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New Spaces Over Modulus Function

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ABSTRACT: Our main aim of this paper is to introduce some new techniques of spaces using modulus function. Some of basic inclusion properties will be taken care of.

Key Words: Modulus function, paranormed sequence, infinite matrices.

Contents

1. Introduction

We represent the set of all sequences with complex terms by $\mho$. By a *sequence space* we define a linear subspace of $\mho$ i.e., the sequence space is the set of scalar sequences(real or complex) which is closed under co-ordinate wise addition and scalar multiplication. Here we symbolize $\mathbf{N}$ and $\mathbf{C}$ to represent the set of non-negative integers and the set of complex numbers, respectively. By ℓ_∞, c and c_0, respectively, we shall mean the set of all bounded sequences, the set of all convergent sequences and those sequences having limit as zero. Note that ℓ_1, $\ell(p)$, cs and bs will specify the spaces of all absolutely, p-absolutely convergent, convergent and bounded series, respectively [10], [14]- [17], [32].

For an infinite matrix $\Lambda=(w_{i,j})$ and $\eta=(\eta_k)\in\Psi$, the Λ-transform of η is $\Lambda\eta=\{(\Lambda\eta)_i\}$ provided it exists $\forall\ i\in\mathbb{N}$, where $(\Lambda\eta)_i=\sum_{j=0}^{\infty}w_{i,j}\eta_j$.

For the matrix $\Lambda=(w_{i,j})$, the set G_Λ, where

$$G_\Lambda=\{\eta=(\eta_j)\in\Psi\ :\ \Lambda\eta\in G\}, \tag{1.1}$$

is known as the matrix domain of Λ in G (see, [18], [21], [27]- [29]).

Consider the sequence of positive numbers (q_k) and write $S_n=\sum\limits_{k=0}^{n}q_k$ for $n\in\mathbf{N}$.

Then the matrix $S^q=(s^q_{nk})$ of the Riesz mean (S,q_n) is given by

$$s^q_{nk}=\begin{cases}\frac{q_k}{S_n}, & \text{if } 0\le k\le n,\\ 0 & \text{if } k>n\end{cases}$$

The Riesz mean (S,q_n) is regular if and only if $S_n\to\infty$ as $n\to\infty$ (see, [24], [30]).

In [22], the author introduced the concept of modulus function. We call a function $\mathcal{G}:[0,\infty)\to[0,\infty)$ to be modulus function if

(i) $\mathcal{G}(\zeta)=0$ if and only if $\zeta=0$,

(ii) $\mathcal{G}(\zeta+\eta)\le\{(\zeta)+\{(\eta)\ \forall\ \zeta\ge0,\eta\ge0$

(iii) $\mathcal{G}$ is increasing, *and*

(iv) $\mathcal{G}$ is continuous from the right at 0.

One can easily see that if $\mathcal{G}_1$ and $\mathcal{G}_2$ are modulus functions then so is $\mathcal{G}_1+\mathcal{G}_2$; and the function $\mathcal{G}^j$ $(j\in\mathbf{N})$, the composition of a modulus function $\mathcal{G}$ with itself j times is also modulus function. It has also been studied in [11].

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Recently, in [25] the new space was introduced by using notion of modulus function as follows:

$$L(\mathcal{F})=\left\{\zeta=(\zeta_r):\ \sum_r|\mathcal{F}(\zeta_r|)|<\infty\right\}$$

In this direction of forming a new sequence space by virtue of matrix domain method has been given by several authors viz., $(\ell_p)_{R^t}=r_p^t$ (see, [2]), $[c_0(u,p)]_{A^r}=a_0^r(u,p)$ and $[c(u,p)]_{A^r}=a_c^r(u,p)$ (see, [3], $(\ell_\infty)_{R^t}=r_\infty^t$, $(c)_{R^t}=r_c^t$ and $(c_o)_{R^t}=r_0^t$ (see, [19]), $(\ell_p)_{C_1}=X_p$ and $(\ell_\infty)_{C_1}=X_\infty$(see, [23]), $r^q(u,p)=\{l(p)\}_{R_u^q}$ (see, [27]), $(\ell_\infty)_{N_q}$ and c_{N_q}(see, [31]), and etc.

