Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–20.
ISSN-0037-8712 in press
doi:10.5269/bspm.53068

A Generalization of Lucas Sequence and Associated Identities*

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ABSTRACT: In this paper, we attempt to generalize Lucas sequence by generating certain number of sequences whose terms are obtained by adding the last two generated terms of the preceding sequence. Lucas sequence is obtained as a particular case of generating only one sequence. Moreover we prove some of the results which can be seen as generalized form of the results which hold for Lucas sequence. We obtain Cassini-like identity for these generalized Lucas sequences.

Key Words: Lucas sequence, Fibonacci sequence, Cassini's identity, Cassini-like identity.

Contents

1. Introduction

Fibonacci sequence (F_k) is generated in [2,6] by the recursive formula $F_k = F_{k-1} + F_{k-2}$ for $k \geq 3$ with $F_1 = 1, F_2 = 1$. That is, each term in the sequence (third term onwards) is the sum of the two that immediately precede it. The Fibonacci sequence is the first known recursive sequence in mathematical work. First few terms of the sequence are 1,1,2,3,5,8,13,21,.... Many generalizations of the sequence and hence its properties are available in the literature. Some of these can be found in [3,4,8,9,12].

Younseok Choo [3] derived identities for a generalized Fibonacci sequence defined by the recurrence relation $F_k = aF_{k-1} + bF_{k-2}, n \geq 2$ with initial conditions F_0 and F_1. $F_0 = 0, F_1 = 1, a = 1, b = 1$ generate the classical Fibonacci sequence whereas $F_0 = 2, F_1 = 1, a = 1, b = 1$ generate the classical Lucas sequence.

Miles [9] defined *k-generalized Fibonacci numbers* $(k \geq 2)$ in such a way that for $k = 2$, ordinary Fibonacci numbers are generated. The *k-generalized Fibonacci numbers* $f_{j,k}$ are defined as

$$f_{j,k} = 0,\ 0 \leq j \leq k-2, \qquad f_{k-1,k} = 1, \qquad f_{j,k} = \sum_{n=1}^{k} f_{j-n,k},\ j \geq k.$$

When $k = 2$, the numbers $f_{j,2}$ or simply f_j are the ordinary Fibonacci numbers.

Stakhov [12] mentioned so-called *Fibonacci p-numbers* which are given by the recurrence relation

$$F_p(n) = F_p(n-1) + F_p(n-p-1) \text{ for } p = 0, 1, 2, 3, \ldots \text{ with } n > p+1$$

with the initial terms

$$F_p(1) = F_p(2) = \ldots = F_p(p) = F_p(p+1) = 1.$$

For $p = 1$, the recurrence relation generates the classical Fibonacci numbers $F_1(n)$ or simply F_n.

In 2016, Kwon [8] introduced a new sequence, called the *modified k-Fibonacci-like sequence* $(M_{k,n})$, defined by the recurrence relation

* This research work is supported by University Grants Commission of India.
2010 *Mathematics Subject Classification:* 11B39.
Submitted April 09, 2020. Published November 27, 2020

$$M_{k,n} = kM_{k,n-1} + M_{k,n-2} \text{ for } n \geq 2$$

with $M_{k,0} = 2$ and $M_{k,1} = 2$ where k is any positive real number.

The above generalizations available in the literature give us a single sequence. Akbulak et al. [1] mentioned some generalizations of Fibonacci sequence along with a generalization which generates multiple sequences. k sequences of the *generalized order-k Fibonacci numbers* are generated [5] by the recurrence relation

$$g_n^i = c_1 g_{n-1}^i + c_2 g_{n-2}^i + \ldots + c_k g_{n-k}^i \text{ for } n > 0 \text{ and } 1 \leq i \leq k$$

with initial conditions

$$g_n^i = \begin{cases} 1 \text{ if } i = 1-n, \\ 0 \text{ otherwise,} \end{cases} \quad \text{for } 1-k \leq n \leq 0$$

where $c_1, c_2, \ldots, c_k$ are constant coefficients and g_n^i is the n^{th} term of the i^{th} sequence.

Akbulak et al. [1] defined m sequences of the *generalized order-m Fibonacci k-numbers* for $n > 0, k, t \geq 1$ and $1 \leq i \leq m$

$$F_{k,n}^i = kF_{k,n-1}^i + tF_{k,n-2}^i + F_{k,n-3}^i + \ldots + F_{k,n-m}^i$$

with initial conditions

$$F_{k,n}^i = \begin{cases} 1 \text{ if } n+i = 1, \\ 0 \text{ otherwise,} \end{cases} \quad \text{for } 1-m \leq n \leq 0$$

where $F_{k,n}^i$ is the n^{th} term of the i^{th} generalized Fibonacci k-sequence.

In [10], m number of sequences are generated following certain recursive rules as follows. When the number of sequence is one, i.e. $m = 1$, these rules coincide with those generating Fibonacci numbers and we get the Fibonacci sequence.

We consider m interconnected sequences

$$(S_{1,k}), (S_{2,k}), (S_{3,k}), \ldots, (S_{m,k})$$

which can be generated according to the following rule

$$\begin{array}{ll} S_{1,1} = S_{2,1} = S_{3,1} = \ldots = S_{m,1} = 1, & S_{1,2} = 1, \\ S_{i,k} = S_{i-1,k-1} + S_{i-1,k}, & 1 < i \leq m, k \geq 2 \\ S_{1,k} = S_{m,k-1} + S_{m,k-2}, & k \geq 3 \end{array}$$

Table 1: Columns show the terms in six sequences (i.e. $m = 6$).

k	$S_{1,k}$	$S_{2,k}$	$S_{3,k}$	$S_{4,k}$	$S_{5,k}$	$S_{6,k}$
1	1	1	1	1	1	1
2	1	2	3	4	5	6
3	7	8	10	13	17	22
4	28	35	43	53	66	83
5	105	133	168	211	264	330
6	413	518	651	819	1030	1294
7	1624	2037	2555	3206	4025	5055
8	6349	7973	10010	12565	15771	19796
...	...	...	...	...	...	...

To illustrate the rule, we consider $m = 6$ and generate six sequences. We start with 1 as the first term of all the sequences and 1 as the second term of the first sequence. Hereafter the terms (rowwise) are obtained by adding two latest terms of the sequence immediately preceding the sequence where the former term belongs. When the term in the first sequence is to be obtained two latest terms of the last sequence are added.

For $m = 1$, we identify the only sequence $(S_{1,k})$ as the Fibonacci sequence (F_k). We obtain it by starting with 1 as the first term in the only sequence and 1 as the second term of the first sequence (which is the only sequence). Hereafter the terms are obtained by adding two latest terms of the sequence immediately preceding the sequence (that is the same sequence) where the former term belongs.

Notation: $S_{p,q}$ denotes the q^{th} term of the p^{th} sequence.

With this definition, the following results for Fibonacci sequence are generalized in [10], along with others.

- $F_k + F_{k+1} = F_{k+2}$
  Generalization: Sum of k^{th} terms in all the sequences and $(k+1)^{th}$ term in the first sequence equals the $(k+2)^{th}$ term in the first sequence.

- $\sum_{k=1}^{n} F_k = F_{n+2} - 1$
  Generalization: Sum of first n terms of all the sequences is one less than the $(n+2)^{th}$ term of the first sequence.

- $\sum_{k=0}^{n-1} F_{2k+1} = F_{2n}$
  Generalization: Sum of all the terms in odd positions (upto $(2n+1)^{th}$ position) in all the sequences equals the $(2n)^{th}$ term of the last sequence.

- $\sum_{k=1}^{n} F_{2k} = F_{2n+1} - 1$
  Generalization: Sum of all the terms in even positions (upto $(2n)^{th}$ position) in all the sequences is one less than the $(2n+1)^{th}$ term of the last sequence.

- $\lim_{k\to\infty} \frac{S_{i,k+1}}{S_{i,k}}$ is a root of the equation $x^{m+1} - (x+1)^m = 0$ for $i = 1, 2, \ldots, m$, which is the generalization of the fact that the golden ratio is a root of the equation $x^2 - x - 1 = 0$.

In [11], a recurrence relation is obtained to generate one of the m sequences without using the terms of the remaining sequences. Consecutive pair of Fibonacci numbers are relatively prime is generalized as gcd of $m+1$ consecutive numbers in any of the generated sequences is one. A generalized Cassini's identity was also obtained along with the generalization of the identity $F_{p+q} = F_{p+1}F_q + F_pF_{q-1}$ related to Fibonacci numbers.

In this paper, we generalize Lucas sequence in a similar fashion and generalize some of the identities related to Lucas numbers. Lucas sequence (L_k) is generated by the recursive formula $L_k = L_{k-1} + L_{k-2}$ for $k \geq 3$ with $L_1 = 1, L_2 = 3$. That is, each term in the sequence (third term onwards) is the sum of the two that immediately precede it. An identity that establishes a relation between Fibonacci and Lucas numbers is $L_k = F_{k-1} + F_{k+1}$. Using this identity and extending the Fibonacci sequence backwards to negative indices, the first few terms of the Lucas sequence for $n \geq 0$ are 2,1,3,4,7,11,18,29,. . . .

Using the identity $L_k = F_{k-1} + F_{k+1}$, we generalize Lucas sequence by generating m sequences

$$(L_{1,k}), (L_{2,k}), (L_{3,k}), \ldots, (L_{m,k})$$

by defining $L_{i,k} = S_{i,k-1} + S_{i,k+1}\ \forall\ k$ and $1 \leq i \leq m$.

