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On the Existence Solutions for some Nonlinear Elliptic Problem

Abdelmoujib Benkirane, Badr El Haji and Mostafa El Moumni

ABSTRACT: In the present paper, we study the existence and regularity of positive solutions for the following boundary value problem : $-\text{div}\left(|\nabla u|^{p-2}\nabla u\right)+u^s=\dfrac{f}{u^\alpha}$ in Ω and $u=0$ on $\partial\Omega$, where Ω is an open and bounded subset of $\mathbb{R}^N$ $(N>p>1)$, $0<\alpha\leq 1$, $s\geq 1$ and f is a nonnegative function that belongs to some Lebesgue space.

Key Words: Semilinear elliptic problem, Nonlinear singular term, Existence, Regularity effets, Sobolev space.

Contents

1. Introduction and Main Result

In this paper, we are concerned with the existence and regularity results for the positive solution to the following problem :

$$\begin{cases} -\text{div}\left(|\nabla u|^{p-2}\nabla u\right)+u^s=\dfrac{f(x)}{u^\alpha} & \text{in} \quad \Omega, \\ u\geq 0 & \text{in} \quad \Omega, \\ u=0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open and bounded subset of $\mathbb{R}^N$ $(N>p)$, $0<\alpha\leq 1$, $s\geq 1$ and f be in $L^1(\Omega)$ function.

Problem (1.1) has been applied in chemical heterogeneous catalysts, non-Newtonian fluids and also the theory of heat conduction in electrically conducting materials, see [19,2,8,17] for detailed discussion. In this work, we are dealing with absorption zero order terms, that usually has a regularizing effect on the solutions to (1.1), by starting from measure data for regularity results on the Lebesgue scale (in [6,10]) and on the Marcinkiewicz one ([4]). We refer the reader to [20,21,12] for another approach using results on elliptic and parabolic problems in the setting of Sobolev spaces. See also [1,3,15,16] for related topics.

In [6] the authors studied the regularizing effect of the term u^s on the solution to the following classical problem

$$\begin{cases} -\Delta u=f(x) & \text{in} \quad \Omega, \\ u=0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

when the term u^s is added in the left-hand side of (1.2), we obtain the following problem

$$\begin{cases} -\Delta u+u^s=f(x) & \text{in} \quad \Omega, \\ u=0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.3}$$

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In [7] the authors study the following problem

$$\begin{cases} -\Delta u = \frac{f(x)}{u^\gamma} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.4}$$

Recently in [14] the authors study the regularity of the solution the following problem

$$\begin{cases} -\Delta u + u^s = \frac{f(x)}{u^\gamma} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.5}$$

Definition 1.1. *A function $u \in W_0^{1,p}(\Omega)$ is a distributional solution to problem (1.1) in case $\alpha \le 1$, $s \ge 1$ and $f \in L^r(\Omega)$ with $r \ge 1$ if*

$$\forall w \subset\subset \Omega \text{ exists } c_w > 0 \text{ s.t. } u \ge c_w \text{ a.e. in } w, \tag{1.6}$$

$$u^s \in L^1(\Omega),$$

and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi + \int_\Omega u^s\varphi = \int_\Omega \frac{f}{u^\alpha}\varphi \quad \forall\varphi \in \mathcal{C}_c^1(\Omega). \tag{1.7}$$

Our purpose is to establish the following result .

Theorem 1.2. *Let $\alpha \le 1$, $s \ge 1$ and $0 \le f \in L^r(\Omega)$ with $r \ge 1$. Then the problem (1.1) has at least one distributional solution u in the sense of the Definition 1.1 . Moreover u belongs to $W_0^{1,p}(\Omega) \cap L^{s+1}(\Omega)$ if*

(i) $\alpha = 1$, $f \in L^1(\Omega)$ or

(ii) $\alpha < 1$, $f \in L^r(\Omega)$ for some $r > 1$ and $s \ge \frac{1-r\alpha}{r-1}$ or

(iii) $\alpha < 1$, $f \in L^{\frac{s+1}{s+\alpha}}(\Omega)$

while if

(iv) $\alpha < 1$, $f \in L^1(\Omega)$, then $u \in W_0^{1,\frac{p(s+\alpha)}{s+1}}(\Omega) \cap L^{s+\alpha}(\Omega)$.

