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Independence and Inverse Domination in Complete z-Ary Tree and Jahangir Graphs

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ABSTRACT: This article includes different properties of the independence and domination (total domination, independent domination, co-independent domination) number of the complete z-ray root and Jahangir graphs. Also, the inverse domination number of these graphs of variant dominating sets (total dominating, independent dominating, co-independent dominating) is determined.

Key Words: Independent set, dominating set, total dominating set, connected dominating set, co-independent dominating set, complete z-ray root graph and Jahangir graph.

Contents

1. Introduction

For a vertex $v \in V(G)$, *the open neighborhood* $N(v)$ is the set of all vertices adjacent to v, and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. A subgraph H of a graph G is said to be induced (or full) if, for any pair of vertices x and y of H, xy is an edge of H if and only if xy is an edge of G. If H is an induced of G with S is a set of its vertices then H is said to be induced by S and denoted by $G[S]$. An independent set or stable set is a set of vertices in a graph G, where no two of which are adjacent. An independence number denoted by $\beta(G)$ of a graph G is the cardinality of a maximum independent set of G. There are many parameters of the domination number as shown below and these parameters have contributed to solving many problems in the graph as in the topological graph [9], fuzzy graph [13,14] and [17,18,19], soft graph [3], and labeled graph [1,2], etc. A set $D \subseteq V(G)$ is a dominating set in G if every vertex v; $v \in V(G) - D$ adjacent with at least one vertex in D. The domination number of G, denoted $\gamma(G)$, is the cardinality of a minimum dominating set of G. A dominating set $D \subseteq V(G)$ is an independent dominating set in G if D is an independent set in G.The independent domination number of G, which denoted by $\gamma_i(G)$, is the cardinality of a minimum independent dominating set of G.
A dominating set $D \subseteq V(G)$ is a *total dominating set* in G if for every vertex v; $v \in V(G)$, adjacent with at least one vertex in D. That is mean $G[D]$ has no isolated vertex. the cardinality of a minimum total dominating set in G is the *total domination number* of G and is denoted by $\gamma_t(G)$. A dominating set $D \subseteq V(G)$ is a connected dominating set in G if $G[D]$ is connected set. The connected domination number of G , denoted $\gamma_c(G)$, is the cardinality of a minimum connected dominating set of G. A dominating set $D \subseteq V(G)$ is a co-independent dominating set in G if the complement of D is an independent set. The co-independent domination number of G, denoted $\gamma_{coi}(G)$, is the cardinality of a minimum co-independent dominating set of G. Various types of domination of graph G have been defined and studied by several authors and more than 75 models of domination are listed in the appendix of Haynes [6]. All definition above about parameters of domination number and for more details, we refer to [8] ,[10,11,12], [15,16]. Mojdeh and Ghameshlou [7] study some results on the number of domination, total domination, independent domination, and connected domination in Jahangir graphs $J_{2,m}$. Here, we study independence number and various types of domination (domination, total domination, independence domination, co-independence domination) number of a complete z-ray root, $z \geq 2$ and Jahangir graph $J_{n,m}, n \geq 3$. Let $D \subseteq V(G)$ be a minimum cardinal of dominating (independent dominating, total dominating, connected

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dominating, co-independent dominating) set in graph G. If $V-D$ contains a dominating (independent dominating, total dominating, connected dominating, co-independent dominating) set, then this set is called an inverse set of D in G and denoted by ID. The symbol $\gamma^{-1}(G)$, $\gamma_i^{-1}(G)$, $\gamma_t^{-1}(G)$, $\gamma_c^{-1}(G)$ and $\gamma_{coi}^{-1}(G)$ is refer to the minimum cardinality over all inverse dominating (independent dominating, total dominating, connected dominating, co-independent dominating) set of G.

2. Complete z-ray trees

A tree T is a connected graph with no cycles. In a tree, a vertex of degree one is referred to as a pendant (leaf) and a vertex which is adjacent to a pendant is a support vertex. A tree is called a rooted tree if one vertex has been designated the root. In a rooted tree, the parent of a vertex is the vertex connected to it on the path to the root; every vertex except the root has a unique parent. A child of a vertex v is a vertex of which v is the parent. In a rooted tree, the depth r is the longest length of a path from the root to a vertex v. An internal vertex in a rooted tree is any vertex that has at least one child. A z-ray tree ,$z \geq 2$ is a rooted tree in which every vertex has z or fewer children. A complete z-ray tree ($T_{c,z,r}$) is a z-ray tree in which every internal vertex has exactly z children and all pendant vertices have the same depth. We label the root vertex by v_0, as shown in Figure 1; $T_{c,2,5}$.