For $m = 1$, we identify the only sequence $(L_{1,k})$ as the Lucas sequence (L_k).

Notation: $L_{p,q}$ denotes the q^{th} term of the p^{th} sequence.

With this definition, we generalize some of the identities related to Lucas numbers, as obtained for generalized Fibonacci numbers including Cassini-like identity. The gcd of $m+1$ consecutive terms of any

Table 2: Columns show the terms in six sequences (i.e. $m = 6$).

k	$L_{1,k}$	$L_{2,k}$	$L_{3,k}$	$L_{4,k}$	$L_{5,k}$	$L_{6,k}$
1	1	2	3	4	5	6
2	8	9	11	14	18	23
3	29	37	46	57	71	89
4	112	141	178	224	281	352
5	441	553	694	872	1096	1377
6	1729	2170	2723	3417	4289	5385
7	6762	8491	10661	13384	16801	21090
8	26475	33237	41728	52389	65773	82574
...	...	...	...	...	...	...

of the m sequences is also found to be 1 as in case of generalized Fibonacci numbers. These sequences can also be generated by the definition used to generate generalized Fibonacci sequences with a change in the initial terms of the sequences. That is, terms are obtained by adding the last two generated terms of the preceding sequence.

2. Preliminary Results

We restate here the definitions for the generalized Fibonacci sequences and Lucas sequences.

Definition 2.1 (Generalized Fibonacci sequences)**.** *m generalized Fibonacci sequences $(S_{i,k})$, viz. $(S_{1,k}),(S_{2,k}),(S_{3,k}),\ldots,(S_{m,k})$ are generated by the following rule*

$$\begin{aligned} For\ i=1, \qquad & S_{1,k}=S_{m,k-1}+S_{m,k-2},\ k\geq 3 \\ For\ 1<i\leq m,\ & S_{i,k}=S_{i-1,k-1}+S_{i-1,k},\ k\geq 2 \end{aligned}$$

with $S_{1,1}=S_{2,1}=S_{3,1}=\ldots=S_{m,1}=1,\ S_{1,2}=1.$

It is shown in [11] that each sequence of the generalized Fibonacci sequences can be generated independently by a recurrence relation given by

$$S_{i,k}=\ {}^{m}C_0S_{i,k-1}+\ {}^{m}C_1S_{i,k-2}+\ {}^{m}C_2S_{i,k-3}+\ldots+\ {}^{m}C_mS_{i,k-m-1}$$

with initial terms being
for $1\leq i\leq m$ and $-(m+1)\leq k\leq -1$

$$S_{i,k}=(-1)^{i+k+m+1}\left({}^{-k-1}C_{m-i}\right)$$

Also $S_{i,0}=0$ for $1\leq i\leq m$. These initial terms and the recurrence relation are used to establish what follows.

Definition 2.2 (Generalized Lucas sequences)**.** *m generalized Lucas sequences $(L_{i,k})$, viz. $(L_{1,k}),(L_{2,k}),(L_{3,k}),\ldots,(L_{m,k})$ are generated by the following rule*

$$For\ 1\leq i\leq m,\ L_{i,k}=S_{i,k-1}+S_{i,k+1}\ for\ k\geq -m.$$

Proposition 2.1. *First term of i^{th} sequence is i, i.e. for $i=1,2,3,\ldots,m,\ L_{i,1}=i$.*

Proof. By definition, $S_{1,2} = 1$ and for $i = 2, 3, \ldots, m$,

$$\begin{aligned} S_{i,2} &= S_{i-1,1} + S_{i-1,2} \\ &= S_{i-1,1} + S_{i-2,1} + S_{i-2,2} \\ &= \ldots \\ &= S_{i-1,1} + S_{i-2,1} + S_{i-3,1} + \ldots + S_{2,1} + S_{1,1} + S_{1,2} \\ &= 1 + 1 + 1 + \ldots + 1 + 1 + 1 \\ &= i \end{aligned}$$

Thus $L_{i,1} = S_{i,0} + S_{i,2} = 0 + S_{i,2} = S_{i,2} = i$. □

Proposition 2.2. *Second term of first sequence is $m + 2$, i.e. $L_{1,2} = m + 2$.*

Proof.

$$\begin{aligned} L_{1,2} &= S_{1,1} + S_{1,3} \\ &= 1 + ({}^mC_0 S_{1,2} + {}^mC_1 S_{1,1} + {}^mC_2 S_{1,0} + \ldots + {}^mC_m S_{1,2-m}) \\ &= 1 + (S_{1,2} + mS_{1,1} + 0 + \ldots + 0) \\ &= 1 + 1 + m.1 \\ &= m + 2 \end{aligned}$$

□

Theorem 2.3. *For $1 \leq i \leq m$ and $\forall k \geq 0$,*

$$L_{i,k} = {}^mC_0 L_{i,k-1} + {}^mC_1 L_{i,k-2} + {}^mC_2 L_{i,k-3} + \ldots + {}^mC_m L_{i,k-m-1}.$$

Proof.

$$\begin{aligned} L_{i,k} =& S_{i,k-1} + S_{i,k+1} \\ =& ({}^mC_0 S_{i,k-2} + {}^mC_1 S_{i,k-3} + {}^mC_2 S_{i,k-4} + \ldots + {}^mC_m S_{i,k-m-2}) \\ &+ ({}^mC_0 S_{i,k} + {}^mC_1 S_{i,k-1} + {}^mC_2 S_{i,k-2} + \ldots + {}^mC_m S_{i,k-m}) \\ =& {}^mC_0 (S_{i,k-2} + S_{i,k}) + {}^mC_1 (S_{i,k-3} + S_{i,k-1}) + {}^mC_2 (S_{i,k-4} + S_{i,k-2}) \\ &+ \ldots + {}^mC_m (S_{i,k-m-2} + S_{i,k-m}) \\ =& {}^mC_0 L_{i,k-1} + {}^mC_1 L_{i,k-2} + {}^mC_2 L_{i,k-3} + \ldots + {}^mC_m L_{i,k-m-1} \end{aligned}$$

□

Theorem 2.4. *For $1 < i \leq m$, each term in i^{th} sequence of the generalized Lucas sequences is sum of four consecutive terms in $(i-1)^{th}$ sequence of the generalized Fibonacci sequences. That is,*

$$L_{i,k} = S_{i-1,k-2} + S_{i-1,k-1} + S_{i-1,k} + S_{i-1,k+1}$$

Also each term in first sequence of the generalized Lucas sequences is sum of four consecutive terms in last sequence of the generalized Fibonacci sequences. That is,

$$L_{1,k} = S_{m,k-3} + S_{m,k-2} + S_{m,k-1} + S_{m,k}$$

Proof. For $1 < i \leq m$,

$$\begin{aligned} L_{i,k} =& S_{i,k-1} + S_{i,k+1} \\ =& S_{i-1,k-2} + S_{i-1,k-1} + S_{i-1,k} + S_{i-1,k+1} \end{aligned}$$

Also for $i = 1$,

$$\begin{aligned} L_{1,k} =& S_{1,k-1} + S_{1,k+1} \\ =& S_{m,k-3} + S_{m,k-2} + S_{m,k-1} + S_{m,k} \end{aligned}$$

□

Corollary 2.5. *$L_{1,k} = L_{m,k-2} + L_{m,k-1}$ and for $1 < i \leq m, L_{i,k} = L_{i-1,k-1} + L_{i-1,k}$.*

Proof. For $i = 1$,

$$\begin{aligned} L_{1,k} =& (S_{m,k-3} + S_{m,k-2}) + (S_{m,k-1} + S_{m,k}) \\ =& (S_{m,k-3} + S_{m,k-1}) + (S_{m,k-2} + S_{m,k}) \\ =& L_{m,k-2} + L_{m,k-1} \end{aligned}$$

For $1 < i \leq m$,

$$\begin{aligned} L_{i,k} =& (S_{i-1,k-2} + S_{i-1,k-1}) + (S_{i-1,k} + S_{i-1,k+1}) \\ =& (S_{i-1,k-2} + S_{i-1,k}) + (S_{i-1,k-1} + S_{i-1,k+1}) \\ =& L_{i-1,k-1} + L_{i-1,k} \end{aligned}$$

□

Corollary 2.5 gives an alternative definition to generate generalized Lucas sequences. Here terms are obtained by adding the last two generated terms of the preceding sequence. This definition coincides with the rule to generate generalized Fibonacci sequences.