2. The space $s_\infty^q(\mathcal{F},p)$, $s_c^q(\mathcal{F},p)$ and $s_0^q(\mathcal{F},p)$

In this section, we shall introduce the space $s_\infty^q(\mathcal{F},p)$, $s_c^q(\mathcal{F},p)$ and $s_0^q(\mathcal{F},p)$ of Riesz type and show that they are complete.

Let Λ be a real or complex linear space, define the function $\tau:\Lambda\to\mathbb{R}$ with $\mathbb{R}$ as set of real numbers. Then, the paranormed space is a pair $(\Lambda;\tau)$ and τ is a paranorm for Λ, if the following axioms are satisfied for all $\zeta,\ \eta\in\Lambda$ and for all scalars β:

(i) $\tau(\theta)=0$,

(ii) $\tau(-\zeta)=\tau(\zeta)$,

(iii) $\tau(\zeta+\eta)\leq\tau(\zeta)+\tau(\eta), and$

(iv) scalar multiplication is continuous, that is,

$|\beta_n-\beta|\to0$ and $h(\zeta_n-\zeta)\to0$ imply $\tau(\beta_n\zeta_n-\beta\zeta)\to0$ for all $\beta's$ in $\mathbb{R}$ and $\zeta's$ in Λ , where θ is a zero vector in the linear space Λ. Assume here and after that (p_k) be a bounded sequence of strictly positive real numbers with $\sup_k p_k=H$and $M=max\{1,H\}$. Then, the linear space $\ell_\infty(p)$ was defined by Maddox [18] as follows :

$$\ell_\infty(p)=\{\zeta=(\zeta_k):\sup_k|\zeta_k|^{p_k}<\infty\}$$

which is complete space paranormed by

$$\tau_1(\zeta)=\left[\sup_k|\zeta_k|^{p_k}\right]^{1/M}.$$

We shall assume throughout that $p_k^{-1}+\{p_k^{'}\}^{-1}$ provided $1<inf p_k\leq H<\infty$, and we denote the collection of all finite subsets of N by F, where N=$\{0,\ 1,\ 2,\ \dots\ \}$.

Following Altay (see, [2]), Başarir and Öztürk (see, [6]), Choudhary and Mishra (see, [5]), Ganie et al. (see, [6]-[13], Mursaleen (see, [20]), Ruckle [25], Sengönül [26], we define the spaces $s_\infty^q(\mathcal{F},p)$, $s_c^q(\mathcal{F},p)$ and $s_0^q(\mathcal{F},p)$ as the set of all sequences whose $R_{\mathcal{F}}^q$-transform are in the spaces $c(p)$ and $c_0(p)$, respectively i.e.,

$$s_\infty^q(\mathcal{F},p)=\left\{x\in\omega:\sup_k\left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^k q_jx_j\right)\right|^{p_k}<\infty\right\},$$

$$s_c^q(\mathcal{F},p)=\left\{x\in\omega:\lim_k\left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^k q_jx_j-l\right)\right|^{p_k}=0\text{ for some }l\in\mathbf{R}\right\},$$

$$s_0^q(\mathcal{F},p)=\left\{x\in\omega:\lim_k\left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^k q_jx_j\right)\right|^{p_k}=0\right\}.$$

These spaces can be written with the help of (2) as follows:

$$s_\infty^q(\mathcal{F},p)=\{l_\infty(p)\}_{S_\mathcal{F}^q},\ s_c^q(\mathcal{F},p)=\{c(p)\}_{S_\mathcal{F}^q} \text{ and } s_0^q(\mathcal{F},p)=\{c_0(p)\}_{S_\mathcal{F}^q},$$

where, $0<p_k\le H<\infty$.

Define the sequence $y=(y_k)$, which will be used, by the $S_\mathcal{F}^q$-transform of a sequence $x=(x_k)$, i.e.,

$$y_k=\mathcal{F}\,\frac{1}{Q_k}\sum_{j=0}^{k}q_jx_j \text{ for all } k\in\mathbb{N}. \tag{2.1}$$

Theorem 2.1. *The spaces $s_\infty^q(\mathcal{F},p)$, $s_c^q(\mathcal{F},p)$ and $s_0^q(\mathcal{F},p)$ are complete linear metric space paranormed by $\mathcal{G}$ defined*

$$\mathcal{G}(x)=\sup_k\left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^{k}q_jx_j\right)\right|^{\frac{p_k}{M}}.$$