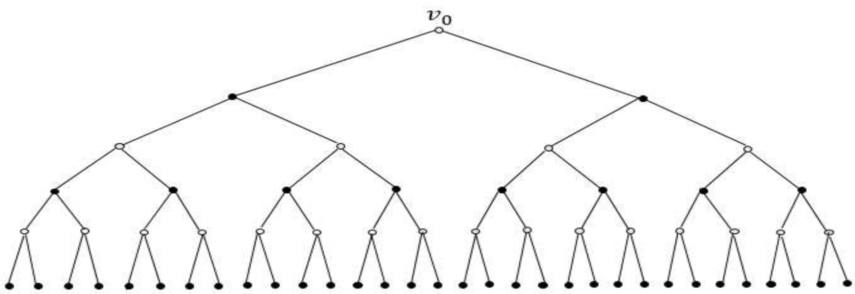

Figure 1: $T_{c,2,5}$

Remark 2.1. *(I) Every pendant vertex in a tree of n vertices is a member in the maximal independent set in G when $n \geq 3$.*
(II) If G is connected graph, then $\beta(G) = n-1$ if and only if G is a star graph of n vertices. For a complete z-ray tree $G \equiv T_{c,z,r}$ with n vertices, have the following properties for independence number and variant domination numbers:

Theorem 2.2.

$$\beta(T_{c,z,r}) = \frac{z^{r+2}(1 - z^{-2(\lfloor \frac{r}{2} \rfloor + 1)})}{z^2 - 1}. \tag{2.1}$$

Proof. Consider $I = \cup_{i=0}^{\lfloor \frac{r}{2} \rfloor} I_i$, where I_0 be the set of all pendant vertices of G,
$I_i = \{v : v$ is a pendant vertex of $G[V - \cup_{j=0}^{i-1}(I_j \cup S_j)]$; $i = 1, 2, .., \lfloor \frac{r}{2} \rfloor\}$ and
$S_k = \{v : v$ is a support vertex of $G[V - \cup_{j=0}^{k-1}(I_j \cup S_j)]; k = 1, 2, .., \lfloor \frac{r-1}{2} \rfloor\}$, where S_0 be the set of all support vertices of G. It is clear that, I_0 is an independent set and it is a member in the maximal independent set of G by Observations 2.1(I). If we add any vertex from S_0 to the set I_0, the result set is not an independent (as an instant, see Figure 1). Thus the set I_0 is the maximum independent set in the induced subgraph $G[I_0 \cup S_0]$ and $|I_0| = z^r$. Now I_i is the set of all pendant vertices of the complete z-ray tree $(G[V - \cup_{j=0}^{i-1}(I_j \cup S_j)]; i = 1, 2, .., \lceil \frac{r}{2} \rceil)$, of depth $(r - 2i)$. It is clear that the set I_i is an independent set for all i and $|I_i| = z^{r-2i}$. To keep the independency, we cannot include any vertex of S_i to I_i. Thus I_i represent the maximum independent set in $G[I_i \cup S_i]$, $i = 0, 1, 2, .., k$, where

$k = \left\{ \begin{array}{ll} \lfloor \frac{r}{2} \rfloor & ; r \textit{ is odd} \\ \lfloor \frac{r}{2} \rfloor - 1 & ; r \textit{ is even} \end{array} \right\}$ and $I_{\lfloor \frac{r}{2} \rfloor} = v_0$; r is even.
Thus $I = \cup_{i=0}^{\lfloor \frac{r}{2} \rfloor} I_i$ is the independent set in G , and $\beta(G) \geqslant |I|$. If we assume that there is a set F such that $|F| > |I|$, then F must contains adjacent vertices, since it is contains a support vertices.

Thus $\beta(G) = |I| = \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} z^{r-2i} = \frac{z^{r+2}(1-z^{-2(\lfloor \frac{r}{2} \rfloor+1)})}{z^2-1}$. □

Theorem 2.3.

$$\gamma(T_{c,z,r}) = \gamma_i(T_{c,z,r}) = \frac{z^{r+2}(1-z^{-3(\lfloor \frac{r-1}{3} \rfloor+1)})}{z^3-1} + \lfloor \frac{r}{3} \rfloor - \lceil \frac{r}{3} \rceil + 1. \tag{2.2}$$

Proof. Consider $D = \cup_{i=0}^{\lfloor \frac{r-1}{3} \rfloor} D_i$, where D_i is the set of all vertices of depth $(r-1-3i)$ in G, and $E_i = \{v : v$is a vertex of depth $r-3i, r-1-3i$ and $r-2-3i$ in G;$i = 0, 1, \ldots, \lfloor \frac{r-1}{3} \rfloor$. We see that D_i is a dominating set in the induced subgraph $G[E_i]$ and $|D_i| = z^{r-1-3i}$. For any set F with $|F| < |D_i|$, we have that, F cannot be dominate some of vertices in E_i. Thus D_i is the minimum dominating set in the induced subgraph $G[E_i]$. Now, we have the following cases that depend on r:
(a) If $r \equiv 0(mod 3)$, then the root vertex is only vertex in G which is not dominated by the set D, so $D \cup \{v_0\}$ is the minimum dominating set in G. Therefore, we have
$\gamma(G) = 1 + \sum_{i=0}^{\lfloor \frac{r-1}{3} \rfloor} z^{r-1-3i} = 1 + \frac{z^{r+2}(1-z^{-3(\lfloor \frac{r-1}{3} \rfloor+1)})}{z^3-1}$.
(b) If $r \equiv 1, 2(mod 3)$, then the set D is the minimum dominating set in G. Therefore we have
$\gamma(G) = \sum_{i=0}^{\lfloor \frac{r-1}{3} \rfloor} z^{r-1-3i} = \frac{z^{r+2}(1-z^{-3(\lfloor \frac{r-1}{3} \rfloor+1)})}{z^3-1}$. We note that if $r \equiv 1(mod 3)$, then$E_{\lfloor \frac{r-1}{3} \rfloor} = \{v_0\} \cup \{v : v$ is a vertex of depth one $\}$.
We combine the formulas in (a) and (b) as one formula for any r, we get:

$\gamma(G) = \frac{z^{r+2}(1-z^{-3(\lfloor \frac{r-1}{3} \rfloor+1)})}{z^3-1} + \lfloor \frac{r}{3} \rfloor - \lceil \frac{r}{3} \rceil + 1.$

We see that in the two cases (a) and (b) the minimum dominating set in G is an independent set, so $\gamma(G) = \gamma_i(G)$. □

Theorem 2.4.

$$\gamma^{-1}(T_{c,z,r}) = \gamma_i^{-1}(T_{c,z,r}) = \left\{ \begin{array}{ll} z + \frac{z^{r+3}(1-z^{-3(\lfloor \frac{r-1}{3} \rfloor+1)})}{z^3-1}, & \textit{if } r \equiv 0(mod\ 3)(a) \\ \lceil \frac{r-1}{3} \rceil - \lfloor \frac{r-1}{3} \rfloor + \frac{z^{r+3}(1-z^{-3(\lfloor \frac{r-1}{3} \rfloor+1)})}{z^3-1}, & \textit{if } r \equiv 1, 2(mod\ 3)(b) \end{array} \right\}. \tag{2.3}$$

Proof. Consider the set $D^{-1} = \cup_{i=0}^{\lfloor \frac{r-1}{3} \rfloor} D_i$, where
$D_i = \{v : v$ is a vertex of depth$(r-3i)$ in $G; i = 0, 1, 2, \ldots, \lfloor \frac{r-1}{3} \rfloor\}$.
Also consider $H_i = \{v : v$ is a vertex of depth $r+1-3i$,$r-3i$ and $r-1-3i$ in G ;$i = 1, 2, \ldots, \lfloor \frac{r-1}{3} \rfloor$ $\}$, where $H_0 = \{v : v$ is a vertex of depth r and $r-1$ in $G\}$. It is clear that D_0 is the minimum dominating set in the induced subgraph $G[H_0]$ and $|D_0| = z^r$, since all vertices of depth r-1 contain in the minimum dominating set (D in Theorem 2.3) in G. The set D_1 is the minimum dominating set in the induced subgraph $G[H_1]$ and $|D_1| = z^{r-3}$. If we assume that a set $F \subseteq H_1$ and $|F| < |D_1|$, then F cannot dominate at least two vertices. Continue with the same manner for the others D_i, we obtain the following three cases:
(a) If $r \equiv 0(mod 3)$, we cannot take v_0 since it is belong to the set D, so we take the vertices of depth one to dominate all vertices of depth one plus v_0 in D^{-1}.
Thus $\gamma^{-1}(G) = z + \sum_{i=0}^{\lfloor \frac{r-1}{3} \rfloor} z^{r-3i} = z + \frac{z^{r+3}(1-z^{-3(\lfloor \frac{r-1}{3} \rfloor+1)})}{z^3-1}$.
(b) If $r \equiv 1(mod 3)$, so D^{-1} the minimum dominating set in $T_{c,z,r}$. Thus $\gamma^{-1}(G) = \sum_{i=0}^{\lfloor \frac{r-1}{3} \rfloor} z^{r-3i}$
(c) If $r \equiv 2(mod 3)$, the vertices not dominate by the set $D^{-1}$is the only vertices of depth one plus v_0, so we can take v_0 to dominate all these vertices . Thus $\gamma^{-1}(T_{c,z,r}) = 1 + \sum_{i=0}^{\lfloor \frac{r-1}{3} \rfloor} z^{r-3i}$.

We combine the formulas in (b) and (c) as one formula for any r we get, $\gamma^{-1}(G) = \lceil \frac{r-1}{3} \rceil - \lfloor \frac{r-1}{3} \rfloor + \sum_{i=0}^{\lfloor \frac{r-1}{3} \rfloor} z^{r-3i} = \lceil \frac{r-1}{3} \rceil - \lfloor \frac{r-1}{3} \rfloor + \frac{z^{r+3}(1-z^{-3(\lfloor \frac{r-1}{3} \rfloor+1)})}{z^3-1}$ We see that in the three cases (a), (b) and (c) the minimum inverse dominating set in G is an independent set, so $\gamma^{-1}(G) = \gamma_i^{-1}(G)$. □

Theorem 2.5.