The paper is organized as follows: Section 2 is devoted to describing the approximated problems and we prove some properties that we need in the proof of our main results. Finally, Section 3, we shall give the complete proof of Theorem 1.2.

2. Preliminary results

For a fixed $k > 0$, we define the truncation functions $T_k : \mathbb{R} \to \mathbb{R}$ and $G_k : \mathbb{R} \to \mathbb{R}$ as follows

$$T_k(s) := \max(-k; \min(s; k)) \text{ and } G_k(s) := (|s| - k)^+ \text{sign}(s).$$

We will also use the following functions

$$S_{\delta,k}(s) = 1 - V_{\delta,k}(s) \tag{2.1}$$

with

$$V_{\delta,k}(s) = \begin{cases} 1 & \text{if} \quad s \le k, \\ \frac{k+\delta-s}{\delta} & \text{if } k < s < k+\delta, \\ 0 & \text{if } s \ge k+\delta, \end{cases}$$

we will denote with $\mathbb{R}^*$ the set $\mathbb{R}\backslash\{0\}$, with $\mathbb{R}^+$ the set $\{t \in \mathbb{R} \text{ s.t. } t > 0\}$, with r^* the Sobolev conjugate of $1 \le r < N$, given by $\frac{Nr}{N-r}$, and with $r' = \frac{r}{r-1}$ the Hölder conjugate of $1 < r < \infty$ (if $r = 1$ we define $r' = \infty$, if $r = \infty$ we define $r' = 1$). Moreover, if no otherwise Specified, we will denote by c several positive constants whose value may change from line to line and, sometimes, on the same line. These values will only depend on the data (for instance c can depend on Ω, α, s, p) but they will never depend on the indexes of the sequences we will introduce.

2.1. Approximating problems

Let us consider the following approximating problems,

$$\begin{cases} -\mathrm{div}\left(|\nabla u_{n,k}|^{p-2}\nabla u_{n,k}\right)+T_k(|u_{n,k}|^{s-1}u_{n,k})=\dfrac{f_n(x)}{(|u_{n,k}|+\frac{1}{n})^{\alpha}} & \text{in} \quad \Omega, \\ u_{n,k}=0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.2}$$

where $n,k\in\mathbb{N}$, $0\le f_n(x)=T_n(f(x))\in L^\infty(\Omega)$, $\alpha\le 1$ and $s\ge 1$.
There exists $u_{n,k}$ weak solution to (2.2), for each $n,k\in\mathbb{N}$ fixed (see [[18], Theorem 2]). Moreover $u_{n,k}\in L^\infty(\Omega)$ for all $n,k\in\mathbb{N}$, since if $m\ge 1$ is fixed, taking $G_m(u_{n,k})\in W_0^{1,p}(\Omega)$ as test function in (2.2) and using that $G_m(u_{n,k})$ and $T_k(|u_{n,k}|^{s-1}u_{n,k})$ have the same sign of $u_{n,k}$, we have taht

$$\int_\Omega |\nabla G_m(u_{n,k})|^p\le\int_\Omega f_n G_m(u_{n,k}),$$

and so we can proceed as in [22] to end up with $u_{n,k}\in L^\infty(\Omega)$.
Moreover the previous L^∞ estimate is independent from $k\in\mathbb{N}$.
Now by chossing $u_{n,k}$ as a test function in the weak formulation of (2.2), we obtain

$$u_{n,k} \text{ is bounded in } W_0^{1,p}(\Omega) \text{ with respect to } k \text{ for } n\in\mathbb{N} \text{ fixed }.$$

Since $u_{n,k}$ is bounded in $L^\infty(\Omega)$ independently on k, for each $n\in\mathbb{N}$ fixed we choose k_n large enough in order to get the following scheme of approximation

$$\begin{cases} -\mathrm{div}\left(|\nabla u_n|^{p-2}\nabla u_n\right)+|u_n|^{s-1}u_n=\dfrac{f_n(x)}{(|u_n|+\frac{1}{n})^{\alpha}} & \text{in} \quad \Omega, \\ u_n=0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.3}$$

where $u_n\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ is given by u_{n,k_n}.

As concerns the sign of u_n, by chosing $u_n^-:=\min(u_n,0)\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ as test function in (2.3), we obtain

$$\int_\Omega|\nabla u_n^-|^p+\int_\Omega|u_n|^{s-1}(u_n^-)^2=\int_\Omega\frac{f_n}{(|u_n|+\frac{1}{n})^\alpha}u_n^-\le 0,$$

and so that $u_n\ge 0$ almost everywhere in Ω.