Proposition 2.6. *$L_{m,0} = 2,\ L_{m,-1} = -1$ and for $i = 1, 2, 3, \ldots, m-1,\ L_{i,0} = 1$.*

Proof. $L_{m,0} = S_{m,-1} + S_{m,1} = 1 + 1 = 2$
$L_{m,-1} = L_{1,1} - L_{m,0} = 1 - 2 = -1$
$L_{i,0} = S_{i,-1} + S_{i,1} = 0 + 1 = 1$ □

Theorem 2.7. *For each of the m sequences, gcd of $(m+1)$ consecutive terms is one.*

Proof. We first write $m+1$ consecutive terms of a particular sequence as below.

$$\begin{array}{llllllll} L_{i,k} & = {}^mC_0 L_{i,k-1} & + {}^mC_1 L_{i,k-2} & + {}^mC_2 L_{i,k-3} & + & \ldots & + {}^mC_m L_{i,k-m-1} \\ L_{i,k+1} & = {}^mC_0 L_{i,k} & + {}^mC_1 L_{i,k-1} & + {}^mC_2 L_{i,k-2} & + & \ldots & + {}^mC_m L_{i,k-m} \\ L_{i,k+2} & = {}^mC_0 L_{i,k+1} & + {}^mC_1 L_{i,k} & + {}^mC_2 L_{i,k-1} & + & \ldots & + {}^mC_m L_{i,k-m+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & & \ldots \\ L_{i,k+m-1} & = {}^mC_0 L_{i,k+m-2} & + {}^mC_1 L_{i,k+m-3} & + {}^mC_2 L_{i,k+m-4} & + & \ldots & + {}^mC_m L_{i,k-2} \\ L_{i,k+m} & = {}^mC_0 L_{i,k+m-1} & + {}^mC_1 L_{i,k+m-2} & + {}^mC_2 L_{i,k+m-3} & + & \ldots & + {}^mC_m L_{i,k-1} \end{array}$$

Suppose g divides all the above terms. Then from the last expression, we can write

$${}^mC_m L_{i,k-1} = L_{i,k+m} - {}^mC_0 L_{i,k+m-1} - {}^mC_1 L_{i,k+m-2} - {}^mC_2 L_{i,k+m-3} - \ldots - {}^mC_{m-1} L_{i,k}$$

which implies that g divides $L_{i,k-1}$.
Now considering the fact that g divides the $m+1$ consecutive terms $L_{i,k-1},\ L_{i,k},\ L_{i,k+1},\ L_{i,k+2},\ \ldots,\ L_{i,k+m-1}$, we proceed as above to get g divides $L_{i,k-2}$. Continuing in similar fashion, we obtain that g divides $L_{i,0} = 1$ for $i = 1, 2, 3, \ldots, m-1$ and g divides $L_{m,-1} = -1$. This implies $g = 1$. □

Theorem 2.7 is generalization of the fact that pair of consecutive Lucas numbers are relatively prime.

Lemma 2.8. *Zeroes of the polynomial $x(x+1)^m - 1$ are simple.*

Proof. Suppose α is a multiple root of $f(x) = x(x+1)^m - 1$. Then $f'(\alpha) = 0$. That is,

$$(\alpha+1)^m + m\alpha(\alpha+1)^{m-1} = 0$$
$$\text{or} \qquad (\alpha+1)^{m-1}(\alpha+1+m\alpha) = 0$$

so that $\alpha + 1 = 0$ or $1 + (m+1)\alpha = 0$.
But neither $\alpha = -1$ nor $\alpha = -\dfrac{1}{m+1}$ satisfies $f(x) = 0$. Hence there is no multiple root. □

Consider $f(x) = x(x+1)^m - 1$ and $g(x) = x^{m+1} - (x+1)^m$, then it is easy to see that $g\left(\frac{1}{x}\right) = f(x)$. Therefore the zeroes of g are also simple.

Theorem 2.9. *For $i = 1, 2, \ldots, m$, $\lim\limits_{k\to\infty} \dfrac{L_{i,k}}{L_{i,k-1}}$ is a root of the equation $x^{m+1} - (x+1)^m = 0$.*

Proof. The equation can be written as

$$x^{m+1} + a_1 x^m + a_2 x^{m-1} + \ldots + a_m x - 1 = 0 \text{ where } a_j = -\ {}^mC_{j-1}.$$

By Bernoulli's Iteration in [7], the ratio $\dfrac{\mu_k}{\mu_{k-1}}$ tends to the largest root in magnitude, where

$$\begin{aligned} &\mu_k + a_1\mu_{k-1} + a_2\mu_{k-2} + \ldots + a_m\mu_{k-m} - \mu_{k-m-1} = 0 \\ \text{or,} \quad &\mu_k = -a_1\mu_{k-1} - a_2\mu_{k-2} - \ldots - a_m\mu_{k-m} + \mu_{k-m-1} \\ \text{or,} \quad &\mu_k = {}^mC_0\mu_{k-1} + {}^mC_1\mu_{k-2} + \ldots + {}^mC_{m-1}\mu_{k-m} + {}^mC_m\mu_{k-m-1} \end{aligned}$$

Identifying μ_k by $L_{i,k}$, we get

$$L_{i,k} = {}^mC_0 L_{i,k-1} + {}^mC_1 L_{i,k-2} + \ldots + {}^mC_{m-1} L_{i,k-m} + {}^mC_m L_{i,k-m-1}$$

which is true by theorem 2.3. Since $L_{i,k}$ are positive for $k > 0$, $\dfrac{L_{i,k}}{L_{i,k-1}}$ tends to the only positive root (by Descartes' rule of signs) of the equation. □

Remark 2.10. *For Lucas sequence, $m = 1$, $i = 1$ and therefore $\lim\limits_{k\to\infty} \dfrac{L_{1,k}}{L_{1,k-1}} = \lim\limits_{k\to\infty} \dfrac{L_k}{L_{k-1}} =$ Golden ratio, which is a root of the equation $x^2 - (x+1) = 0$.*

Theorem 2.11. *For $i = 1, 2, \ldots, m$, $\lim\limits_{k\to\infty} \dfrac{L_{i,k}}{L_{i,k+1}}$ is a root of the equation $x(x+1)^m - 1 = 0$.*

Proof. Since $\lim\limits_{k\to\infty} \dfrac{L_{i,k}}{L_{i,k-1}}$ exists and is nonzero, $\lim\limits_{k\to\infty} \dfrac{L_{i,k-1}}{L_{i,k}}$ also exists and equals $\dfrac{1}{\lim_{k\to\infty} \frac{L_{i,k}}{L_{i,k-1}}} = l$, say.
Now $\dfrac{1}{l}$ is a zero of $g(x) = x^{m+1} - (x+1)^m$. Hence l is a zero of $g(\frac{1}{x}) = f(x) = x(x+1)^m - 1$. Also $\lim\limits_{k\to\infty} \dfrac{L_{i,k}}{L_{i,k+1}} = \lim\limits_{k\to\infty} \dfrac{L_{i,k-1}}{L_{i,k}} = l$. Thus, $\lim\limits_{k\to\infty} \dfrac{L_{i,k}}{L_{i,k+1}}$ is a root of the equation $x(x+1)^m - 1 = 0$.
Alternatively,

$$\lim_{k\to\infty} \frac{L_{i+1,k}}{L_{i+1,k+1}} = \lim_{k\to\infty} \frac{L_{i,k-1} + L_{i,k}}{L_{i,k} + L_{i,k+1}} = \lim_{k\to\infty} \frac{\frac{L_{i,k-1}}{L_{i,k}} + 1}{1 + \frac{L_{i,k+1}}{L_{i,k}}} = \frac{l+1}{1+\frac{1}{l}} = l$$

so that $\lim\limits_{k\to\infty} \dfrac{L_{1,k}}{L_{1,k+1}} = \lim\limits_{k\to\infty} \dfrac{L_{2,k}}{L_{2,k+1}} = \ldots = \lim\limits_{k\to\infty} \dfrac{L_{m,k}}{L_{m,k+1}} = l.$
Now combining the rules for the interconnected sequences, we get,

$$\begin{aligned} L_{1,k+1} &= L_{m,k-1} + L_{m,k} \\ &= (L_{m-1,k-2} + L_{m-1,k-1}) + (L_{m-1,k-1} + L_{m-1,k}) \\ &= L_{m-1,k-2} + 2L_{m-1,k-1} + L_{m-1,k} \\ &= \ldots \\ &= a_m L_{1,k} + a_{m-1} L_{1,k-1} + a_{m-2} L_{1,k-2} + \ldots + a_0 L_{1,k-m} \end{aligned}$$

where $a_j = {}^mC_j$.
Thus,

$$\begin{aligned} &\lim_{k\to\infty} \frac{L_{1,k-1}}{L_{1,k}} = \lim_{k\to\infty} \frac{L_{1,k}}{L_{1,k+1}} \\ \Rightarrow &\lim_{k\to\infty} \frac{L_{1,k-1}}{L_{1,k}} = \lim_{k\to\infty} \frac{L_{1,k}}{a_m L_{1,k} + a_{m-1} L_{1,k-1} + a_{m-2} L_{1,k-2} + \ldots + a_0 L_{1,k-m}} \\ \Rightarrow &\lim_{k\to\infty} \frac{L_{1,k-1}}{L_{1,k}} = \lim_{k\to\infty} \frac{1}{a_m \frac{L_{1,k}}{L_{1,k}} + a_{m-1} \frac{L_{1,k-1}}{L_{1,k}} + a_{m-2} \frac{L_{1,k-2}}{L_{1,k}} + \ldots + a_0 \frac{L_{1,k-m}}{L_{1,k}}} \end{aligned}$$

or,

$$l = \frac{1}{a_m + a_{m-1} l + a_{m-2} l^2 + \ldots + a_0 l^m} \qquad \Rightarrow l = \frac{1}{(1+l)^m} \qquad \Rightarrow l\,(1+l)^m = 1$$

Therefore l is a root of the equation $x(x+1)^m - 1 = 0$. □

Remark 2.12. *For Lucas sequence, $m = 1,\ i = 1$ and therefore $\lim\limits_{k\to\infty} \dfrac{L_{1,k}}{L_{1,k+1}} = \lim\limits_{k\to\infty} \dfrac{L_k}{L_{k+1}} =$ (Golden ratio)$^{-1}$, which is a root of the equation $x(x+1) - 1 = 0$.*