$$\gamma_t(T_{c,z,r}) = \left\{ \begin{array}{ll} \frac{(z^{r+3}+z^{r+2})\left(1-z^{-4\left(\lfloor \frac{r-1}{4} \rfloor+1\right)}\right)}{z^4-1} + \lfloor \frac{r-1}{4} \rfloor + \lceil \frac{r-1}{4} \rceil + 1 & ,if\ r \not\equiv 1(mod\ 4)\ (a) \\ \frac{(z^{r+3}+z^{r+2})\left(1-z^{-4\left(\lfloor \frac{r-1}{4} \rfloor+1\right)}\right)}{z^4-1} + 2 & ,if\ r \equiv 1(mod\ 4)\ (b) \end{array} \right\}. \quad (2.4)$$

Proof. Consider $D^t = \cup_{i=0}^{\lfloor \frac{r-1}{4} \rfloor} \{A_i \cup B_i\}$, where
$A_i = \{v : v$ is a vertex of depth $r-1-4i$ in $G\}$ and
$B_i = \{v : v$ is a vertex of depth $r-2-4i$ in $G\}$, $i = 0, 1, \ldots, \lfloor \frac{r-1}{4} \rfloor$. We see that A_0 is a dominating set in the induced subgraph generated by the vertices of depth $r, r-1$and $r-2$, but it is not total dominating set, since each vertex of A_0 is isolated vertex in the set $G[A_0]$. Therefore we must choose the vertices of B_0 which are adjacent to the vertices of A_0. Let's consider that $E_i = \{v : v$ is a vertex of depth $r-4i, r-1-4i, r-2-4i$ and $r-3-4i$ in $G\}$, then $A_0 \cup B_0$ is the minimum total dominating set in $G[E_0]$, where $|A_0 \cup B_0| = z^{r-1} + z^{r-2}$. Also $A_1 \cup B_1$ is the minimum dominating set in $G[E_1]$ where $|A_1 \cup B_1| = z^{r-5} + z^{r-6}$, so $A_0 \cup A_1 \cup B_0 \cup B_1$ is the minimum total dominating set in $E_0 \cup E_1$ where $|A_0 \cup B_0 \cup A_1 \cup B_1| = z^{r-1} + z^{r-2} + z^{r-5} + z^{r-6}$. Continue with this procedure, we exit the following two cases that depend on r.
(I) If $r \not\equiv 1(mod\ 4)$, then there are two states as follows. (a) If $r \equiv 0(mod\ 4)$, then v_0 is the only vertex which is not totally dominated by the set D^t. So we must choose only one vertex from the vertices of depth one, and include it in the set D^t to conserve the total dominating set in G. Thus $\gamma_t(G) = \sum_{i=0}^{\lfloor \frac{r-1}{4} \rfloor} (z^{r-1-4i} + z^{r-2-4i}) + 1$ ..
(b) If $r \equiv 2, 3(mod 4)$, then the set D^t is a minimum total dominating set in G.
Thus $\gamma_t(G) = \sum_{i=0}^{\lfloor \frac{r-1}{4} \rfloor} (z^{r-1-4i} + z^{r-2-4i})$.
Now if we combine the formulas in (a) and (b) as one formula for any r, we obtain

$$\begin{aligned} \gamma_t(G) &= \sum_{i=0}^{\lfloor \frac{r-1}{4} \rfloor} (z^{r-1-4i} + z^{r-2-4i}) + \left\lfloor \frac{r-1}{4} \right\rfloor - \left\lceil \frac{r-1}{4} \right\rceil + 1 \\ &= \frac{(z^{r+3} + z^{r+2})\left(1 - z^{-4\left(\lfloor \frac{r-1}{4} \rfloor+1\right)}\right)}{z^4 - 1} + \left\lfloor \frac{r}{4} \right\rfloor - \left\lceil \frac{r}{4} \right\rceil + 1. \end{aligned}$$

We note that where $r \equiv 2(mod\ 4), E_{\lfloor \frac{r-1}{4} \rfloor} = \{v : v$ is a vertex of depths 2,1 and 0.

(II) If $r \equiv 1(mod\ 4)$, consider $C^t = \cup_{i=0}^{\lfloor \frac{r-1}{4} \rfloor - 1} \{A_i \cup B_i\}$, then as the set D^t, C^t is the minimum total dominating set in $G[C^t]$. The vertices which are not dominated by the set C^t are v_0 andvertices of depth one. So we include v_0 and one vertex of depth one in the set C^t to get a minimum total dominating set in G. Thus $\gamma_t(G) = \sum_{i=0}^{\lfloor \frac{r-5}{4} \rfloor} (z^{r-1-4i} + z^{r-2-4i}) + 2 = \frac{(z^{r+3}+z^{r+2})\left(1-z^{-4\left(\lfloor \frac{r-5}{4} \rfloor+1\right)}\right)}{z^4-1} + 2$. □

Remark 2.6. *The inverse total dominating set in G is not exist, since all pendant vertices are isolated in $G[V - D^t]$ where D^t is a minimum total dominating set in G.*

Theorem 2.7.

$$\gamma_{coi}(T_{c,z,r}) = \frac{z^{r+1}\left(1 - z^{-2\left(\lfloor \frac{r-1}{2} \rfloor+1\right)}\right)}{z^2 - 1}. \quad (2.5)$$

Proof. Consider $D^{coi} = \cup_{i=0}^{\lfloor \frac{r-1}{2} \rfloor} D_i$, where D_i is the set of all vertices with depth $r-1-2i$ in $T_{c,z,r}$, and $E_i = \{v : v$ is a vertex of depth $r-2i, r-1-2i$ and $r-2-2i$ in $G\}, i = 0, 1, \ldots, \lfloor \frac{r-1}{2} \rfloor$. It is clear that D_0 is the minimum dominating set in $G[E_0]$ and $E_0 - D_0$ is an independent set in $G[E_0]$. Also D_1 is the minimum dominating set in $G[E_1]$ and $E_1 - D_1$ is an independent set in $G[E_1]$ and so on. . . .