Proof. We only prove the theorem for the space $s_\infty^q(\mathcal{F},p)$. The linearity of $s_\infty^q(\mathcal{F},p)$ with respect to the co-ordinate wise addition and scalar multiplication follows from from the inequalities which are satisfied for $z,x\in s_\infty^q(\mathcal{F},p)$(see [18], p.30])

$$\begin{aligned}\sup_k&\left|\mathcal{F}\left(\tfrac{1}{Q_k}\textstyle\sum_{j=0}^{k}q_j(z_j+x_j)\right)\right|^{\frac{p_k}{M}}\\ &\le\sup_k\left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^{k}q_jz_j\right)\right|^{\frac{p_k}{M}}+\sup_k\left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^{k}q_jx_j\right)\right|^{\frac{p_k}{M}}\end{aligned} \tag{2.2}$$

and for any $\alpha\in\mathbb{R}$ (see, [17])

$$|\alpha|^{p_k}\le max\{1,|\alpha|^M\}. \tag{2.3}$$

It is clear that, $\mathcal{G}(\theta)$=0 and $\mathcal{G}(x)=\mathcal{G}(-x)$ for all $x\in s_0^q(\mathcal{F},p)$. Again the inequality (4) and (5), yield the subadditivity of g and

$$\mathcal{G}(\alpha x)\le max\{1,|\alpha|\}\mathcal{G}(x).$$

Let $\{x^n\}$ be any sequence of points of the space $s_0^q(\mathcal{F},p)$ such that $g(x^n-x)\to0$ and (α_n) is a sequence of scalars such that $\alpha_n\to\alpha$. Then, since the inequality,

$$\mathcal{G}(x^n)\le\mathcal{G}(x)+\mathcal{G}(x^n-x)$$

holds by subadditivity of $\mathcal{G}$, $\{\mathcal{G}(x^n)\}$ is bounded and we thus have

$$\begin{aligned}g(\alpha_nx^n-\alpha x)&=\sup_k\left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^{k}q_j(\alpha_nx_j^n-\alpha x_j)\right)\right|^{\frac{p_k}{M}}\\ &\le|\alpha_n-\alpha|\,g(x^n)+|\alpha|\,g(x^n-x)\end{aligned}$$

which tends to zero as $n\to\infty$. That is to say that the scalar multiplication is continuous. Hence, g is paranorm on the space $s_\infty^q(\mathcal{F},p)$.

It remains to prove the completeness of the space $s_\infty^q(\mathcal{F},p)$. Let $\{x^i\}$ be any Cauchy sequence in the space $s_\infty^q(\mathcal{F},p)$, where $x^i=\{x_0^i,x_2^i,...\}$. Then, for a given $\epsilon>0$ there exists a positive integer $n_0(\epsilon)$ such that

$$\mathcal{G}(x^i-x^j)<\epsilon \tag{2.4}$$

for all $i, j \geq n_0(\epsilon)$. Using definition of $\mathcal{G}$ and for each fixed $k \in \mathbb{N}$ that

$$\left|(S^q_{\mathcal{F}}x^i)_k - (S^q_{\mathcal{F}}x^j)_k\right| \leq \sup_k \left|(S^q_{\mathcal{F}}x^i)_k - (S^q_{\mathcal{F}}x^j)_k\right|^{\frac{p_k}{M}} < \epsilon$$

for $i, j \geq n_0(\epsilon)$, which leads us to the fact that $\{(S^q_{\mathcal{F}}x^0)_k, (S^q_{\mathcal{F}}x^1)_k, \ldots\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbf{N}$. Since $\mathbf{R}$ is complete, it converges,say, $(S^q_{\mathcal{F}}x^i)_k \to ((S^q_{\mathcal{F}}x)_k$ as $i \to \infty$. Using these infinitely many limits $(S^q_{\mathcal{F}}x)_0, (S^q_{\mathcal{F}}x)_1, \ldots$, we define the sequence $\{(S^q_{\mathcal{F}}x)_0, (S^q_{\mathcal{F}}x)_1, \ldots\}$. From (6) for with $j \to \infty$ we have

$$\left|(S^q_{\mathcal{F}}x^i)_k - (S^q_{\mathcal{F}}x)_k\right| \leq \epsilon, \tag{2.5}$$