Now we prove some local positivity property that will guarantee that the limit of the approximations (2.3) satisfies (1.6).

Proposition 2.1. *For each $n\in\mathbb{N}$ fixed, the nonnegative $u_n\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ weak solution to (2.3) is nondecreasing in $n\in\mathbb{N}$ and it results*

$$\forall w\subset\subset\Omega, \exists c_w>0 \text{ (independent of } n\in\mathbb{N}) \text{ s.t. } u_n\ge c_w \text{ in } w\ \forall n\in\mathbb{N} \tag{2.4}$$

Proof:

We can prove that the sequence u_n is nondecreasing in $n\in\mathbb{N}$ proceeding precisely as in [[7], Lemma 2.2], namely taking $(u_n-u_{n+1})^+:=\max(u_n-u_{n+1},0)\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ as test function in the difference between the problem solved by u_n and the one solved by u_{n+1}, so we will omit the details. To prove (2.4), we will instead use that

$$u_n\ge u_1 \quad \forall n\in\mathbb{N} \text{ a.e. in } \Omega, \tag{2.5}$$

and we will apply the strong maximum principle to $u_1\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, that solves

$$\begin{cases} -\mathrm{div}\left(|\nabla u_1|^{p-2}\nabla u_1\right)+u_1^s=\dfrac{f_1(x)}{(u_1+1)^\alpha}\ge\dfrac{f_1(x)}{(\|u_1\|_{L^\infty(\Omega)}+1)^\alpha}\ge 0 & \text{in} \quad \Omega, \\ u_1=0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.6}$$

Indeed, since $u_1, \text{div}\left(|\nabla u_1|^{p-2}\nabla u_1\right) \in L^1_{loc}(\Omega)$, $u_1 \geq 0$ almost everywhere in Ω,

$$\text{div}\left(|\nabla u_1|^{p-2}\nabla u_1\right) \leq u_1^s$$

and

$$\int_0^{\cdot} (t^{s+1})^{-\frac{1}{2}} = \infty \Longleftrightarrow s \geq 1,$$

we can apply [[23],Theorem 1] and deduce that

$$\forall w \subset\subset \Omega, \exists c_w > 0 \text{ s.t. } u_1 \geq c_w \text{ in } w.$$

Then (2.4) follows from (2.5). □

2.2. A priori estimates

Now we need some compactness results on the sequence of approximating solutions u_n, at least up to subsequences.

Proposition 2.2. *Let $n \in \mathbb{N}$ and $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a solution to (2.3) where $s \geq 1$.*

a) If one of the following holds

$$\begin{cases} \alpha = 1,\ f \in L^1(\Omega), \\ \alpha < 1,\ f \in L^r(\Omega) \text{ for some } r > 1 \text{ and } s \geq \frac{1-r\alpha}{r-1}, \\ \alpha < 1,\ f \in L^{\frac{s+1}{s+\alpha}}(\Omega), \end{cases}$$

then u_n is bounded in $W_0^{1,p}(\Omega) \cap L^{s+1}(\Omega)$.

b) If $\alpha < 1$ and $f \in L^1(\Omega)$ then u_n is bounded in $W_0^{1,\frac{p(s+\alpha)}{s+1}}(\Omega) \cap L^{s+\alpha}(\Omega)$.

Proof:

a) The first case. Let us take $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ as test function in (2.3). We obtain

$$\int_\Omega |\nabla u_n|^p + \int_\Omega u_n^{s+1} \leq \int_\Omega f_n u_n^{1-\alpha}. \tag{2.7}$$

- If $\alpha = 1$, we immediately find that u_n is bounded in $W_0^{1,p}(\Omega)$ and in $L^{s+1}(\Omega)$.
- If $\alpha < 1$, we apply Young's inequality with weights $(\epsilon, c(\epsilon))$ and exponents (r, r') on the right hand side of the previous, obtaining

$$\int_\Omega |\nabla u_n|^p + \int_\Omega u_n^{s+1} \leq \frac{1}{c(\epsilon)} \int_\Omega f_n^r + \epsilon \int_\Omega u_n^{(1-\alpha)r'} \leq \frac{1}{c(\epsilon)} \int_\Omega f^r + \epsilon c \int_\Omega u_n^{s+1}.$$