3. Generalized Identities

Theorem 3.1. *Sum of k^{th} terms in all the sequences and the following term, i.e. $(k+1)^{th}$ term in the first sequence equals the next i.e. $(k+2)^{th}$ term in the first sequence.*

Symbolically, $\sum\limits_{i=1}^{m} L_{i,k} + L_{1,k+1} = L_{1,k+2}$ *Lucas equivalent:* $L_k + L_{k+1} = L_{k+2}$

Proof.

$$\begin{aligned} \sum_{i=1}^{m} L_{i,k} + L_{1,k+1} &= (L_{1,k} + L_{2,k} + L_{3,k} + \ldots + L_{m-1,k} + L_{m,k}) + L_{1,k+1} \\ &= (L_{m,k} + L_{m-1,k} + \ldots + L_{3,k} + L_{2,k} + L_{1,k}) + L_{1,k+1} \\ &= (L_{m,k} + L_{m-1,k} + \ldots + L_{3,k} + L_{2,k}) + (L_{1,k} + L_{1,k+1}) \\ &= (L_{m,k} + L_{m-1,k} + \ldots + L_{3,k} + L_{2,k}) + L_{2,k+1} \\ &= (L_{m,k} + L_{m-1,k} + \ldots + L_{3,k}) + (L_{2,k} + L_{2,k+1}) \\ &= \ldots \\ &= L_{m,k} + L_{m,k+1} \\ &= L_{1,k+2} \end{aligned}$$

Columns in the table below show the terms in six sequences (i.e. $m = 6$). The result is illustrated for $k = 2$ and $k = 6$, where the summands are bold and the sum is italicized.

1	2	3	4	5	6
8	**9**	**11**	**14**	**18**	**23**
29	37	46	57	71	89
112	141	178	224	281	352
441	553	694	872	1096	1377
1729	**2170**	**2723**	**3417**	**4289**	**5385**
6762	8491	10661	13384	16801	21090
26475	33237	41728	52389	65773	82574
103664	130139	163376	205104	257493	323266
...	...	...	...	...	...

When $m = 1$, we get,

$$\sum_{i=1}^{m} L_{i,k} + L_{1,k+1} = L_{1,k+2} \quad \text{or} \quad \sum_{i=1}^{1} L_{i,k} + L_{1,k+1} = L_{1,k+2} \quad \text{or} \quad L_{1,k} + L_{1,k+1} = L_{1,k+2}$$

i.e. $L_k + L_{k+1} = L_{k+2}$ $\square$

Theorem 3.2. *Sum of first n terms of all the sequences is $m+2$ less than the $(n+2)^{th}$ term of the first sequence.*

Symbolically, $\sum_{i=1}^{m}\sum_{k=1}^{n} L_{i,k} = L_{1,n+2} - (m+2)$ *Lucas equivalent:* $\sum_{k=1}^{n} L_k = L_{n+2} - 3$

Proof.

$$\begin{array}{lll}
L_{1,n} = L_{2,n+1} - L_{1,n+1} & L_{1,n-1} = L_{2,n} - L_{1,n} & L_{1,n-2} = L_{2,n-1} - L_{1,n-1} \\
L_{2,n} = L_{3,n+1} - L_{2,n+1} & L_{2,n-1} = L_{3,n} - L_{2,n} & L_{2,n-2} = L_{3,n-1} - L_{2,n-1} \\
L_{3,n} = L_{4,n+1} - L_{3,n+1} & L_{3,n-1} = L_{4,n} - L_{3,n} & L_{3,n-2} = L_{4,n-1} - L_{3,n-1} \\
\dots & \dots & \dots \\
\dots & \dots & \dots \\
L_{m-2,n} = L_{m-1,n+1} - L_{m-2,n+1} & L_{m-2,n-1} = L_{m-1,n} - L_{m-2,n} & L_{m-2,n-2} = L_{m-1,n-1} - L_{m-2,n-1} \\
L_{m-1,n} = L_{m,n+1} - L_{m-1,n+1} & L_{m-1,n-1} = L_{m,n} - L_{m-1,n} & L_{m-1,n-2} = L_{m,n-1} - L_{m-1,n-1} \\
L_{m,n} = L_{1,n+2} - L_{m,n+1} & L_{m,n-1} = L_{1,n+1} - L_{m,n} & L_{m,n-2} = L_{1,n} - L_{m,n-1}
\end{array}$$

$$\begin{array}{llllll}
\dots & \dots & \dots & L_{1,2} = L_{2,3} - L_{1,3} & & L_{1,1} = L_{2,2} - L_{1,2} \\
\dots & \dots & \dots & L_{2,2} = L_{3,3} - L_{2,3} & & L_{2,1} = L_{3,2} - L_{2,2} \\
\dots & \dots & \dots & L_{3,2} = L_{4,3} - L_{3,3} & & L_{3,1} = L_{4,2} - L_{3,2} \\
\dots & & & \dots & & \dots \\
\dots & & & \dots & & \dots \\
\dots & \dots & \dots & L_{m-2,2} = L_{m-1,3} - L_{m-2,3} & & L_{m-2,1} = L_{m-1,2} - L_{m-2,2} \\
\dots & \dots & \dots & L_{m-1,2} = L_{m,3} - L_{m-1,3} & & L_{m-1,1} = L_{m,2} - L_{m-1,2} \\
\dots & \dots & \dots & L_{m,2} = L_{1,4} - L_{m,3} & & L_{m,1} = L_{1,3} - L_{m,2}
\end{array}$$

Thus,

$$\sum_{i=1}^{m} L_{i,n} = L_{1,n+2} - L_{1,n+1} \qquad \sum_{i=1}^{m} L_{i,n-1} = L_{1,n+1} - L_{1,n} \qquad \sum_{i=1}^{m} L_{i,n-2} = L_{1,n} - L_{1,n-1}$$

$$\dots \quad \dots \quad \dots \quad \sum_{i=1}^{m} L_{i,2} = L_{1,4} - L_{1,3} \qquad \sum_{i=1}^{m} L_{i,1} = L_{1,3} - L_{1,2}$$

Therefore $\sum_{i=1}^{m}\sum_{k=1}^{n} L_{i,k} = L_{1,n+2} - L_{1,2} = L_{1,n+2} - (m+2)$.

The result is illustrated in the table for $m = 6, n = 6$, where the summands are bold and the sum is 26467, which is $(6 + 2 = 8)$ less than the italicized term.

1	**2**	**3**	**4**	**5**	**6**
8	**9**	**11**	**14**	**18**	**23**
29	**37**	**46**	**57**	**71**	**89**
112	**141**	**178**	**224**	**281**	**352**
441	**553**	**694**	**872**	**1096**	**1377**
1729	**2170**	**2723**	**3417**	**4289**	**5385**
6762	8491	10661	13384	16801	21090
26475	33237	41728	52389	65773	82574
103664	130139	163376	205104	257493	323266
...	...	...	...	...	...

When $m = 1$, we get,

$$\sum_{i=1}^{m}\sum_{k=1}^{n} L_{i,k} = L_{1,n+2} - (m+2) \quad \text{or} \quad \sum_{i=1}^{1}\sum_{k=1}^{n} L_{i,k} = L_{1,n+2} - 3 \quad \text{or} \quad \sum_{k=1}^{n} L_{1,k} = L_{1,n+2} - 3$$

i.e. $\sum_{k=1}^{n} L_k = L_{n+2} - 3$ □

Theorem 3.3. *Sum of all the terms in odd positions (upto $(2n-1)^{th}$ position) in all the sequences is two less than the $(2n)^{th}$ term of the last sequence.*

Symbolically, $\sum_{i=1}^{m}\sum_{k=0}^{n-1} L_{i,2k+1} = L_{m,2n} - 2$ *Lucas equivalent:* $\sum_{k=0}^{n-1} L_{2k+1} = L_{2n} - 2$

Proof.

$L_{1,1} = L_{2,2} - L_{1,2}$ $L_{1,3} = L_{2,4} - L_{1,4}$ $L_{1,5} = L_{2,6} - L_{1,6}$ ... ... $L_{1,2n-1} = L_{2,2n} - L_{1,2n}$

$L_{2,1} = L_{3,2} - L_{2,2}$ $L_{2,3} = L_{3,4} - L_{2,4}$ $L_{2,5} = L_{3,6} - L_{2,6}$ ... ... $L_{2,2n-1} = L_{3,2n} - L_{2,2n}$

$L_{3,1} = L_{4,2} - L_{3,2}$ $L_{3,3} = L_{4,4} - L_{3,4}$ $L_{3,5} = L_{4,6} - L_{3,6}$ ... ... $L_{3,2n-1} = L_{4,2n} - L_{3,2n}$

... ...
... ...