Thus D^{coi} is the co- independent dominating set in G with $|D^{coi}| = \sum_{i=0}^{\lfloor \frac{r-1}{2} \rfloor} z^{r-1-2i}$. Let's consider that there is a set F of vertices such that $|F| < |D^{coi}|$, F is not co-independent dominating set in G, since $V - F$ is not an independent set (it contains at least two adjacent vertices).

Thus $\gamma_{coi}(G) = \sum_{i=0}^{\lfloor \frac{r-1}{2} \rfloor} z^{r-1-2i} = \frac{z^{r+1}\left(1-z^{-2\left(\left\lfloor \frac{r-1}{2} \right\rfloor+1\right)}\right)}{z^2-1}$. □

Theorem 2.8.

$$\gamma_{coi}^{-1}(T_{c,z,r}) = \frac{z^{r+2}\left(1-z^{-2\left(\left\lfloor \frac{r}{2} \right\rfloor+1\right)}\right)}{z^2-1}. \tag{2.6}$$

Proof. Consider $(D^{coi})^{-1} = \cup_{i=0}^{\lfloor \frac{r-1}{2} \rfloor} D_i$, where D_i is the set of all vertices with depth $r-2i$ in $T_{c,z,r}$, and $E_i = \{v : v$ is a vertex of depth $r+1-2i, r-2i$ and $r-1-2i$ in $T_{c,z,r}\}, i = 0, 1, \ldots, \lfloor \frac{r}{2} \rfloor$. $E_0 = \{v : v$ is a vertex of depth r and $r-1$ in $T_{c,z,r}\}$. It is clear that D_0 is the minimum co-independent set in $G[E_0]$. As same the manner in the previous theorem $(D^{coi})^{-1}$ is the minimum dominating set in G where, $(D^{coi})^{-1} \subseteq V - D^{coi}$. Thus $\gamma_{coi}^{-1}(T_{c,z,r}) = \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} z^{r-2i} = \frac{z^{r+2}\left(1-z^{-2\left(\left\lfloor \frac{r}{2} \right\rfloor+1\right)}\right)}{z^2-1}$.
We note that if $r \equiv 0 (mod\ 2)$, then $E_{\lfloor \frac{r-1}{4} \rfloor} = \{v : v$ is a vertex of depths 1 or 0 $\}$. □

Theorem 2.9. [6] *For any tree T of n vertices with p pendant vertices,*

$$\gamma_c(T) = n - p; n \geq 2. \tag{2.7}$$

Remark 2.10. *The inverse connected dominating set in $T_{c,z,r}$ is not exist, since $V - D^c$ represent all pendant vertices and these vertices are isolated in $G[V - D^c]$ where D^c is a minimum connected dominating set in G.*

3. Jahangir graph

For n and $m; m \geq 3$ and $n \geq 2$ the Jahangir graph $J_{n,m}$, is a graph on $nm+1$ vertices consisting of a cycle C_{nm} with one additional central vertex which is adjacent to certain m vertices of C_{nm} where these vertices at distance n in order (sequence) on C_{nm}. Consider v_0 be the center vertex of $J_{(n,m)}$ and v_1 be one vertex in C_{nm} which is adjacent to v_0, and $v_1, v_2, v_3, \ldots, v_{mn}$ are the other vertices that incident clockwise in C_{nm}. In this section, we take v_1 is the first vertex adjacent to the center v_0 (see Figure 2) for $J_{4,4}$.

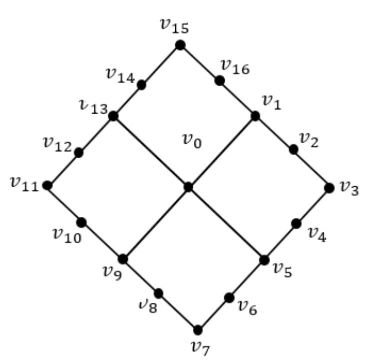

Figure 2: $J_{4,4}$

Mojdeh and Ghameshlou [5] study some results on domination number, total domination number, independence domination number and connected domination number in Jahangir graph $J_{2,m}$. They proved that $\beta(J_{2,m}) = \beta_c(J_{2,m}) = \beta_t(J_{2,m}) = \lfloor \frac{n}{2} \rfloor + 1$ and $\beta_t(J_{2,m}) = \lfloor \frac{2m}{3} \rfloor$.

