{k=2}^{\infty}(\psi_{k-1})(B[k]_q + A)|b_k| r^{k-1}} < 1.$$

which proves for C_k and D_k defined, respectively, by (2.4) and (2.5), that

$$\sum_{k=2}^{\infty}\left(C_k|a_k| + D_k|b_k|\right) r^{k-1} < (B-A) \quad (0 \le r < 1). \tag{2.8}$$

Let σ_k be the sequence of partial sums of the series

$$\sum_{k=2}^{\infty}\left(C_k|a_k| + D_k|b_k|\right).$$

Then σ_k is a non decreasing sequence and by (2.8) it is bounded above. Thus, as $r \to 1^-$, it is convergent and

$$\sum_{k=2}^{\infty}[C_k|a_k| + D_k|b_k|] = \lim_{k\to\infty} \sigma_k \leq (B - A).$$

This gives the condition (2.3).
$\square$

Theorem 2.6. *The class $\mathcal{T}S_H^0(n, q, A, B)$ is a convex and compact subclass of the class of functions $f = h + \bar{g} \in \mathcal{H}^0$, where h and g are of the form (1.7).*

Proof. Let for $t = 1, 2, f_t \in \mathcal{T}S_H^0(n, q, A, B)$, and let for this n it is of the form

$$f_t(z) = z - \sum_{k=2}^{\infty} |a_{t,k}|z^k + (-1)^n \sum_{k=2}^{\infty} |b_{t,k}|\overline{z}^k \quad (z \in \mathbb{D}). \tag{2.9}$$

Then for $0 \leq \rho \leq 1$,

$$\begin{aligned} F(z) &= \rho f_1(z) + (1 - \rho)f_2(z) \\ &= z - \sum_{k=2}^{\infty} \{\rho|a_{1,k}| + (1 - \rho)|a_{2,k}|\}\, z^k + (-1)^n \sum_{k=2}^{\infty} \{\rho|b_{1,k}| + (1 - \rho)|b_{2,k}|\}\, \overline{z}^k \end{aligned}$$

and by Theorem (2.5), we get for C_k and D_K defined by (2.4), that

$$\begin{aligned} &\sum_{k=2}^{\infty} [C_k \{\rho|a_{1,k}| + (1 - \rho)|a_{2,k}|\} + D_k \{\rho|b_{1,k}| + (1 - \rho)|b_{2,k}|\}] \\ &= \rho \sum_{k=2}^{\infty}\{C_k|a_{1,k}| + D_k|b_{1,k}|\} + (1 - \rho) \sum_{k=2}^{\infty}\{C_k|a_{2,k}| + D_k|b_{2,k}|\} \\ &\leq \rho(B - A) + (1 - \rho)(B - A) = (B - A). \end{aligned}$$

This proves that the function $F \in \mathcal{T}S_H^0(n, q, A, B)$. Hence, the class $\mathcal{T}S_H^0(n, q, A, B)$ is convex. On the other hand, if we consider a sequence of functions $f_t \in \mathcal{T}S_H^0(n, q, A, B)$, $t \in \mathbb{N} = \{1, 2, 3, ...\}$ of the form (2.9), then by Theorem (2.5), we get for C_k and D_k defined by (2.4),

$$\sum_{k=2}^{\infty}\{C_k|a_{t,k}| + D_k|b_{t,k}|\} \leq (B - A). \tag{2.10}$$

Hence, for $|z| \leq r$ $(0 < r < 1)$,

$$\begin{aligned} |f_t(z)| &\leq r + \sum_{k=2}^{\infty}\{|a_{t,k}| + |b_{t,k}|\}r^k \\ &\leq r + \sum_{k=2}^{\infty}\{C_k|a_{t,k}| + D_k|b_{t,k}|\}r^k \\ &< r + (B - A)r^2. \end{aligned}$$