$L_{m-1,1} = L_{m,2} - L_{m-1,2}$ $L_{m-1,3} = L_{m,4} - L_{m-1,4}$ $L_{m-1,5} = L_{m,6} - L_{m-1,6}$ ... ... $L_{m-1,2n-1} = L_{m,2n} - L_{m-1,2n}$

$L_{m,1} = L_{1,3} - L_{m,2}$ $L_{m,3} = L_{1,5} - L_{m,4}$ $L_{m,5} = L_{1,7} - L_{m,6}$ ... ... $L_{m,2n-1} = L_{1,2n+1} - L_{m,2n}$

Thus,

$$\sum_{i=1}^{m} L_{i,1} = L_{1,3} - L_{1,2} = L_{m,1} + L_{m,2} - L_{1,2} = m + L_{m,2} - (m+2)$$

$$\sum_{i=1}^{m} L_{i,3} = L_{1,5} - L_{1,4} = L_{m,3} + L_{m,4} - L_{m,2} - L_{m,3} = L_{m,4} - L_{m,2}$$

$$\sum_{i=1}^{m} L_{i,5} = L_{1,7} - L_{1,6} = L_{m,5} + L_{m,6} - L_{m,4} - L_{m,5} = L_{m,6} - L_{m,4}$$

... ...

$\dots \quad \dots \quad \dots \quad \dots$

$$\sum_{i=1}^{m} L_{i,2n-1} = L_{1,2n+1} - L_{1,2n} = L_{m,2n-1} + L_{m,2n} - L_{m,2n-2} - L_{m,2n-1} = L_{m,2n} - L_{m,2n-2}$$

Therefore $\sum_{i=1}^{m}\sum_{k=0}^{n-1} L_{i,2k+1} = L_{m,2n} - 2.$

The result is illustrated in the table for $m = 6, n = 4$, where the summands are bold and the sum is 82572, which is two less than the italicized term.

1	**2**	**3**	**4**	**5**	**6**
8	9	11	14	18	23
29	**37**	**46**	**57**	**71**	**89**
112	141	178	224	281	352
441	**553**	**694**	**872**	**1096**	**1377**
1729	2170	2723	3417	4289	5385
6762	**8491**	**10661**	**13384**	**16801**	**21090**
26475	33237	41728	52389	65773	*82574*
103664	130139	163376	205104	257493	323266
...	...	...	...	...	...

When $m = 1$, we get,

$$\sum_{i=1}^{m}\sum_{k=0}^{n-1} L_{i,2k+1} = L_{m,2n} \quad \text{or} \quad \sum_{i=1}^{1}\sum_{k=0}^{n-1} L_{i,2k+1} = L_{1,2n} \quad \text{or} \quad \sum_{k=0}^{n-1} L_{1,2k+1} = L_{1,2n}$$

i.e. $\sum_{k=0}^{n-1} L_{2k+1} = L_{2n}$ □

Theorem 3.4. *Sum of all the terms in even positions (upto $(2n)^{th}$ position) in all the sequences is m less than the $(2n+1)^{th}$ term of the last sequence.*

Symbolically, $\sum_{i=1}^{m}\sum_{k=1}^{n} L_{i,2k} = L_{m,2n+1} - m$ *Lucas equivalent:* $\sum_{k=1}^{n} L_{2k} = L_{2n+1} - 1$

Proof. $\sum_{i=1}^{m}\sum_{k=1}^{n} L_{i,2k} = \sum_{i=1}^{m}\sum_{k=1}^{2n} L_{i,k} - \sum_{i=1}^{m}\sum_{k=0}^{n-1} L_{i,2k+1} = (L_{1,2n+2} - (m+2)) - (L_{m,2n} - 2) = L_{m,2n+1} - m$

The result is illustrated in the table for $m = 6, n = 3$, where the summands are bold and the sum is 21084, which is six less than the italicized term.

1	2	3	4	5	6
8	**9**	**11**	**14**	**18**	**23**
29	37	46	57	71	89
112	**141**	**178**	**224**	**281**	**352**
441	553	694	872	1096	1377
1729	**2170**	**2723**	**3417**	**4289**	**5385**
6762	8491	10661	13384	16801	*21090*
26475	33237	41728	52389	65773	82574
103664	130139	163376	205104	257493	323266
...	...	...	...	...	...

When $m = 1$, we get,

$$\sum_{i=1}^{m}\sum_{k=1}^{n} L_{i,2k} = L_{m,2n+1} - m \quad \text{or} \quad \sum_{i=1}^{1}\sum_{k=1}^{n} L_{i,2k} = L_{1,2n+1} - 1 \quad \text{or} \quad \sum_{k=1}^{n} L_{1,2k} = L_{1,2n+1} - 1$$

i.e. $\sum_{k=1}^{n} L_{2k} = L_{2n+1} - 1$ $\square$

4. Cassini-like Identity For Generalized Lucas Sequence

Cassini's identity for Fibonacci numbers is given by $F_{k+1}F_{k-1} - F_k^2 = (-1)^k$. Cassini-like identity for Lucas numbers is given by $L_{k+1}L_{k-1} - L_k^2 = 5(-1)^{k+1}$. In determinant notation, this identity can be put as $\begin{vmatrix} L_{k+1} & L_k \\ L_k & L_{k-1} \end{vmatrix} = 5(-1)^{k+1}$. Below we establish a theorem which generalizes this identity for generalized Lucas sequences dealt in this paper.

Theorem 4.1 (Generalized Cassini-like identity for generalized Lucas sequences)**.** *For i^{th} sequence,*

$$\begin{vmatrix} L_{i,k+1} & L_{i,k} & L_{i,k-1} & \dots & L_{i,k-(m-1)} \\ L_{i,k} & L_{i,k-1} & L_{i,k-2} & \dots & L_{i,k-m} \\ L_{i,k-1} & L_{i,k-2} & L_{i,k-3} & \dots & L_{i,k-(m+1)} \\ \dots & \dots & \dots & \dots & \dots \\ L_{i,k-(m-2)} & L_{i,k-(m-1)} & L_{i,k-m} & \dots & L_{i,k-(2m-2)} \\ L_{i,k-(m-1)} & L_{i,k-m} & L_{i,k-(m+1)} & \dots & L_{i,k-(2m-1)} \end{vmatrix} = (-1)^{m(i-1)}|B| \begin{cases} (-1)^{\frac{m}{2}} & \text{when } m \text{ is even,} \\ (-1)^{\frac{m-1}{2}} & \text{when } m \text{ is odd, and } k \text{ is even,} \\ (-1)^{\frac{m+1}{2}} & \text{when both } m \text{ and } k \text{ are odd.} \end{cases}$$

where
$$B = \begin{bmatrix} {}^mC_0 & {}^mC_1+1 & {}^mC_2 & {}^mC_3 & {}^mC_4 & \dots & {}^mC_{m-1} & {}^mC_m \\ 1 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & -{}^mC_0 & -{}^mC_1 & -{}^mC_2 & -{}^mC_3 & \dots & -{}^mC_{m-2}+1 & -{}^mC_{m-1} \end{bmatrix}$$

Proof. We write

$$\begin{aligned} L_{i,k+m} &= S_{i,k+m-1} + S_{i,k+m+1} \\ &= S_{i,k+m-1} + ({}^mC_0 S_{i,k+m} + {}^mC_1 S_{i,k+m-1} + \ldots + {}^mC_{m-1} S_{i,k+1} + {}^mC_m S_{i,k}) \\ &= {}^mC_0 S_{i,k+m} + ({}^mC_1 + 1) S_{i,k+m-1} + \ldots + {}^mC_{m-1} S_{i,k+1} + {}^mC_m S_{i,k} \\ L_{i,k+m-1} &= S_{i,k+m} + S_{i,k+m-2} \\ L_{i,k+m-2} &= S_{i,k+m-1} + S_{i,k+m-3} \\ \dots & \quad \dots \\ \dots & \quad \dots \\ L_{i,k+1} &= S_{i,k+2} + S_{i,k} \\ L_{i,k} &= S_{i,k+1} + S_{i,k-1} \\ &= S_{i,k+1} + (S_{i,k+m} - {}^mC_0 S_{i,k+m-1} - \ldots - {}^mC_{m-2} S_{i,k+1} - {}^mC_{m-1} S_{i,k}) \\ &= S_{i,k+m} - {}^mC_0 S_{i,k+m-1} - \ldots - ({}^mC_{m-2} - 1) S_{i,k+1} - {}^mC_{m-1} S_{i,k} \end{aligned}$$

so that

$$\begin{bmatrix} L_{i,k+m} \\ L_{i,k+m-1} \\ \dots \\ L_{i,k+2} \\ L_{i,k+1} \\ L_{i,k} \end{bmatrix} = \begin{bmatrix} {}^mC_0 & {}^mC_1+1 & {}^mC_2 & {}^mC_3 & {}^mC_4 & \dots & {}^mC_{m-1} & {}^mC_m \\ 1 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & -{}^mC_0 & -{}^mC_1 & -{}^mC_2 & -{}^mC_3 & \dots & -{}^mC_{m-2}+1 & -{}^mC_{m-1} \end{bmatrix} \begin{bmatrix} S_{i,k+m} \\ S_{i,k+m-1} \\ \dots \\ S_{i,k+2} \\ S_{i,k+1} \\ S_{i,k} \end{bmatrix}$$

or $\begin{bmatrix} L_{i,k+m} \\ L_{i,k+m-1} \\ \dots \\ L_{i,k+2} \\ L_{i,k+1} \\ L_{i,k} \end{bmatrix} = B \begin{bmatrix} S_{i,k+m} \\ S_{i,k+m-1} \\ \dots \\ S_{i,k+2} \\ S_{i,k+1} \\ S_{i,k} \end{bmatrix} = BA \begin{bmatrix} S_{i,k+m-1} \\ S_{i,k+m-2} \\ \dots \\ S_{i,k+1} \\ S_{i,k} \\ S_{i,k-1} \end{bmatrix} = BA^2 \begin{bmatrix} S_{i,k+m-2} \\ S_{i,k+m-3} \\ \dots \\ S_{i,k} \\ S_{i,k-1} \\ S_{i,k-2} \end{bmatrix}$

where $A = \begin{bmatrix} {}^mC_0 & {}^mC_1 & {}^mC_2 & \dots & {}^mC_{m-1} & {}^mC_m \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$