For Jahangir $J_{n,m}$ with $n \geq 3$, we have the following properties for independence number and variant domination number:

Theorem 3.1.

$$\beta(J_{n,m}) = \left\lfloor \frac{mn}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{n}{2} \right\rceil + 1. \tag{3.1}$$

Proof. Consider $D = \{v_2, v_4, \ldots, v_k\}$ where $k = \left\{ \begin{array}{cc} mn & ; if\ mn\ is\ even \\ mn-1 & ; if\ mn\ is\ odd \end{array} \right\}$,

then we have the following two cases:
(i) If n is even , $D \cup \{v_0\}$ is an independent set in $J_{n,m}$, where $|D \cup \{v_0\}| = \lfloor \frac{mn}{2} \rfloor + 1$, since D does not contain any adjacent vertex to v_0, then $\beta(J_{n,m}) \geq \lfloor \frac{mn}{2} \rfloor + 1$. If F is a vertex set such that $|F| > \lfloor \frac{mn}{2} \rfloor + 1$, this mean that B contains at least two adjacent vertices. Thus $D \cup v_0$ is the maximum independent set in $J_{n,m}$, and $\beta(J_{n,m}) = \lfloor \frac{mn}{2} \rfloor + 1$.
(ii) If n is odd, D is an independent set in $J_{n,m}$ with $|D| = \lfloor \frac{mn}{2} \rfloor$, then $\beta(J_{n,m}) \geq \lfloor \frac{mn}{2} \rfloor$. If there is a set F of vertices; $|F| > \lfloor \frac{mn}{2} \rfloor$, then F must contains at least two adjacent vertices. Thus $D \cup \{v_0\}$ is a maximum independent set in $J_{n,m}$, and $\beta(J_{n,m}) = \lfloor \frac{mn}{2} \rfloor$.
We combine the formulas in (i) and (ii) as one formula, then $\beta(J_{n,m}) = \lfloor \frac{mn}{2} \rfloor + \lfloor \frac{n}{2} \rfloor - \lceil \frac{n}{2} \rceil + 1$. □

Theorem 3.2.

$$\gamma(J_{n,m}) = \gamma_i(J_{n,m}) = \left\lceil \frac{mn}{3} \right\rceil. \tag{3.2}$$

Proof. Let $D = \{v_{3i+1}; i = 0, 1, \ldots, \lceil \frac{mn}{3} \rceil - 1\}, D$ is a dominating set in $J_{n,m}$, with $|D| = \lceil \frac{mn}{3} \rceil$, then $\gamma(J_{n,m}) \leq \lceil \frac{mn}{3} \rceil$. If we assume there is a set F of vertices with $|F| = \lceil \frac{mn}{3} \rceil - 1$, then the maximum number of vertices which are dominated by the set F at most $3(\lceil \frac{mn}{3} \rceil - 1) + 1 = 3\lceil \frac{mn}{3} \rceil - 2$, but $3\lceil \frac{mn}{3} \rceil - 2 < mn+1$. Therefore the set F cannot be a dominating set in $J_{n,m}$. Thus D is minimum dominating set in $J_{n,m}$ and $\gamma(J_{n,m}) = \lceil \frac{mn}{3} \rceil$. But D is independent set in $J_{n,m}$, then we have $\gamma(J_{n,m}) = \gamma_i(J_{n,m})$. □

Theorem 3.3.

$$\gamma^{-1}(J_{n,m}) = \gamma_i^{-1}(J_{n,m}) = \left\{ \begin{array}{cc} \lceil \frac{mn}{3} \rceil + 1 & , if n \equiv 0 (mod\ 3)(a) \\ \lceil \frac{mn}{3} \rceil & , if n \equiv 1, 2 (mod\ 3)(b) \end{array} \right\}. \tag{3.3}$$

Proof. Let $D^{-1} = \{v_{3i+2}; i = 0, 1, \ldots, \lceil \frac{mn}{3} \rceil - 1\}$, there are three cases that depend on n as follows.
(i) If $n \equiv 0 (mod\ 3)$, then the set D^{-1} is the minimum dominating set in C_{nm}. The set D^{-1} not dominate v_0, so we include v_0 to the set D^{-1} It is clear that $\gamma^{-1}(J_{n,m}) = |D^{-1}| + 1 = \lceil \frac{mn}{3} \rceil + 1$.
(ii) If $n \equiv 1 (mod 3)$, then the set D^{-1} is dominating set in $J_{n,m}$, and $\gamma(J_{n,m}) \leq \lceil \frac{mn}{3} \rceil$. Since v_{n+1} is adjacent to v_0. It is clear that $\gamma^{-1}(J_{n,m}) = \lceil \frac{mn}{3} \rceil$ and D^{-1} is independent set. Thus we have $\gamma^{-1}(J_{n,m}) = \gamma_i^{-1}(J_{n,m})$.
(iii) If $n \equiv 2 (mod\ 3)$, then D^{-1} is a dominating set in $J_{n,m}$ and $|D^{-1}| = \lceil \frac{mn}{3} \rceil$. Therefore $\gamma(J_{n,m}) \leq \lceil \frac{mn}{3} \rceil$, since v_{1+2n} is adjacent to v_0. With the same manner as (ii), we have $\gamma^{-1}(J_{n,m}) = \lceil \frac{mn}{3} \rceil$. It is clear that D^{-1} is independent set in $J_{n,m}$ and then $\gamma^{-1}(J_{n,m}) = \gamma_i^{-1}(J_{n,m})$. □

