



Marshall-Olkin Family of Distributions: Additional Properties and Comparative Studies

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ABSTRACT: In this paper we will study the relationship between the baseline pdf and the corresponding MO pdf, we will show the effectiveness on the percentile points, also we will give a closed form for the MLE estimator of the new parameter and will study some examples.

Key Words: Marshall-Olkin (MO), Hazard rate, Reliability function, Maximum likelihood MLE.

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1. General Theory of Marshall-Olkin Distributions

Marshall-Olkin family of distribution are defined to create new distributions from old distribution in a way that can control the reliability and hazard rate which is introduced by Marshall and Olkin first in 1997 [15] as follows

Definition 1.1. *If X is a random variable with pdf $f(x)$, cdf $F(x)$, reliability function $R(x)$ and hazard function $h(x)$, then the MO cdf is defined by:*

$$F_{\alpha}(x) = \frac{F(x)}{F(x) + \alpha R(x)}, \quad \alpha \geq 1.$$

The MO reliability function $R_{\alpha}(x)$, pdf $f_{\alpha}(x)$ and hazard function $h_{\alpha}(x)$ are:

$$R_{\alpha}(x) = \frac{\alpha R(x)}{F(x) + \alpha R(x)},$$

$$f_{\alpha}(x) = \frac{\alpha f(x)}{(F(x) + \alpha R(x))^2}, \text{ and}$$

$$h_{\alpha}(x) = \frac{h(x)}{F(x) + \alpha R(x)}, \text{ respectively.}$$

Marshall and Olkin [15] also mentioned that $R_{\alpha}(x) \geq R(x)$ for $\alpha > 1$ and $R_{\alpha}(x) \leq R(x)$ for $0 < \alpha < 1$, also $h_{\alpha}(x) \leq h(x)$ for $\alpha > 1$ and $h_{\alpha}(x) \geq h(x)$ for $0 < \alpha < 1$.

Many researchers like Jose and Alice [12,13], Alice and Jose [1]-[5], Ghitany et al. [7], Jayakumar and Mathew [11], Jayakumar and Kuttikrishnan [10], Ghitany and Kotz [8], Jose and Uma [7], Gupta et al. [9] and Jose et al. [14] studied Marshall-Olkin extended family of distributions and showed that it can be used to model real situation in a better manner than the basic distribution depending on the fact that it could have (depending on the value of the added parameter) an interesting hazard function.

Cox, D.R. [6], considered the hazard function (age-specific failure rate) to be a function of the explanatory variables and unknown regression coefficients multiplied by an arbitrary and unknown function of time.

2. Some Comparisons

In this section we will compare between the original pdf and the corresponding MO pdf, also we will show the effectiveness of MO on the percentile point and on the symmetry of the original distribution.

Theorem 2.1. $f_\alpha(x) = f(x)$ if and only if $\alpha = 1$ or $X = F^{-1}\left(\frac{\sqrt{\alpha}}{1+\sqrt{\alpha}}\right)$.

Proof. \Rightarrow if $\alpha = 1$, then $f_\alpha(x) = f(x)$,

if $\alpha \neq 1$ and $X = F^{-1}\left(\frac{\sqrt{\alpha}}{1+\sqrt{\alpha}}\right)$, then

$$f_\alpha\left(F^{-1}\left(\frac{\sqrt{\alpha}}{1+\sqrt{\alpha}}\right)\right) = \frac{\alpha f\left(F^{-1}\left(\frac{\sqrt{\alpha}}{1+\sqrt{\alpha}}\right)\right)}{\left(\frac{\sqrt{\alpha}}{1+\sqrt{\alpha}} + \alpha\left(1 - \frac{\sqrt{\alpha}}{1+\sqrt{\alpha}}\right)\right)^2},$$

$$f_\alpha\left(F^{-1}\left(\frac{\sqrt{\alpha}}{1+\sqrt{\alpha}}\right)\right) = \frac{\alpha f\left(F^{-1}\left(\frac{\sqrt{\alpha}}{1+\sqrt{\alpha}}\right)\right)}{\left(\frac{\sqrt{\alpha}}{1+\sqrt{\alpha}} + \alpha\left(\frac{1}{1+\sqrt{\alpha}}\right)\right)^2},$$

$$f_\alpha\left(F^{-1}\left(\frac{\sqrt{\alpha}}{1+\sqrt{\alpha}}\right)\right) = \frac{\alpha f\left(F^{-1}\left(\frac{\sqrt{\alpha}}{1+\sqrt{\alpha}}\right)\right)}{\left(\frac{\sqrt{\alpha}(1+\sqrt{\alpha})}{1+\sqrt{\alpha}}\right)^2},$$

$$f_\alpha\left(F^{-1}\left(\frac{\sqrt{\alpha}}{1+\sqrt{\alpha}}\right)\right) = f\left(F^{-1}\left(\frac{\sqrt{\alpha}}{1+\sqrt{\alpha}}\right)\right).$$