Now

$$A^k = \begin{bmatrix} \sum_{j=0}^{m} {}^mC_j S_{1,k-j} & \sum_{j=1}^{m} {}^mC_j S_{1,k+1-j} & \dots & \sum_{j=m-1}^{m} {}^mC_j S_{1,k+(m-1)-j} & S_{1,k} \\ \sum_{j=0}^{m} {}^mC_j S_{1,k-1-j} & \sum_{j=1}^{m} {}^mC_j S_{1,k-j} & \dots & \sum_{j=m-1}^{m} {}^mC_j S_{1,(k-1)+(m-1)-j} & S_{1,k-1} \\ \sum_{j=0}^{m} {}^mC_j S_{1,k-2-j} & \sum_{j=1}^{m} {}^mC_j S_{1,k-1-j} & \dots & \sum_{j=m-1}^{m} {}^mC_j S_{1,(k-2)+(m-1)-j} & S_{1,k-2} \\ \dots & \dots & \dots & \dots & \dots \\ \sum_{j=0}^{m} {}^mC_j S_{1,k-(m-1)-j} & \sum_{j=1}^{m} {}^mC_j S_{1,k-(m-2)-j} & \dots & \sum_{j=m-1}^{m} {}^mC_j S_{1,k-j} & S_{1,k-(m-1)} \\ \sum_{j=0}^{m} {}^mC_j S_{1,k-m-j} & \sum_{j=1}^{m} {}^mC_j S_{1,k-(m-1)-j} & \dots & \sum_{j=m-1}^{m} {}^mC_j S_{1,k-(m-1)+m-j} & S_{1,k-m} \end{bmatrix}$$

Therefore

$$BA^k = \begin{bmatrix} \sum_{j=0}^{m} {}^mC_j L_{1,k-j} & \sum_{j=1}^{m} {}^mC_j L_{1,k+1-j} & \dots & \sum_{j=m-1}^{m} {}^mC_j L_{1,k+(m-1)-j} & L_{1,k} \\ \sum_{j=0}^{m} {}^mC_j L_{1,k-1-j} & \sum_{j=1}^{m} {}^mC_j L_{1,k-j} & \dots & \sum_{j=m-1}^{m} {}^mC_j L_{1,(k-1)+(m-1)-j} & L_{1,k-1} \\ \sum_{j=0}^{m} {}^mC_j L_{1,k-2-j} & \sum_{j=1}^{m} {}^mC_j L_{1,k-1-j} & \dots & \sum_{j=m-1}^{m} {}^mC_j L_{1,(k-2)+(m-1)-j} & L_{1,k-2} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \sum_{j=0}^{m} {}^mC_j L_{1,k-m-j} & \sum_{j=1}^{m} {}^mC_j L_{1,k-(m-1)-j} & \dots & \sum_{j=m-1}^{m} {}^mC_j L_{1,k-(m-1)+m-j} & L_{1,k-m} \end{bmatrix}$$

or,

$$BA^k = \begin{bmatrix} L_{1,k+1} & \sum_{j=1}^{m} {}^mC_j L_{1,k+1-j} & \cdots & \sum_{j=m-1}^{m} {}^mC_j L_{1,k+(m-1)-j} & L_{1,k} \\ L_{1,k} & \sum_{j=1}^{m} {}^mC_j L_{1,k-j} & \cdots & \sum_{j=m-1}^{m} {}^mC_j L_{1,(k-1)+(m-1)-j} & L_{1,k-1} \\ L_{1,k-1} & \sum_{j=1}^{m} {}^mC_j L_{1,k-1-j} & \cdots & \sum_{j=m-1}^{m} {}^mC_j L_{1,(k-2)+(m-1)-j} & L_{1,k-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ L_{1,k-(m-2)} & \sum_{j=1}^{m} {}^mC_j L_{1,k-(m-2)-j} & \cdots & \sum_{j=m-1}^{m} {}^mC_j L_{1,k-j} & L_{1,k-(m-1)} \\ L_{1,k-(m-1)} & \sum_{j=1}^{m} {}^mC_j L_{1,k-(m-1)-j} & \cdots & \sum_{j=m-1}^{m} {}^mC_j L_{1,k-(m-1)+m-j} & L_{1,k-m} \end{bmatrix}$$

[Note: The entries in the matrix are from the first sequence. We verify few entries of the matrix, for the reader, in a remark to this theorem.]

so that

$$|BA^k| = \begin{vmatrix} L_{1,k+1} & \sum_{j=1}^{m} {}^mC_j L_{1,k+1-j} & \cdots & \sum_{j=m-1}^{m} {}^mC_j L_{1,k+(m-1)-j} & L_{1,k} \\ L_{1,k} & \sum_{j=1}^{m} {}^mC_j L_{1,k-j} & \cdots & \sum_{j=m-1}^{m} {}^mC_j L_{1,(k-1)+(m-1)-j} & L_{1,k-1} \\ L_{1,k-1} & \sum_{j=1}^{m} {}^mC_j L_{1,k-1-j} & \cdots & \sum_{j=m-1}^{m} {}^mC_j L_{1,(k-2)+(m-1)-j} & L_{1,k-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ L_{1,k-(m-2)} & \sum_{j=1}^{m} {}^mC_j L_{1,k-(m-2)-j} & \cdots & \sum_{j=m-1}^{m} {}^mC_j L_{1,k-j} & L_{1,k-(m-1)} \\ L_{1,k-(m-1)} & \sum_{j=1}^{m} {}^mC_j L_{1,k-(m-1)-j} & \cdots & \sum_{j=m-1}^{m} {}^mC_j L_{1,k-(m-1)+m-j} & L_{1,k-m} \end{vmatrix}$$

Applying column operations, we get

$$\begin{vmatrix} L_{1,k+1} & L_{1,k-(m-1)} & L_{1,k-(m-2)} & \cdots & L_{1,k-1} & L_{1,k} \\ L_{1,k} & L_{1,k-m} & L_{1,k-(m-1)} & \cdots & L_{1,k-2} & L_{1,k-1} \\ L_{1,k-1} & L_{1,k-(m+1)} & L_{1,k-m} & \cdots & L_{1,k-3} & L_{1,k-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ L_{1,k-(m-2)} & L_{1,k-(2m-2)} & L_{1,k-(2m-3)} & \cdots & L_{1,k-m} & L_{1,k-(m-1)} \\ L_{1,k-(m-1)} & L_{1,k-(2m-1)} & L_{1,k-(2m-2)} & \cdots & L_{1,k-(m+1)} & L_{1,k-m} \end{vmatrix} = |B||A|^k = (-1)^{mk}|B|$$

Rearranging the columns, we obtain

$$\begin{vmatrix} L_{1,k+1} & L_{1,k} & L_{1,k-1} & \cdots & L_{1,k-(m-1)} \\ L_{1,k} & L_{1,k-1} & L_{1,k-2} & \cdots & L_{1,k-m} \\ L_{1,k-1} & L_{1,k-2} & L_{1,k-3} & \cdots & L_{1,k-(m+1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ L_{1,k-(m-2)} & L_{1,k-(m-1)} & L_{1,k-m} & \cdots & L_{1,k-(2m-2)} \\ L_{1,k-(m-1)} & L_{1,k-m} & L_{1,k-(m+1)} & \cdots & L_{1,k-(2m-1)} \end{vmatrix} = |B| \begin{cases} (-1)^{\frac{m}{2}}(-1)^{mk} & \text{when } m \text{ is even,} \\ (-1)^{\frac{m-1}{2}}(-1)^{mk} & \text{when } m \text{ is odd.} \end{cases}$$

Thus,

$$\begin{vmatrix} L_{1,k+1} & L_{1,k} & L_{1,k-1} & \dots & L_{1,k-(m-1)} \\ L_{1,k} & L_{1,k-1} & L_{1,k-2} & \dots & L_{1,k-m} \\ L_{1,k-1} & L_{1,k-2} & L_{1,k-3} & \dots & L_{1,k-(m+1)} \\ \dots & \dots & \dots & \dots & \dots \\ L_{1,k-(m-2)} & L_{1,k-(m-1)} & L_{1,k-m} & \dots & L_{1,k-(2m-2)} \\ L_{1,k-(m-1)} & L_{1,k-m} & L_{1,k-(m+1)} & \dots & L_{1,k-(2m-1)} \end{vmatrix} = |B| \begin{cases} (-1)^{\frac{m}{2}} & \text{when } m \text{ is even,} \\ (-1)^{\frac{m-1}{2}} & \text{when } m \text{ is odd, } k \text{ is even,} \\ (-1)^{\frac{m+1}{2}} & \text{when } m \text{ and } k \text{ are odd.} \end{cases}$$

