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Global Existence and Blow-up of Solutions for a Class of Steklov Parabolic Problems

A. Lamaizi, A. Zerouali, O. Chakrone and B. Karim

abstract: In this paper, we study weak solutions to the following Steklov parabolic problem:

$$\begin{cases} u_t - \Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega,\ t > 0, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^q u & \text{on } \partial\Omega,\ t > 0, \\ u(x;0) = u_0(x) & \text{in } \Omega. \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain for $n \geq 2$ with smooth boundary $\partial\Omega$, $\lambda > 0$. Here, u_t denote the partial derivative with respect to the time variable t and ∇u denotes the one with respect to the space variable x. We prove theorems of existence of weak solutions, via Galerkin approximation. Moreover, we show the existence of solutions which blow up in a finite time.

Key Words: Steklov parabolic problem, p-Laplacian, global existence, blow-up.

Contents

1. Introduction and main results

The motivation of this paper contains several aspects. The first one is that in parabolic problems are fundamental to the modeling of space and time-dependent problems such as problems from physics or biology. To be specific , evolutionary equations and systems are likely to be used to model physical processes like heat conduction or diffusion processes. Let's take as an illustration the Navier- strokes equation, the basic equation in fluid mechanics. What's more, we would like to refer to [14], where fluids in motion are studied. Applications involve climate modeling and climatology as well.(see [9,10]).

The second interesting aspect of this article is that the p-Laplacian equation has a strong basis in mathematical physics, and it is very important in many mathematical models. Generally is a mathematical model for various fields such as biological sciences, population dynamics, and heat transfer theory, such as thermoelastic distortion, diffusion phenomena, heat transfer in two media, heat transfer in a solid in contact with a moving fluid, (see, [1,15,13]).

The third distinctive feature in the present context is that nonlinear boundary condition have arisen in many physical fields, such as elasticity, fluid mechanics, electromagnetism (see [4]), and have attracted attention many researchers. These problems are also important in inverse problems and conformal geometry [2,5]. Moreover, they also have many applications, one finds them for example in the study of the waves of surface [6] the study of the modes of vibration of a structure in contact with an incompressible fluid (see [7]), in the study of surface waves, the analysis of the stability of mechanical oscillators immersed in a viscous fluid, and in the analysis of the stability of mechanical oscillators immersed in a viscous fluid (see [12], and the references it contains).

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In [13], we have recently established the global existence and blow-up of weak solutions to the following parabolic problem with nonlinear boundary conditions:

$$\begin{cases} u_t - \Delta u + u = 0 & \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial \nu} = \lambda |u|^{p-1} u & \text{on } \partial\Omega \times (0,T), \\ u(x;0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where Ω is an open bounded domain of $\mathbb{R}^n$, $n \geq 1$ with smooth boundary $\partial\Omega$, $\lambda > 0$, and p satisfies

$$(P) \ \begin{cases} 1 \leq p \leq \frac{n}{n-2} \text{ if } n > 2, \\ 1 \leq p < \infty \text{ if } n = 1; 2. \end{cases}$$

More precisely, we have proved the following results
Theorem 1 : *Let p satisfy* (P), $u_0(x) \in H^1(\Omega)$. *Assume that* $A(u_0) < k, B(u_0) > 0$. *Then, there exists a global weak solution* $u(t) \in L^\infty\left(0, \infty; H^1(\Omega)\right) \cap C\left([0,T]; L^2(\Omega) \times L^2(\partial\Omega, \rho)\right)$ *of* (1.1) *with* $u_t(t) \in L^2\left(0, \infty; L^2(\Omega)\right)$ *and* $u(t) \in X$ *for* $0 \leq t < \infty$.
Theorem 2 : *Let p satisfy (P), $u_0(x) \in H^1(\Omega)$. Then the weak solution $u(x,t)$ of problem (1.1) must blows up in finite time provided that:*