Conversely, let $f_\alpha(x) = f(x) \Rightarrow \frac{\alpha f(x)}{(F(x) + \alpha R(x))^2} = f(x)$,

$$\Rightarrow (F(x) + \alpha R(x))^2 = \alpha,$$

$$\Rightarrow F(x) + \alpha R(x) = \sqrt{\alpha},$$

$$\Rightarrow F(x) = \frac{\sqrt{\alpha} - \alpha}{1 - \alpha} = \frac{\sqrt{\alpha}}{1 + \sqrt{\alpha}},$$

$$\text{Hence } X = F^{-1}\left(\frac{\sqrt{\alpha}}{1 + \sqrt{\alpha}}\right). \quad \square$$

Theorem 2.2. If P_k is the $(100k)^{th}$ percentile for the random variable X , then $F_\alpha(P_k) = \frac{k}{k + \alpha(100 - k)}$.

Proof. Obvious. □

Corollary 2.3. Since $F_\alpha(P_k)$ is a decreasing function of α for a fixed k , then the modified distribution transforms the percentile point into a smaller one.

For example, if Q_2 is the median for the random variable X under the original distribution, then $F_\alpha(Q_2) = \frac{1}{1 + \alpha}$ which is a decreasing function of α .

The following tables shows the new percentile points for different values of α .

Table 1:

α	$F_\alpha(Q_2)$	New percentile point
1	0.50	50 th
2	0.33	33 rd
3	0.25	25 th (lower quartile)
4	0.20	20 th

From the above table, one can see that as alpha increases, the data will be shifted to the left which means that we can affect the skewness of the original distribution.

Theorem 2.4. *If $f(x)$ is symmetric about $X = \mu$, then $f_\alpha(x)$ is not symmetric about $X = \mu$.*

Proof. By definition, $f(\mu - x) = f(\mu + x)$ and $F(\mu - x) = R(\mu + x)$, so

$$\begin{aligned} f_\alpha(\mu - x) &= \frac{\alpha f(\mu - x)}{(F(\mu - x) + \alpha R(\mu - x))^2} \\ &= \frac{\alpha f(\mu + x)}{(R(\mu + x) + \alpha F(\mu + x))^2} \\ &\neq f_\alpha(\mu + x). \end{aligned}$$

□

3. MLE

In this section we will find the MLE of the new parameter and will show by examples that some distributions will remain belong to the same family under MO while other will be totally different.

Theorem 3.1. *The maximum likelihood estimator for the parameter is the solution for the equation:*

$$\sum_{i=1}^n R_\alpha(x_i) = \frac{n}{2}$$

Proof. Let

$$L(\alpha) = f_\alpha(x_1, x_2, x_3, \dots, x_n) = \prod_{i=1}^n f_\alpha(x_i)$$

$$\begin{aligned} &= \prod_{i=1}^n \frac{\alpha f(x_i)}{(F(x_i) + \alpha R(x_i))^2} \\ &= \frac{\alpha^n \prod_{i=1}^n f(x_i)}{\prod_{i=1}^n (F(x_i) + \alpha R(x_i))^2} \end{aligned}$$

$$\log(L(\alpha)) = n \log(\alpha) + \sum_{i=1}^n \log(f(x_i)) - 2 \sum_{i=1}^n \log(F(x_i) + \alpha R(x_i))$$

Differentiating both sides with respect to α and equating the derivative to zero will complete the proof. □

The following example will show that the modified cdf sometimes will belong to the same family of distributions as the original one.

Example 3.2. *If X is a random variable such that $X \sim \text{Pareto}(1, 1)$, then the modified distribution will be $\text{Pareto}(1, \alpha)$.*

Proof. Let $X \sim \text{Pareto}(1, 1)$, then the pdf, cdf and survival functions are

$$f(x) = \frac{1}{(1+x)^2}, x > 0$$

$$F(x) = \frac{x}{1+x}, \text{ and}$$

$$R(x) = \frac{1}{1+x}, \text{ respectively.}$$

$$F(x) + \alpha R(x) = \frac{x}{1+x} + \frac{\alpha}{1+x} = \frac{x+\alpha}{1+x}.$$

The modified cdf will be $F_\alpha(x) = \frac{x}{\alpha+x}$ and the corresponding pdf will be $f_\alpha(x) = \frac{\alpha}{(\alpha+x)^2}$ which is the pdf for the Pareto distribution $\text{Pareto}(1, \alpha)$.