Thus far the identity has been established for the first sequence. Now we proceed to establish the identity for i^{th} sequence.
Suppose

$$\begin{vmatrix} L_{i,k+1} & L_{i,k} & L_{i,k-1} & \dots & L_{i,k-(m-2)} & L_{i,k-(m-1)} \\ L_{i,k} & L_{i,k-1} & L_{i,k-2} & \dots & L_{i,k-(m-1)} & L_{i,k-m} \\ L_{i,k-1} & L_{i,k-2} & L_{i,k-3} & \dots & L_{i,k-m} & L_{i,k-(m+1)} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ L_{i,k-(m-2)} & L_{i,k-(m-1)} & L_{i,k-m} & \dots & L_{i,k-(2m-3)} & L_{i,k-(2m-2)} \\ L_{i,k-(m-1)} & L_{i,k-m} & L_{i,k-(m+1)} & \dots & L_{i,k-(2m-2)} & L_{i,k-(2m-1)} \end{vmatrix} = \Delta$$

By $C_j \to C_j + C_{j+1}$, for $j = 1, 2, \dots, m$

$$\begin{vmatrix} L_{i+1,k+1} & L_{i+1,k} & L_{i+1,k-1} & \dots & L_{i+1,k-(m-2)} & L_{i,k-(m-1)} \\ L_{i+1,k} & L_{i+1,k-1} & L_{i+1,k-2} & \dots & L_{i+1,k-(m-1)} & L_{i,k-m} \\ L_{i+1,k-1} & L_{i+1,k-2} & L_{i+1,k-3} & \dots & L_{i+1,k-m} & L_{i,k-(m+1)} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ L_{i+1,k-(m-2)} & L_{i+1,k-(m-1)} & L_{i+1,k-m} & \dots & L_{i+1,k-(2m-3)} & L_{i,k-(2m-2)} \\ L_{i+1,k-(m-1)} & L_{i+1,k-m} & L_{i+1,k-(m+1)} & \dots & L_{i+1,k-(2m-2)} & L_{i,k-(2m-1)} \end{vmatrix} = \Delta$$

Also

$$\begin{aligned} L_{i+1,k} = \; & L_{i+1,k+m} \\ & - (1 +^m C_0) L_{i+1,k+m-1} \\ & + (1 +^m C_0 -^m C_1) L_{i+1,k+m-2} \\ & - (1 +^m C_0 -^m C_1 +^m C_2) L_{i+1,k+m-3} \\ & + \dots \\ & + (-1)^{m-1} (1 +^m C_0 -^m C_1 +^m C_2 - \dots +^m C_{m-2}) L_{i+1,k+1} \\ & + (-1)^m L_{i,k} \end{aligned}$$

Applying column operation on the last column as per the above formula, we get

$$\begin{vmatrix} L_{i+1,k+1} & L_{i+1,k} & L_{i+1,k-1} & \dots & L_{i+1,k-(m-2)} & L_{i+1,k-(m-1)} \\ L_{i+1,k} & L_{i+1,k-1} & L_{i+1,k-2} & \dots & L_{i+1,k-(m-1)} & L_{i+1,k-m} \\ L_{i+1,k-1} & L_{i+1,k-2} & L_{i+1,k-3} & \dots & L_{i+1,k-m} & L_{i+1,k-(m+1)} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ L_{i+1,k-(m-2)} & L_{i+1,k-(m-1)} & L_{i+1,k-m} & \dots & L_{i+1,k-(2m-3)} & L_{i+1,k-(2m-2)} \\ L_{i+1,k-(m-1)} & L_{i+1,k-m} & L_{i+1,k-(m+1)} & \dots & L_{i+1,k-(2m-2)} & L_{i+1,k-(2m-1)} \end{vmatrix} = (-1)^m \Delta$$

Thus we generalize Cassini-like identity as

$$\begin{vmatrix} L_{i,k+1} & L_{i,k} & L_{i,k-1} & \dots & L_{i,k-(m-1)} \\ L_{i,k} & L_{i,k-1} & L_{i,k-2} & \dots & L_{i,k-m} \\ L_{i,k-1} & L_{i,k-2} & L_{i,k-3} & \dots & L_{i,k-(m+1)} \\ \dots & \dots & \dots & \dots & \dots \\ L_{i,k-(m-2)} & L_{i,k-(m-1)} & L_{i,k-m} & \dots & L_{i,k-(2m-2)} \\ L_{i,k-(m-1)} & L_{i,k-m} & L_{i,k-(m+1)} & \dots & L_{i,k-(2m-1)} \end{vmatrix} = (-1)^{m(i-1)} |B| \begin{cases} (-1)^{\frac{m}{2}} & \text{when } m \text{ is even,} \\ (-1)^{\frac{m-1}{2}} & \text{when } m \text{ is odd, and } k \text{ is even,} \\ (-1)^{\frac{m+1}{2}} & \text{when both } m \text{ and } k \text{ are odd.} \end{cases}$$

□

Remark 4.2. *While forming the matrix B, we first write the first and last rows and fill the remaining rows with 0s and 1s according to the mentioned pattern. 1 is added to the entry in the second column of the first row and to the entry in the m^{th} column of the last row.*

$$\text{For } m=3,\ B=\begin{bmatrix} {}^3C_0 & {}^3C_1+1 & {}^3C_2 & {}^3C_3 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & -{}^3C_0 & -{}^3C_1+1 & -{}^3C_2 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 3 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & -2 & -3 \end{bmatrix}$$

$$\text{For } m=2,\ B=\begin{bmatrix} {}^2C_0 & {}^2C_1+1 & {}^2C_2 \\ 1 & 0 & 1 \\ 1 & -{}^2C_0+1 & -{}^2C_1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & -2 \end{bmatrix}$$

$$\text{For } m=1,\ B=\begin{bmatrix} {}^1C_0 & {}^1C_1+1 \\ 1+1 & -{}^1C_0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

Remark 4.3. *Here we verify few entries of the matrix BA^k for $m=6$ and $k=1$.*

1. Entry in the first row and second column:

From the product BA, the entry is ${}^mC_0\ {}^mC_1 + {}^mC_2 = {}^6C_0\ {}^6C_1 + {}^6C_2 = 1.6 + 15 = 21.$

By substituting $m=6$ and $k=1$ in the corresponding entry in BA^k, we get the entry as

$$\begin{aligned}\sum_{j=1}^{m} {}^mC_j L_{1,k+1-j} &= \sum_{j=1}^{6} {}^6C_j L_{1,2-j} \\ &= {}^6C_1L_{1,1} + {}^6C_2L_{1,0} + {}^6C_3L_{1,-1} + {}^6C_4L_{1,-2} + {}^6C_5L_{1,-3} + {}^6C_6L_{1,-4} \\ &= ({}^6C_0L_{1,2} + {}^6C_1L_{1,1} + {}^6C_2L_{1,0} + {}^6C_3L_{1,-1} + {}^6C_4L_{1,-2} + {}^6C_5L_{1,-3} \\ &\qquad + {}^6C_6L_{1,-4}) - {}^6C_0L_{1,2} \\ &= L_{1,3} - L_{1,2} = 29 - 8 = 21\end{aligned}$$

2. Entry in the first row and third column:

From the product BA, the entry is ${}^mC_0\ {}^mC_2 + {}^mC_3 = {}^6C_0\ {}^6C_2 + {}^6C_3 = 1.15 + 20 = 35.$

By substituting $m=6$ and $k=1$ in the corresponding entry in BA^k, we get the entry as

$$\begin{aligned}\sum_{j=2}^{m} {}^mC_j L_{1,k+2-j} &= \sum_{j=2}^{6} {}^6C_j L_{1,3-j} \\ &= {}^6C_2L_{1,1} + {}^6C_3L_{1,0} + {}^6C_4L_{1,-1} + {}^6C_5L_{1,-2} + {}^6C_6L_{1,-3} \\ &= ({}^6C_0L_{1,3} + {}^6C_1L_{1,2} + {}^6C_2L_{1,1} + {}^6C_3L_{1,0} + {}^6C_4L_{1,-1} \\ &\qquad + {}^6C_5L_{1,-2} + {}^6C_6L_{1,-3}) - {}^6C_0L_{1,3} - {}^6C_1L_{1,2} \\ &= L_{1,4} - L_{1,3} - 6L_{1,2} \\ &= 112 - 29 - 6.8 = 35\end{aligned}$$

3. Entry in the third row and fourth column:

From the product BA, the entry is $0\ {}^mC_3 + 1.0 + 0.0 + 1.0 = 0.$

By substituting $m = 6$ and $k = 1$ in the corresponding entry in BA^k, we get the entry as

$$\begin{aligned}\sum_{j=3}^{m} {}^{m}C_j L_{1,k+1-j} &= \sum_{j=3}^{6} {}^{6}C_j L_{1,2-j} \\ &= {}^{6}C_3 L_{1,-1} + {}^{6}C_4 L_{1,-2} + {}^{6}C_5 L_{1,-3} + {}^{6}C_6 L_{1,-4} \\ &= ({}^{6}C_0 L_{1,2} + {}^{6}C_1 L_{1,1} + {}^{6}C_2 L_{1,0} + {}^{6}C_3 L_{1,-1} + {}^{6}C_4 L_{1,-2} \\ &\qquad + {}^{6}C_5 L_{1,-3} + {}^{6}C_6 L_{1,-4}) - {}^{6}C_0 L_{1,2} - {}^{6}C_1 L_{1,1} - {}^{6}C_2 L_{1,0} \\ &= L_{1,3} - L_{1,2} - 6L_{1,1} - 15L_{1,0} \\ &= 29 - 8 - 6.1 - 15.1 = 0\end{aligned}$$

Remark 4.4. *For $m = 1$, generalized Cassini-like identity reduces to a 2x2 determinant with $i = 1$. That is,*

$$\begin{vmatrix} L_{1,k+1} & L_{1,k} \\ L_{1,k} & L_{1,k-1} \end{vmatrix} = |B| \begin{cases} 1 & \text{when } k \text{ is even,} \\ -1 & \text{when } k \text{ is odd.} \end{cases}$$

where $|B| = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -5$ which is the Cassini-like identity $L_{k+1}L_{k-1} - L_k^2 = 5(-1)^{k+1}$ for Lucas numbers. Note that for $m = 1$, $L_{1,k} = L_k$.