Theorem 3.4. [5] *If P_n be a path of order n , then*

$$\gamma_t(P_n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor, n > 2. \tag{3.4}$$

Theorem 3.5.

$$\gamma_c(J_{n,m}) = m(n-2) + 1. \tag{3.5}$$

Proof. Consider $A = \{v_k; k = \lfloor \frac{n+1}{2} \rfloor + ni, \lfloor \frac{n+1}{2} \rfloor + ni + 1; i = 0, 1, \ldots, m-1\}$, and $D = V - A$.
It is clear that D is dominating set in $J_{n,m}$ and $J_{n,m}[D]$ is connected graph, since all vertices of A located in middle vertices between any successive two vertices that joined with the center vertex. Therefore D is connected dominating set in $J_{n,m}$ and $|D| = m(n-2)+1$, then $\gamma_c(J_{n,m}) \leq m(n-2)+1$. Now if there is a set B of vertices; $|B| < m(n-2)+1$, then we have two cases as follows.
(i) If we remove one vertex from the adjacent vertices of the set A, then there are three vertices v_j, v_{j+1},and v_{j+2} not belonging to D. Therefore we cannot dominate the vertex v_{j+1} by vertices of D. Therefore this set does not dominating set in $J_{n,m}$.
(ii) If we remove one vertex from vertices which are not adjacent to the vertices of the set A, then $J_{n,m}[B]$ becomes disconnected.
From i and ii, we get D is the minimum connected dominating set in $J_{n,m}$. Thus $\gamma_c(J_{n,m}) = m(n-2)+1$. □

Theorem 3.6.

$$\gamma_t(J_{n,m}) = \left\{ \begin{array}{ll} 2\lceil \frac{mn}{4} \rceil + \lceil \frac{mn-1}{4} \rceil - \lfloor \frac{mn-1}{4} \rfloor - 1 & , if\ n \not\equiv 3(mod\ 4)(a) \\ \frac{1}{2}m(n-1)+1 & , if\ n \equiv 3(mod\ 4)(b) \end{array} \right\}. \quad (3.6)$$

Proof. Let $D = \{v_{4i+1}, v_{4i+2} : i = 0, 1, \ldots, \lfloor \frac{mn}{4} \rfloor - 1\}$, there are two cases that depend on mn as follows.
(I) If $n \not\equiv 3(mod\ 4)$, we have four cases as follows.
(i) If $mn \equiv 0(mod\ 4)$, then D is the total dominating set in $J_{n,m}$, where $|D| = \frac{mn}{2}$, so $\gamma_t(J_{n,m}) \leq \frac{mn}{2}$. If F is a set of vertices; $|F| < |D|$, then there are at least two vertices of $J_{n,m}$ are not dominated by F. Therefore D is minimum total domination set. Thus $\gamma_t(J_{n,m}) = \frac{mn}{2}$.
(ii) If $mn \equiv 1(mod\ 4)$, with the same manner as (i), D is minimum total domination set in $J_{n,m}$ except one vertex v_{mn}, where $|D| = \frac{mn-1}{2}$. To obtain the minimum of the total dominating set we cannot add the vertex v_{mn} to the set D , since v_{mn} is an isolated vertex in $G[D \cup \{v_{mn}\}]$. Since the vertex v_{mn-1} is adjacent to the vertices v_{mn} and v_{mn-2}, where $v_{mn-2} \in A$. Therefore $D \cup \{v_{mn-1}\}$ is the minimum total dominating set in $J_{n,m}$, then $\gamma_t(J_{n,m}) = 2\lceil \frac{mn}{4} \rceil - 1$.
(iii) If $mn \equiv 2(mod\ 4)$, with the same manner in part (i), D is minimum total dominating set in $J_{n,m}$ except the two vertices v_{mn-1} and v_{mn} , where $|D| = \frac{mn-2}{2}$. To obtain the minimum total dominating set, we cannot add one vertex v_i; $i = mn-1, mn$ to the set D, since it is become an isolated vertex in $G[D \cup \{v_i\}]$. Adding these vertices to the set D,then we have $\gamma_t(J_{n,m}) = \frac{mn-2}{2} + 2 = 2\lceil \frac{mn}{4} \rceil$.
(iv) If $mn \equiv 3(mod\ 4)$, again as part (i), D is a minimum total dominating set in $J_{n,m}$ except the three vertices v_{mn-2}, v_{mn-1} and v_{mn} with $|D| = \frac{mn-3}{2}$, so we include any two adjacent vertices from these vertices to the set D to obtain the total dominating set, as in (ii) we cannot take one vertex. Then $\gamma_t(J_{n,m}) = \frac{mn-2}{2} + 2 = 2\lceil \frac{mn}{4} \rceil$. We combine the formulas in (i), (ii), (iii) and (iv) as one formula for any mn, we get: $\gamma_t(J_{n,m}) = 2\lceil \frac{mn}{4} \rceil + \lceil \frac{mn-1}{4} \rceil - \lfloor \frac{mn-1}{4} \rfloor - 1$.
(II) If $n \equiv 3(mod\ 4)$, consider $A = \{v_0, v_1, v_{n+1}, v_{2n+1}, \ldots, v_{(m-1)n+1}\}$ and $S = A \cup \{v : v$ is adjacent vertex to the vertices of $A\}$. The set A is the minimum total dominating set in the induced subgraph $H = J_{n,m}[S]$ (as in Figure 3;$n = 7, m = 4$, and H is the thick edges). The set A contains $m+1$ vertices. The graph $J_{n,m} - H$ is union of m disjoint path of order $n-3$. Using Theorem 2.1, we get $\gamma_t(J_{n,m}) = \frac{1}{2}m(n-1)+1$. □