For the above example and for $n = 2$, the MLE of α is $\hat{\alpha} = \sqrt{X_1 X_2}$ (the geometric mean). \square

Example 3.3. If X is a random variable such that $X \sim \text{Pareto}(\kappa, \theta)$ with pdf and cdf

$$f(x) = \frac{\kappa}{\theta \left(1 + \frac{x}{\theta}\right)^{\kappa+1}}, x > 0 \text{ and } F(x) = \frac{\left(1 + \frac{x}{\theta}\right)^\kappa - 1}{\left(1 + \frac{x}{\theta}\right)^\kappa}, \text{ respectively.}$$

Then the modified pdf and cdf are

$$f_\alpha(x) = \frac{\alpha \kappa \left(1 + \frac{x}{\theta}\right)^{\kappa-1}}{\theta \left[\left(1 + \frac{x}{\theta}\right)^\kappa + \alpha - 1\right]^2}, x > 0 \text{ and } F_\alpha(x) = \frac{\left(1 + \frac{x}{\theta}\right)^\kappa - 1}{\left(1 + \frac{x}{\theta}\right)^\kappa + \alpha - 1}, \text{ respectively.}$$

One can notice that the modified cdf is a generalization but similar to the original one.

Now, we will apply the modified distribution on a Uniform distribution to find some moments and to study Shanon's entropy for the modified one.

Example 3.4. Let $X \sim \text{Uniform}(0, 1)$, then the modified cdf, pdf, survival function and hazard function are

$$F_\alpha(x) = \frac{x}{x(1-\alpha) + \alpha}, f_\alpha(x) = \frac{\alpha}{(x(1-\alpha) + \alpha)^2}, R_\alpha(x) = \frac{\alpha(1-x)}{x(1-\alpha) + \alpha},$$

$$h_\alpha(x) = \frac{1}{(x(1-\alpha) + \alpha)(1-x)}, \text{ respectively.}$$

One can show that:

$$E_{f_\alpha}(X) = \frac{\alpha(\alpha - 1 - \ln(\alpha))}{(1-\alpha)^2},$$

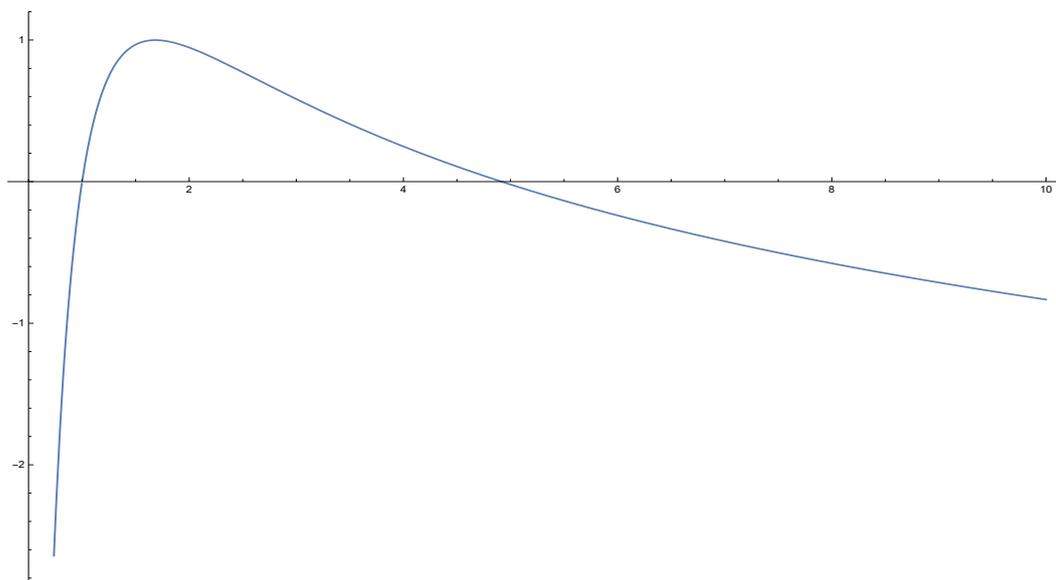
$$E_{f_\alpha}(X^2) = \frac{\alpha(1 - \alpha^2 + 2\alpha \ln(\alpha))}{(1-\alpha)^3}, \text{ and hence,}$$

$$\text{Var}_{f_\alpha}(x) = \frac{\alpha(1 - 3\alpha^2 + 2\alpha^3 + \ln(\alpha))(4\alpha - 4\alpha^2 + \alpha \ln(\alpha))}{(1-\alpha)^4}.$$

The Shanon entropy is

$$S_{f_\alpha}(X) = -E_{f_\alpha}(\ln(f_\alpha(x))) = \frac{4(1 - \alpha^3 + 3 \ln(\alpha))}{3\alpha^2(1-\alpha)} - \ln(\alpha),$$

The following is the graph of the Shanon entropy as a function of α .



The above function is undefined when $\alpha = 1$ and is decreasing for $\alpha > 1$, that is , the distribution becomes better as α increases.

4. Conclusion

From the above discussion one can see that MO can give flexibility with dealing with distributions by controlling the reliability and hazard functions, also we showed that the original pdf and the corresponding MO pdf meet at only one point, an addition to that MO distribution will affect the location of the percentile point and hence the skewness. Finally, there are some distributions that will belong to the same family of distribution under the MO distribution as the original one like the Pareto distribution.

Acknowledgments

The author would like to thank the referees for careful reading of this manuscript and for the thoughtful comments, which help to improve the quality of this manuscript.

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