  If ϵ is small enough, we deduce the following estimate

$$\int_\Omega |\nabla u_n|^p + c(\Omega, \epsilon) \int_\Omega u_n^{s+1} dx \leq \frac{1}{c(\epsilon)} \int_\Omega f^r \leq c.$$

- If $\alpha < 1$ and $f \in L^{\frac{s+1}{s+\alpha}}(\Omega)$, we apply Young's inequality with weights $(\epsilon, c(\epsilon))$ and exponents

$$\left(\frac{s+1}{s+\alpha}, \frac{s+1}{1-\alpha}\right)$$

  on the right hand side of (2.7). Proceeding as before, we can easily prove the last assertion.

b) Here. by choosing $(u_n+\epsilon)^\alpha - \epsilon^\alpha \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ as test function in (2.3), where $0<\epsilon<\frac{1}{n}$. one has

$$\alpha\int_\Omega |\nabla u_n|^p (u_n+\epsilon)^{\alpha-1} + \int_\Omega u_n^s((u_n+\epsilon)^\alpha - \epsilon^\alpha) \leq \int_\Omega f_n.$$

Then, we can deduce that

$$\int_\Omega u_n^s((u_n+\epsilon)^\alpha - \epsilon^\alpha) \leq \int_\Omega f,$$

and by letting $\epsilon \to 0$, we have

$$\int_\Omega u_n^{s+\alpha} \leq \int_\Omega f.$$

It follows that

$$\int_\Omega \frac{|\nabla u_n|^p}{(u_n+\epsilon)^{1-\alpha}} \leq c.$$

Now, if $q<p$, by thanking to Hölder's inequality with exponents $\frac{p}{q}$ and $\frac{p}{p-q}$, we obtain

$$\int_\Omega |\nabla u_n|^q = \int_\Omega \frac{|\nabla u_n|^q}{(u_n+\epsilon)^{(1-\alpha)\frac{q}{p}}}(u_n+\epsilon)^{(1-\alpha)\frac{q}{p}} \leq c\Big(\int_\Omega (u_n+\epsilon)^{\frac{(1-\alpha)q}{p-q}}\Big)^{1-\frac{q}{p}}$$

Now we choose q such that

$$\frac{(1-\alpha)q}{p-q} = s+\alpha.$$

It is not difficult to verify that

$$q = \frac{p(s+\alpha)}{(s+1)} \leq p,$$

which achieve the proof of this proposition.

□

3. Proof of the main results

3.1. Proof of Theorem 1.2

Proof: Let u_n be a solution to (2.2), then it follows from 2.2 that it is bounded in $W_0^{1,p}(\Omega)$ with respect to n. Hence there exists a function $u_p \in W_0^{1,p}(\Omega)$ such that u_n, up to subsequences, converges to u_p in $L^r(\Omega)$ for all $r<\frac{pN}{N-p}$ and weakly in $W_0^{1,p}(\Omega)$. proposition 2.2 also gives that $\dfrac{f_n(x)}{(|u_n|+\frac{1}{n})^\alpha}$ is bounded in $L^1_{loc}(\Omega)$ and clearly, $|u_n|^{s-1}u_n$ is bounded in $L^1(\Omega)$ with respect to n. Hence one can apply Theorem 2,1 of [5] which gives that ∇u_n converges to ∇u_p almost everywhere in Ω.

Now we prove that u_p satisfies 1.7 by passing to the limit in n every term in the weak formulation of (2.2) easily pass to the limit the first term in (2.2) with respect to n; hence we focus on the absorption term u^s, which we show to be equi-integrable.Indeed if we test (2.2) with $S_{\eta,k}(u_n)$(defined in (2.1) where $\eta, k>0$ and we deduce

$$\int_\Omega |\nabla u|^p S'_{\eta,k}(u_n) + \int_\Omega u_n^s S_{\eta,k}(u_n) \leq sup_{s\in[k,\infty)}\frac{1}{s^\alpha}\int_\Omega f_n S_{\eta,k}(u_n).$$

Which, observing that the first term on the left hand side is nonnegative and taking the limit with respect to $\eta \to 0$,implies

$$\int_{\{u_n \geq k\}} u_n^s S_{\eta,k}(u_n) \leq sup_{s\in[k,\infty)} \frac{1}{s^\alpha} \int_{\{u_n \geq k\}} f_n. \tag{3.1}$$