$$0 < A(u_0) < \frac{p-1}{2C(p+1)} \|u_0\|^2,$$

where

$$C = \underset{u \in H^1(\Omega)}{Sup} \frac{\|u\|^2}{\|u\|^2_{H^1}},$$

$$A(u) = \frac{1}{2}\|u\|^2_{H^1} - \frac{\lambda}{p+1}\|u\|^{p+1}_{p+1,\partial\Omega},$$

and

$$B(u) = \|u\|^2_{H^1} - \lambda \|u\|^{p+1}_{p+1,\partial\Omega}.$$

In this paper, we will present a generalization of the previous problem. More precisely, we discuss the existence and blow-up of solutions for the following Steklov parabolic problem involving the p-Laplacian

$$\begin{cases} u_t - \Delta_p u + |u|^{p-2}u = 0 & \text{in } \ \Omega, \ t > 0, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda |u|^q u & \text{on } \ \partial\Omega, \ t > 0, \\ u(x;0) = u_0(x) & \text{in } \ \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain for $n \geq 2$ with smooth boundary $\partial\Omega$, and $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the well known p-Laplacian operator defined in $W^{1,p}(\Omega)$ in a weak setting in the usual way for any real number $p > 1$. Here, u_t denote the partial derivative with respect to the time variable t and ∇u denotes the one with respect to the space variable x. Moreover, $\lambda > 0$, p and q satisfy

$$(H) \qquad \frac{2n}{n+2} \leq p < +\infty \ , \ p < 2+q \quad \text{and} \quad \begin{cases} 1 \leq q+2 \leq p^\partial \text{ if } p \neq n, \\ 1 \leq q+2 < \infty \text{ if } p = n. \end{cases}$$

Recall that

$$p^\partial := \begin{cases} \frac{p(n-1)}{n-p} \text{ if } 1 < p < n, \\ \infty \ \text{ if } p \geq n. \end{cases}$$

Let us introduce some functionals and sets as follows

$$E(u) = \frac{1}{p}\|u\|^p_{1,p} - \frac{\lambda}{2+q}\|u\|^{2+q}_{2+q,\partial\Omega},$$

$$E_\delta(u)=\frac{\delta}{p}\|u\|_{1,p}^p-\frac{\lambda}{2+q}\|u\|_{2+q,\partial\Omega}^{2+q},\quad \forall\delta\in(0,1),$$

the depth of potential well

$$d=\inf_{\substack{u\in W^{1,p}(\Omega)\\ u\neq 0}}\sup_{\beta\geq 0}E(\beta u), \tag{1.2}$$

the depth function of potential wells

$$d(\delta)=\frac{1-\delta}{p}\left(\frac{2+q}{\lambda p C_*^{2+q}}\delta\right)^{\frac{p}{2+q-p}}, \tag{1.3}$$

where C_* is the embedding constant form $W^{1,p}(\Omega)$ into $L^{2+q}(\partial\Omega)$, i.e.,

$$C_*=\sup\frac{\|u\|_{2+q,\partial\Omega}}{\|u\|_{1,p}}, \tag{1.4}$$

$$X_\delta=\left\{u\in W^{1,p}(\Omega)\mid E_\delta(u)>0,E(u)<d(\delta)\right\}\cup\{0\},\quad \forall\ 0<\delta<1,$$

and

$$B_\delta=\left\{u\in W^{1,p}(\Omega)\mid \|u\|_{1,p}<\left(\frac{2+q}{\lambda p C_*^{2+q}}\delta\right)^{\frac{1}{2+q-p}}\right\}.$$

Our main results are stated as follows

Theorem 1.1. *Let $u_0(x)\in W^{1,p}(\Omega)$, p and q satisfy (H). Assume that $0<E(u_0)<d,\delta_1<\delta_2$ are the two roots of equation $d(\delta)=E(u_0)$ and $E_{\delta_2}(u_0)>0$. Then problem (1.1) admits a global solution $u(t)\in L^\infty\left(0,\infty;W^{1,p}(\Omega)\right)\cap C\left(0,\infty;L^p(\Omega)\times L^p(\partial\Omega,\rho)\right)$ with $u_t(t)\in L^2\left(0,\infty;L^2(\Omega)\right)$ and $u(t)\in X_\delta$ for $\delta\in(\delta_1,\delta_2),t\in[0,\infty)$.*