Therefore, class $\mathcal{T}S_H^0(n, q, A, B)$ is locally uniformly bounded. Let $f = h + \bar{g}$, where h and g are given by (1.7). If we assume that $f_t \to f$, then we conclude that $|a_{t,k}| \to |a_k|$ and $|b_{t,k}| \to |b_k|$ as $t \to \infty$ for any k=2, 3, ... Hence, from (2.10), we get

$$\sum_{k=2}^{\infty}\{C_k|a_k| + D_k|b_k|\} \leq (B - A),$$

which proves that $f \in \mathcal{T}S_H^0(n, q, A, B)$ and therefore the class $\mathcal{T}S_H^0(n, q, A, B)$ is closed. This proves the compactness of the class $\mathcal{T}S_H^0(n, q, A, B)$. $\square$

Theorem 2.7. $\mathcal{T}S_H^0(n,q,A,B) = \{h_k : k \in \mathbb{N}\} \cup \{g_k : k \in 2,3\ldots\}$,

$$h_1(z) = z, \qquad h_k(z) = z - \frac{(B-A)}{C_k} z^k,$$
$$g_k(z) = z + (-1)^n \frac{(B-A)}{D_k} \bar{z}^k \quad (z \in \mathbb{D}), \tag{2.11}$$

Proof. Let $g_k = \rho f_1 + (1-\rho) f_2$ where $0 < \rho < 1$ and $f_1, f_2 \in \mathcal{T}S_H^0(n,q,A,B)$ are functions of the form

$$f_t(z) = z - \sum_{k=2}^{\infty} |a_{t,k}| + (-1)^n \sum_{k=2}^{\infty} |b_{t,k}| \bar{z}^k \qquad (z \in \mathbb{D}, t = 1,2),$$

Then, by (2.3), we have

$$|b_{1,k}| = |b_{2,k}| = \frac{(B-A)}{D_k}.$$

and therefore $a_{1,t} = a_{2,t} = 0$ for $t \in \{2,3,\ldots\}$ and $b_{1,t} = b_{2,t} = 0$ for $t \in \{2,3,\ldots\} \setminus \{k\}$. it follows that $g_k(z) = f_1(z) = f_2(z)$ and g_k are in the class of extreme point of the functions class $\mathcal{T}S_H^0(n,q,A,B)$. Similarly, we can verify that the functions $h_k z$ are the extreme point of the class $\mathcal{T}S_H^0(n,q,A,B)$. Now, suppose that a function f of the form (1) is in the family of extreme point of the class $\mathcal{T}S_H^0(n,q,A,B)$ anf f is not in the form (2.11). Then there exists $m \in \{2,3,\ldots\}$ such that

$$0 < |a_m| < \frac{(B-A)}{C_m},$$

or

$$0 < |b_m| < \frac{(B-A)}{D_m},$$

if

$$0 < |a_m| < \frac{(B-A)}{C_m},$$

then putting

$$\rho = \frac{|a_m| C_m}{B-A},$$

and

$$\varphi = \frac{f - \rho h_m}{1-\rho}.$$

and $0 < \rho < 1, h_m \neq \varphi$. Therefore, f is not in the family of extreme point of the class $\mathcal{T}S_H^0(n,q,A,B)$. similarly, if

$$0 < |b_m| < \frac{(B-A)}{D_m},$$

then putting

$$\rho = \frac{|b_m| D_m}{(B-A)},$$

and

$$\varphi = \frac{f - \rho g_m}{1-\rho}.$$

we have $0 < \rho < 1, h_m \neq \varphi$.
It follows that f is not in the family of extreme points of the class $\mathcal{T}S_H^0(n,q,A,B)$ and so the proof is completed. $\square$

Corollary 2.8. *Let* $f \in \mathcal{T}S^0_H(n,q,A,B)$ *be of the form* (1.7)*. Then*

$$|a_k| \leq \frac{B-A}{C_k} \text{ and } |b_k| \leq \frac{B-A}{D_k}, \quad k = 2,3,4,...,$$

where C_k *and* D_k *are defined, respectively, by* (2.4) *and* (2.5)*. Equality occur for the extremal functions* $h_k(z)$ *and* $g_k(z)$ *given in (2.11) for* $k = 2,3,4,...$

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