for all k, i.e.,

$$\mathcal{G}(x^i - x) \leq \epsilon \ (i \geq n_0(\epsilon)).$$

Finally, taking $\epsilon = 1$ in (7) and letting $i \geq n_0(1)$. we have by Minkowski's inequality for each $m \in \mathbf{N}$ that

$$\left|(S^q_{\mathcal{F}}x)_k\right|^{\frac{p_k}{M}} \leq \mathcal{G}(x^i - x) + \mathcal{G}(x^i) \leq 1 + \mathcal{G}(x^i)$$

which implies that $x \in s^q_\infty(\mathcal{F}, p)$. Since $\mathcal{G}(x - x^i) \leq \epsilon$ for all $i \geq n_0(\epsilon)$, it follows that $x^i \to x$ as $i \to \infty$, hence we have shown that $s^q_\infty(\mathcal{F}, p)$ is complete, hence the proof.
□

Remark 2.2. *One can easily see the absolute property does not hold on the spaces $s^q_\infty(\mathcal{F}, p)$, $s^q_c(\mathcal{F}, p)$ and $s^q_0(\mathcal{F}, p)$, that is $\mathcal{G}(x) \neq \mathcal{G}(|x|)$ for atleast one sequence in the spaces $s^q_\infty(\mathcal{F}, p)$,, $s^q_c(\mathcal{F}, p)$ and $s^q_0(\mathcal{F}, p)$ and this says that the spaces $s^q_\infty(\mathcal{F}, p)$, $s^q_c(\mathcal{F}, p)$ and $s^q_0(\mathcal{F}, p)$ are sequence spaces of non-absolute type.*

Theorem 2.3. *If p_k and t_k are bounded sequences of positive real numbers with $0 < p_k \leq t_k < \infty$ for each $k \in \mathbf{N}$, then for any modulus function $\mathcal{F}$, we have*
(i) $s^q_c(\mathcal{F}, p) \subseteq s^q_c(\mathcal{F}, t)$.
(ii) $s^q_c(\mathcal{F}, p) \subseteq s^q_c(\mathcal{F}, t)$.
(iii) $s^q_{c_0}(\mathcal{F}, p) \subseteq s^q_{c_0}(\mathcal{F}, t)$.

Proof. We only prove (i) and the rest can be proven similarly. For $\zeta \in s^q_\infty(\mathcal{F}, p)$ it is obvious that

$$\sup_k \left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^{k} q_j x_j\right)\right| < \infty.$$

Consequently, for sufficiently large values of k say $k \geq k_0$ for some fixed $k_0 \in \mathbf{N}$, we have

$$\left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^{k} q_j x_j\right)\right| < \infty.$$

But $\mathcal{F}$ being increasing and $p_k \leq t_k$, we have

$$\sup_{k \geq k_0} \left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^{k} q_j x_j\right)\right|^{t_k} \leq \sup_{k \geq k_0} \left|\mathcal{F}\left(\frac{1}{Q_k}\sum_{j=0}^{k} q_j x_j\right)\right|^{p_k} < \infty.$$

From this, it is clear that $\zeta \in s^q_\infty(\mathcal{F}, t)$ and the result follows. □

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References

1. B. Altay and F. Başar, *On the paranormed Riesz sequence space of nonabsolute type*, Southeast Asian Bull. Math., 26, 701-715, (2002).

2. B. Altay and F. Başar, *On the paranormed Riesz sequence space of nonabsolute type*, Southeast Asian Bull. Math., 26, 701-715, (2002).

3. C. Aydin and F. Başar, *Some new paranormed sequence spaces*, Inf. Sci., 160, 27-40, (2004).

4. M. Başarir and M. Öztürk, *On the Riesz difference sequence space*, Rendiconti del Cirocolo di Palermo, 57, 377-389, (2008).

5. B. Choudhary and S. K. Mishra, *On Köthe Toeplitz Duals of certain sequence spaces and matrix Transformations*, Indian, J. Pure Appl. Math., 24(4), 291-301, (1993).

6. A. H. Ganie, *Some new approach of spaces of non-integral order*, J. Nonlinear Sci, Appl., 14(2), 89-96, (2021).

7. A. H. Ganie, *Riesz spaces using modulus function*, Int. jour. Math. Mod. Meth. Appl. Sci., 14, 20-23, (2020).

8. A. H. Ganie and A. Antesar, *Certain spaces using $\triangle$- operator*, Adv. Stud. Contemp. Math. (Kyungshang), 30(1)(2020), 17-27.