Theorem 1.2. *Let $u_0(x)\in W^{1,p}(\Omega)$, p and q satisfy (H). Assume that $0<e<d$ and $\delta_1<\delta_2$ are the two roots of equation $d(\delta)=e$. Then,*

1. *Solutions of problem (1.1) with initial condition $0<E(u_0)\leq e$ lie in $\overline{B}_{\delta_1}$, provided $u_0(x)\in B_{\delta_0}$.*
2. *Solutions of problem (1.1) with initial condition $0<E(u_0)\leq e$ lie in $B_{\delta_2}^c\cup\partial B_{\delta_2}$, provided $u_0(x)\in B_{\delta_0}^c$.*

Theorem 1.3. *Let $u_0(x)\in W^{1,p}(\Omega)$, p and q satisfy (H), and $\delta_1<\delta_2$ be the two roots of equation $d(\delta)=E(u_0)$.*

1. *Assume that $E(u_0)<d$ and $E_{\delta_1}(u_0)<0$. Then solutions of problem (1.1) blow up in finite time.*
2. *Assume that $E(u_0)=d$ and $E_{\delta_0}(u_0)<0$. Then the conclusion of (i) remains valid.*

2. Preliminaries

The Lebesgue norm of $L^p(\Omega)$ will be denoted by $\|\cdot\|_p$, and the Lebesgue norm of $L^p(\partial\Omega,\rho)$ by $\|\cdot\|_{p,\partial\Omega}$, for $p\in[1,\infty]$, where $d\rho$ denotes the restriction to $\partial\Omega$, and

$$\langle u,v\rangle=\int_\Omega uv\,dx,\quad \langle u,v\rangle_0=\oint_{\partial\Omega}uv\,d\rho.$$

Moreover, we denote the usual Sobolev space on Ω

$$W^{1,p}(\Omega)=\left\{u\in L^p(\Omega):|\nabla u|\in L^p(\Omega)\right\},$$

equipped by the norm

$$\|u\|_{1,p}=\|u\|_p+\|\nabla u\|_p,$$

or to the equivalent norm

$$\|u\|_{1,p} = \left(\|u\|_p^p + \|\nabla u\|_p^p\right)^{\frac{1}{p}},$$

if $1 \leq p < +\infty$.

It is worth noting that

$$W^{1,p}(\Omega) \hookrightarrow L^2(\Omega) \Longleftrightarrow p \geq p_0 := \frac{2n}{n+2}.$$

Proposition 2.1. *(See [4])*
The trace operator $W^{1,p}(\Omega) \to L^q(\partial\Omega, \rho)$ is continuous if and only if $1 \leq q \leq p^\partial$ if $p \neq n$ and for $1 \leq q < \infty$ if $p = n$. Especially for $q = 2$, the trace operator is well-defined and continuous under the following condition:

$$W^{1,p}(\Omega) \to L^2(\partial\Omega, \rho) \Longleftrightarrow p \geq p_1 := \frac{2n}{n+1}.$$

Let us introduce some functionals and sets as follows

$$X = \left\{u \in W^{1,p}(\Omega) \mid F(u) > 0, E(u) < d\right\} \cup \{0\},$$

where

$$F(u) = \|u\|_{1,p}^p - \lambda \|u\|_{2+q,\partial\Omega}^{2+q},$$

$$E_\delta(u) = \frac{\delta}{p}\|u\|_{1,p}^p - \frac{\lambda}{2+q}\|u\|_{2+q,\partial\Omega}^{2+q}, \quad \forall \delta \in (0,1),$$