Which, since f_n converges to f in $L^r(\Omega), r \geq 1$ easily implies that u_n^s is equi-integrable and so it converges to u_p^s in $L^1(\Omega)$.This is sufficient to pass to the limit in the second term of the weak formulation of (2.2). For what concerns the right hand side, using (2.4), we find

$$\Big|\frac{f_n\varphi}{(u_n + \frac{1}{n})^\alpha}\Big| \leq \Big|\frac{f\varphi}{c^\alpha_{supp\varphi}}\Big| \quad \forall \varphi \in \mathcal{C}^1_c(\Omega).$$

Then, thanks to Lebesgue Theorem, we can pass to the limit also in the right hand side of the distributional formulation of (2.3). This concludes the proof. □

3.2. Some comments on the regularizing effect

Firstly, it is easy to verify that, if

$$\alpha < 1 \text{ and } s > \frac{N+p}{N-p} \tag{3.2}$$

then

$$\tfrac{s+1}{s+\alpha} < \left(\frac{p^*}{1-\alpha}\right)'$$

Since $f \in L^{\left(\frac{p^*}{1-\alpha}\right)'}(\Omega)$ is the weaker assumption on the datum in order to find a priori estimates in $W_0^{1,p}(\Omega)$ for the sequence of approximating solutions to problem below:

$$\begin{cases} -\text{div}\left(|\nabla u|^{p-2}\nabla u\right) = \dfrac{f(x)}{u^\alpha} & \text{in} \quad \Omega, \\ u \geq 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.3}$$

it follows that, if we add the term u^s, with s satisfying (3.2), in the left hand side of (3.3), we find a priori estimates in $W_0^{1,p}(\Omega)$ for the sequence of approximating solutions also for less regular data.

Furthermore, if $f \in L^1(\Omega)$ and $\alpha < 1$, the Sobolev space in which the sequence of approximating solutions to (3.3) is bounded is given by $W_0^{1,\frac{N(\alpha+1)}{N-(1-\alpha)}}(\Omega)$ (see [9,11,13]).

It is easy to verify that, if

$$\alpha < 1 \text{ and } s > \frac{N+\alpha p}{N-p}, \tag{3.4}$$

then

$$\tfrac{N(\alpha+1)}{N-(1-\alpha)} < \tfrac{p(s+\alpha)}{s+1}.$$

So we have another regularizing effect of the lower order term u^s, with s such that (3.4) holds, on the a priori estimates for the approximating solutions.

Finally we recall that, if $f \in L^1(\Omega)$ and $s > \frac{N}{N-p}$, then the sequence of approximating solutions to the following problem:

$$\begin{cases} -\text{div}\left(|\nabla u|^{p-2}\nabla u\right) + u^s = f(x) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.5}$$

is bounded in $W_0^{1,q}(\Omega)$ for all $q \in [1, \frac{ps}{(s+1)})$ (see [18]). Since

$$\frac{ps}{(s+1)} < \frac{p(s+\alpha)}{(s+1)} \Longleftrightarrow \alpha > 0,$$

we immediately obtain the, if we perturb the right hand side of (3.5) through the singular term $\frac{1}{u^\alpha}$ with $\alpha > 0$, we find a priori estimates on the sequence of approximating solutions.