9. A. H. Ganie and S. A. Lone, *Some sequence spaces of sigma means defined by Orlicz function*, Appl. Math. Inf. Sci., (accepted 2020).

10. A. H. Ganie and N. A. Sheikh, *On some new sequence space of non-absolute type and matrix transformations*, J. Egyptain Math. Soc., 21, 34-40, (2013).

11. A. H. Ganie, N. A. Sheikh and T. Jalal, *On some new spaces of invariant means with respect to modulus function*, The inter. Jou. Modern Math. Sciences, USA, 13(3), 210-216, (2015).

12. A. H. Ganie, A. Mobin N. A. Sheikh and T. Jalal, *New type of Riesz sequence space of non-absolute type*, J. Appl. Comput. Math., 5(1), 1-4, (2016).

13. A. H. Ganie, Mobin Ahmad, N. A. Sheikh, T.Jalal and S. A. Gupkari, *Some new type of difference sequence space of non-absolute type*, Int. J. Modern Math.Sci., 14(1), 116-122, (2016).

14. K. G. Gross Erdmann, *Matrix transformations between the sequence spaces of Maddox*, J. Math. Anal. Appl., 180(1993), 223- 238.

15. T. Jalal, S. A. Gupkari and A. H. Ganie, *Infinite matrices and sigma convergent sequences*, Southeast Asian Bull. Math., 36, 825-830, (2012).

16. C. G. Lascarides and I. J. Maddox, *Matrix transformations between some classes of sequences*, Proc. Camb. Phil. Soc., 68, 99-104, (1970).

17. I. J. Maddox, *Paranormed sequence spaces generated by infinite matrices*, Proc. Camb. Phil. Soc., 64, 335-340, (1968).

18. I. J. Maddox, *Elements of Functional Analysis*, 2^{nd}ed., The University Press, Cambridge, (1988).

19. E. Malkowsky, *Matrix transformations between spaces of absolutely and strongly summable sequences*, Habilitationsschrift, Giessen (1988).

20. M. Mursaleen, F. Başar and B. Altay, *On the Euler sequence spaces which include the spaces l_p and l_∞-II*, Nonlinear Anal., 65, 707-717, (2006).

21. M. Mursaleen, A. H. Ganie and N. A. Sheikh, *New type of difference sequence space and matrix transformation*, FILOMAT, 28(7), 1381-1392, (2014).

22. N. Nakano, *Cancave modulars*, J. Math. Soc. Japan, 5, 29-49, (1953).

23. P.-N. Ng and P.-Y. Lee, *Cesàro sequences spaces of non-absolute type*, Comment. Math. Prace Mat. 20(2), 429-433, (1978).

24. G. M. Petersen, *Regular matrix transformations*, Mc Graw-Hill, London, (1966).

25. W. H. Ruckle, *FK spaces in which the sequence of coordinate vectors is bounded*, Canand. J. Math., 25, 973-978, (1973).

26. M. Sengönül and F. Başar, *Some new Cesàro sequences spaces of non-absolute type, which include the spaces c_o and c*, Soochow J. Math., 1, 107-119, (2005).

27. N. A. Sheikh and A. H. Ganie, *A new paranormed sequence space and some matrix transformations*, Acta Math. Acad. Paedago. Nygr., 28, 47-58, (2012).

28. N. A. Sheikh and A. H. Ganie, *New paranormed sequence space and some matrix transformations*, WSEAS Transaction of Math., 8(12), 852-859, (2013).

29. N. A. Sheikh, T. Jalal and A. H. Ganie, *New type of sequence spaces of non-absolute type and some matrix transformations*, Acta Math. Acad. Paedago. Nygr., 29, 51-66, (2013).

30. Ö. Toeplitz, *Über allegemeine Lineare mittelbildungen*, Prace Math. Fiz., 22, 113-119, (1991).

31. C.-S. Wang, *On Nörlund sequence spaces*, Tamkang J. Math., 9, 269-274, (1978).

32. A. Wilansky, *Summability through Functional Analysis*, North Holland Mathematics Studies, Amsterdam - New York - Oxford, (1984).

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