In [11], the identity $F_{p+q} = F_{p+1}F_q + F_pF_{q-1}$ is generalized, for the first sequence, as

$$\begin{aligned}S_{1,p+q} = &\left(\sum_{j=0}^{m} {}^{m}C_j S_{1,p-j}\right) S_{1,q} + \left(\sum_{j=1}^{m} {}^{m}C_j S_{1,p-j+1}\right) S_{1,q-1} + \left(\sum_{j=2}^{m} {}^{m}C_j S_{1,p-j+2}\right) S_{1,q-2} + \dots \\ &+ \dots + \left(\sum_{j=m-1}^{m} {}^{m}C_j S_{1,p-j+(m-1)}\right) S_{1,q-(m-1)} + S_{1,p}S_{1,q-m}\end{aligned}$$

This can be further generalized for the i^{th} sequence as

$$\begin{aligned}S_{i,p+q} = &\left(\sum_{j=0}^{m} {}^{m}C_j S_{1,p-j}\right) S_{i,q} + \left(\sum_{j=1}^{m} {}^{m}C_j S_{1,p-j+1}\right) S_{i,q-1} + \left(\sum_{j=2}^{m} {}^{m}C_j S_{1,p-j+2}\right) S_{i,q-2} + \dots \\ &+ \dots + \left(\sum_{j=m-1}^{m} {}^{m}C_j S_{1,p-j+(m-1)}\right) S_{i,q-(m-1)} + S_{1,p}S_{i,q-m}\end{aligned}$$

Our next theorem generalizes the identity $L_{p+q} = L_{p+1}F_q + L_pF_{q-1}$.

Theorem 4.5. *For i^{th} sequence,*

$$\begin{aligned}L_{i,p+q} = &\left(\sum_{j=0}^{m} {}^{m}C_j L_{1,p-j}\right) S_{i,q} + \left(\sum_{j=1}^{m} {}^{m}C_j L_{1,p-j+1}\right) S_{i,q-1} + \left(\sum_{j=2}^{m} {}^{m}C_j L_{1,p-j+2}\right) S_{i,q-2} + \dots \\ &+ \dots + \left(\sum_{j=m-1}^{m} {}^{m}C_j L_{1,p-j+(m-1)}\right) S_{i,q-(m-1)} + L_{1,p}S_{i,q-m}\end{aligned}$$

Proof. Since $BA^{p+q} = (BA^p)A^q$, the entries in the first row first column of both these matrices are equal.

This gives

$$L_{1,p+q+1} = \left(\sum_{j=0}^{m} {}^{m}C_j L_{1,p-j}\right)\left(\sum_{j=0}^{m} {}^{m}C_j S_{1,q-j}\right) + \left(\sum_{j=1}^{m} {}^{m}C_j L_{1,p-j+1}\right)\left(\sum_{j=0}^{m} {}^{m}C_j S_{1,q-j-1}\right)$$
$$+ \left(\sum_{j=2}^{m} {}^{m}C_j L_{1,p-j+2}\right)\left(\sum_{j=0}^{m} {}^{m}C_j S_{1,q-j-2}\right) + \ldots + \ldots$$
$$+ \left(\sum_{j=m-1}^{m} {}^{m}C_j L_{1,p-j+(m-1)}\right)\left(\sum_{j=0}^{m} {}^{m}C_j S_{1,q-j-(m-1)}\right) + L_{1,p}\left(\sum_{j=0}^{m} {}^{m}C_j S_{1,q-j-m}\right)$$

$$\text{Note: } S_{1,q} = \sum_{j=0}^{m} {}^{m}C_j S_{1,(q-1)-j}.$$

Therefore

$$L_{1,p+q+1} = \left(\sum_{j=0}^{m} {}^{m}C_j L_{1,p-j}\right) S_{1,q+1} + \left(\sum_{j=1}^{m} {}^{m}C_j L_{1,p-j+1}\right) S_{1,q} + \left(\sum_{j=2}^{m} {}^{m}C_j L_{1,p-j+2}\right) S_{1,q-1} + \ldots$$
$$+ \ldots + \left(\sum_{j=m-1}^{m} {}^{m}C_j L_{1,p-j+(m-1)}\right) S_{1,q-(m-2)} + L_{1,p} S_{1,q-(m-1)} \tag{4.1}$$

Writing $q-1$ for q, we get

$$L_{1,p+q} = \left(\sum_{j=0}^{m} {}^{m}C_j L_{1,p-j}\right) S_{1,q} + \left(\sum_{j=1}^{m} {}^{m}C_j L_{1,p-j+1}\right) S_{1,q-1} + \left(\sum_{j=2}^{m} {}^{m}C_j L_{1,p-j+2}\right) S_{1,q-2} + \ldots$$
$$+ \ldots + \left(\sum_{j=m-1}^{m} {}^{m}C_j L_{1,p-j+(m-1)}\right) S_{1,q-(m-1)} + L_{1,p} S_{1,q-m} \tag{4.2}$$

Adding corresponding sides of (4.1) and (4.2), we get

$$L_{2,p+q+1} = \left(\sum_{j=0}^{m} {}^{m}C_j L_{1,p-j}\right) S_{2,q+1} + \left(\sum_{j=1}^{m} {}^{m}C_j L_{1,p-j+1}\right) S_{2,q} + \left(\sum_{j=2}^{m} {}^{m}C_j L_{1,p-j+2}\right) S_{2,q-1} + \ldots$$
$$+ \ldots + \left(\sum_{j=m-1}^{m} {}^{m}C_j L_{1,p-j+(m-1)}\right) S_{2,q-(m-2)} + L_{1,p} S_{2,q-(m-1)}$$

Writing $q-1$ for q,

$$L_{2,p+q} = \left(\sum_{j=0}^{m} {}^{m}C_j L_{1,p-j}\right) S_{2,q} + \left(\sum_{j=1}^{m} {}^{m}C_j L_{1,p-j+1}\right) S_{2,q-1} + \left(\sum_{j=2}^{m} {}^{m}C_j L_{1,p-j+2}\right) S_{2,q-2} + \ldots$$
$$+ \ldots + \left(\sum_{j=m-1}^{m} {}^{m}C_j L_{1,p-j+(m-1)}\right) S_{2,q-(m-1)} + L_{1,p} S_{2,q-m}$$

Proceeding in a similar fashion, we get the required identity as

$$L_{i,p+q} = \left(\sum_{j=0}^{m} {}^{m}C_j L_{1,p-j}\right) S_{i,q} + \left(\sum_{j=1}^{m} {}^{m}C_j L_{1,p-j+1}\right) S_{i,q-1} + \left(\sum_{j=2}^{m} {}^{m}C_j L_{1,p-j+2}\right) S_{i,q-2} + \dots + \dots + \left(\sum_{j=m-1}^{m} {}^{m}C_j L_{1,p-j+(m-1)}\right) S_{i,q-(m-1)} + L_{1,p} S_{i,q-m}$$

□

Remark 4.6. *For $m = 1$, $i = 1$, and therefore*

$$L_{i,p+q} = \left(\sum_{j=0}^{m} {}^{m}C_j L_{1,p-j}\right) S_{i,q} + \left(\sum_{j=1}^{m} {}^{m}C_j L_{1,p-j+1}\right) S_{i,q-1} + \left(\sum_{j=2}^{m} {}^{m}C_j L_{1,p-j+2}\right) S_{i,q-2} + \dots + \dots + \left(\sum_{j=m-1}^{m} {}^{m}C_j L_{1,p-j+(m-1)}\right) S_{i,q-(m-1)} + L_{1,p} S_{i,q-m}$$

reduces to

$$\begin{aligned} L_{1,p+q} &= \left(\sum_{j=0}^{1} {}^{1}C_j L_{1,p-j}\right) S_{1,q} + L_{1,p} S_{1,q-1} \\ or, L_{p+q} &= \left(\sum_{j=0}^{1} {}^{1}C_j L_{p-j}\right) F_q + L_p F_{q-1} \\ or, L_{p+q} &= \left({}^{1}C_0 L_p + {}^{1}C_1 L_{p-1}\right) F_q + L_p F_{q-1} \\ or, L_{p+q} &= \left(L_p + L_{p-1}\right) F_q + L_p F_{q-1} \\ or, L_{p+q} &= L_{p+1} F_q + L_p F_{q-1} \end{aligned}$$

5. Conclusion

The generalization of Lucas sequence as discussed in this paper is used to generalize Cassini-like identity for Lucas numbers in matrix form. In addition to it, we have generalized many identities and also obtained a recurrence relation to generate one of the m sequences. This recurrence relation is used to prove that gcd of $(m+1)$ consecutive terms in any of the sequences is one.

Acknowledgments

Authors thank the referees for their comments and suggestions in improving the work. The first author thanks University Grants Commission of India for its financial support to the work.

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