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References

1. Akdim.Y, Benkirane.A and El Moumni.M, *Solutions of nonlinear elliptic problems withlower order terms,* Ann. Funct. Anal. **6**, no. 1, 34-53 (2015).
2. Ambrosetti.A, Brézis.H and Cerami.G, *Combined effects of concave and convex nonlinearities in some elliptic problems.* J. Funct. Anal. **122**(2), 519-543 (1994).
3. Benkirane.A, El Haji.B and El Moumni.M, *On the existence of solution for degenerate parabolic equations with singular terms.* Pure and Applied Mathematics Quarterly Volume 14, Number 3-4, (2018), 591–606
4. Boccardo.L, *Marcinkiewicz estimates for solutions of some elliptic problems with nonregular data*, Ann Mat. Pura Appl. (4) 188, no. 4, pp. 591–601 (2009).
5. Boccardo.L and F. Murat.F, *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations, Nonlinear Analysis* 19 (1992) 581-597.
6. Boccardo.L, Gallouët.T, Vázquez.J.L, *Nonlinear elliptic equations in* $\mathbb{R}^N$ *without growth restrictions on the data*, J. Differential Equations **105**, no. 2, pp. 334–363 (1993).
7. Boccardo.L and Orsina.L, *Semilinear elliptic equations with singular nonlinearities,* Calc. Var. Partial Differential Equations, **37**, pp. 363–380 (2009).
8. Callegari.A and Nashman.A, *A nonlinear singular boundary-value problem in the theory of pseudoplastic fluids.* SIAM J. Appl. Math. **38**(2), 275-281 (1980).
9. Canino.A, Sciunzi.B and Trombetta.A, *Existence and uniqueness for p-Laplace equations involving singular nonlinearities*, NoDEA Nonlinear Differential Equations Appl. **23**, pp. 8–18 (2016).
10. Cirmi.R, *Regularity of the solutions to nonlinear elliptic equations with a lower-order term*, Nonlinear Anal. T.M.A. **25**, pp. 569–580 (1995).
11. Chu.Y, Gao.R and Sun.y, *Existence and regularity of solutions to a quasilinear elliptic problem involving variable sources,* Boundary Value Problems 2017:155. DOI 10.1186/s13661-017-0888-4.
12. Dall'Aglio. A, Orsina.L and Petitta.F, *Existence of solutions for degenerate parabolic equations with singular terms.* Nonlinear Anal. **131**, 273–288 (2016).
13. De Cave.L, *Nonlinear elliptic equations with singular nonlinearities*, Asymptot. Anal. **84** (3-4), pp. 181–195(2013).
14. De Cave.L and F. Oliva.F, *On the regularizing effet of some absorption and singular lower order terms in classical Dirichlet problems with* L^1 *data*, JEPE. Vol **2**, p. 73-85 (2016).
15. El Haji.B, El Moumni.M and Kouhaila.K, *On a nonlinear elliptic problems having large monotonocity with* L^1 *data in weighted Orlicz-Sobolev spaces.* Moroccan J. of Pure and Appl. Anal. (MJPAA) Vol 5(1), (2019), 104-116.
16. El Moumni.M, *Nonlinear Elliptic Equations Without Sign Condition and* L^1*-Data in Musielak-Orlicz-Sobolev Spaces*, Acta Applicandae Mathematicae DOI: 10.1007/s10440-018-0186-x.
17. Keller .H.B and Cohen.D.S, *Some positive problems suggested by nonlinear heat generators.* J. Math. Mech. **16**(12), 1361-1376 (1967).
18. Leray.J and Lions.J-L, *Quelques résultats de Visik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder*, Bull. Soc. Math. France, **93**, pp. 97–107 (1965).
19. Maso.G. D, Murat.F, Orsina.L and Prignet.A , *Renormalized solutions of elliptic equations with general measure data.* Ann. Sc. Norm. Super. Pisa, Cl. Sci. **28**(4), 741-808 (1999).
20. Oliva.F and Petitta.F, *Finite and infinite energy solutions of singular elliptic problems: existence and uniqueness.* J. Differential Equations **264**, no. 1, 311–340 (2018).
21. Oliva.F and Petitta.F, *On singular elliptic equations with measure sources.* ESAIM Control Optim. Calc. Var. **22**, no. 1, 289–308 (2016).

22. Stampacchia.G, *Equations elliptiques du second ordre a coéficients discontinus,* Les Presses de l'Université de Montréal 1966.

23. Vázquez.J. L, *A Strong Maximum Principle for Some Quasilinear Elliptic Equations,* Appl. Math. Optim., **12**, pp. 191–202 (1984).

Abdelmoujib Benkirane,
Laboratory LAMA, Department of Mathematics,
Faculty of Sciences Fez, University Sidi Mohamed Ben Abdellah,
P. O. Box 1796 Atlas Fez, Morocco.
E-mail address: abd.benkirane@gmail.com

and

Badr El Haji,
Laboratory LAMA, Department of Mathematics,
Faculty of Sciences Fez, University Sidi Mohamed Ben Abdellah,
P. O. Box 1796 Atlas Fez, Morocco.
E-mail address: badr.elhaji@gmail.com

and

Mostafa El Moumni,
Department of Mathematics,
Faculty of Sciences El Jadida, University Chouaib Doukkali,
P. O. Box 20, 24000 El Jadida, Morocco.
E-mail address: mostafaelmoumni@gmail.com