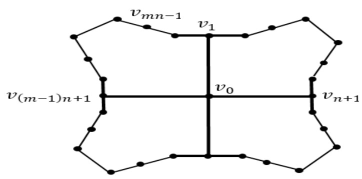

Figure 3: $J_{7,3}$

Remark 3.7. *Since there is not any vertex $v \in V - D^c$ (or $V - D^t$) which is adjacent to the vertex v_0, where D^c (D^t) is a minimum connected (total) dominating set in G, then the inverse connected (total) dominating set in $J_{n,m}$ is not exist.*

Theorem 3.8.

$$\gamma_{coi}\ (J_{n,m}) = \left\lfloor \frac{mn}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor . \tag{3.7}$$

Proof. Consider $D = \left\{v_{2i+1}; i = 0, 1, \ldots, \left\lfloor \frac{mn}{2} \right\rfloor - 1\right\}$, then there are two cases as follows.
(i) *If n is odd*, then D is the minimum dominating set and the set of vertices $V - D$ is not independent set in $J_{n,m}$, since v_0 is adjacent to some vertices in $V - D$. For this reason we add v_0 to D. Therefore $D \cup \{v_0\}$ is a dominating set in $J_{n,m}$, and $V - (D \cup \{v_0\})$ is independent set in $J_{n,m}$, then $\gamma_{coi}\ (J_{n,m}) \leq |D \cup \{v_0\}| = \left\lfloor \frac{mn}{2} \right\rfloor + 1$. If there is a set F of vertices; $|F| < \left\lfloor \frac{mn}{2} \right\rfloor + 1$, then F is not co- independent dominating set, since $G[V - F]$ contains at least two adjacent vertices. Thus $D \cup \{v_0\}$ is minimum co-independent dominating set in $J_{n,m}$, and we have $\gamma_{coi}\ (J_{n,m}) = \left\lfloor \frac{mn}{2} \right\rfloor + 1$.
(ii) *If n is even*, then D is minimum dominating set and the set of vertices $V - D$ is independent set in $J_{n,m}$, since there is no vertex in D adjacent to v_0, then $\gamma_{coi}\ (J_{n,m}) \leq \left\lfloor \frac{mn}{2} \right\rfloor$. Again if F is a set of vertices; $|F| < |D|$, then F is not co- independent dominating set, since $G[V - F]$ contains at least two adjacent vertices. Thus D is minimum co-independence dominating set, and $\gamma_{coi}\ (J_{n,m}) = \frac{mn}{2}$.
We combine the formulas in (i) and (ii) as one formula for any n, we get:
$\gamma_{coi}\ (J_{n,m}) = \left\lfloor \frac{mn}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor$ □

Theorem 3.9. *There is no inverse co-independent dominating set in $J_{n,m}$.*

Proof. Consider $\left(D^{coi}\right)^{-1} = V - D^{coi}$, where D^{coi} is a minimum co-independent dominating set in $J_{n,m}$, there are two cases that depend on n as follows.
(i) *If n is even* there is no any vertex in $\left(D^{coi}\right)^{-1}$ dominate the vertex v_0. Thus there is no any dominating set in $J_{n,m}$ such that the vertices of $\left(D^{coi}\right)^{-1}$contains in $V - D^{coi}$.
(ii) *If n is odd* then $\left(D^{coi}\right)^{-1}$ is not co-independence dominating set since $V - D^{coi}$ not independent (there are some vertices adjacent to v_0) and we cannot include v_0 to the set $\left(D^{coi}\right)^{-1}$ since $v_0 \in D^{coi}$. Thus there is no any dominating set in $J_{n,m}$ such that the vertices of $\left(D^{coi}\right)^{-1}$ contains in $V - D^{coi}$. □

Conclusion 3.1. *For a complete z-ray tree $G = T_{c,z,r}$ or Jahangir graphs $G = J_{n,m}$, we have*

$$\gamma(G) \leq \gamma_i(G) \leq \gamma_{coi}(G) \leq \gamma_t(G) \leq \gamma_c(G).$$

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