$$\begin{aligned}
\overline{X}_\delta &= \left\{u \in W^{1,p}(\Omega) \mid E_\delta(u) \geq 0, E(u) \leq d(\delta)\right\},\\
Y_\delta &= \left\{u \in W^{1,p}(\Omega) \mid E_\delta(u) < 0, E(u) < d(\delta)\right\}, \quad \forall\, 0 < \delta < 1,\\
\overline{Y}_\delta &= Y_\delta \cup \partial Y_\delta = \left\{u \in W^{1,p}(\Omega) \mid E_\delta(u) \leq 0, E(u) \leq d(\delta)\right\},\\
Y &= \left\{u \in W^{1,p}(\Omega) \mid F(u) < 0, E(u) < d\right\},
\end{aligned}$$

$$\overline{B}_\delta = \left\{u \in W^{1,p}(\Omega) \mid \|u\|_{1,p} \leq \left(\frac{2+q}{\lambda p C_*^{2+q}}\delta\right)^{\frac{1}{2+q-p}}\right\},$$

$$B_\delta^c = \left\{u \in W^{1,p}(\Omega) \mid \|u\|_{1,p} > \left(\frac{2+q}{\lambda p C_*^{2+q}}\delta\right)^{\frac{1}{2+q-p}}\right\}.$$

We end this section with the definition of the weak solution of problem (1.1)

Definition 2.2. *A function*

$$u \in L^\infty\left(0,T; W^{1,p}(\Omega)\right) \cap C\left(0,T; L^p(\Omega) \times L^p(\partial\Omega, \rho)\right)$$

with $u_t \in L^2\left(0,T; L^2(\Omega)\right)$, is said to be a weak solution of problem (1.1) on $\Omega \times (0,T)$ if

1. $\langle u_t, v\rangle + \langle |u|^{p-2}u, v\rangle + \langle |\nabla u|^{p-2}\nabla u, \nabla v\rangle = \lambda \langle |u|^q u, v\rangle_0,$ *for all $v \in W^{1,p}(\Omega)$ and a.e. $t \in (0,T]$;*
2. $u(x,0) = u_0(x)$ *in $W^{1,p}(\Omega)$.*

3. Proof of Main Results

3.1. Proof of Theorem 1.1

First, we give some results which will be used to prove the main result.

Lemma 3.1. *As a function of δ, $d(\delta)$ satisfies the following properties on $[0,1]$.*

1. $d(0) = d(1) = 0$;

2. $d(\delta)$ *is increasing for* $0 \leq \delta \leq \delta_0$, *decreasing for* $\delta_0 \leq \delta \leq 1$, *and takes the maximum* $d(\delta_0)$ *at* $\delta_0 = \frac{p}{2+q}$;

3. *The equation* $d(\delta) = e$ *has two roots* $\delta_1 \in (0, \delta_0)$ *and* $\delta_2 \in (\delta_0, 1)$, *for any given* $e \in (0, d(\delta_0))$.

Proof. This lemma follows directly from

$$\begin{aligned} d'(\delta) &= \frac{-1}{p}\left(\frac{2+q}{\lambda p C_*^{2+q}}\delta\right)^{\frac{p}{2+q-p}} + \frac{(2+q)(1-\delta)}{(2+q-p)\lambda p C_*^{2+q}}\left(\frac{2+q}{\lambda p C_*^{2+q}}\delta\right)^{\frac{p}{2+q-p}-1} \\ &= \frac{1}{p}\left(\frac{2+q}{\lambda p C_*^{2+q}}\right)^{\frac{p}{2+q-p}} \delta^{\frac{p}{2+q-p}}\left(\frac{p}{2+q-p}\cdot\frac{1-\delta}{\delta}-1\right) \\ &= \frac{1}{2+q-p}\left(\frac{2+q}{\lambda p C_*^{2+q}}\right)^{\frac{p}{2+q-p}} \delta^{\frac{p}{2+q-p}}\left(\frac{1}{\delta}-\frac{2+q}{p}\right). \end{aligned}$$

□

Proposition 3.2. *If* $u \in W^{1,p}(\Omega)$, $\|u\|_{1,p} \neq 0$ *and* $E_\delta(u) = 0$, *then* $d(\delta) = \inf E(u)$. *Moreover,* $d = d(\delta_0)$.

Proof. By (1.4) and $E_\delta(u) = 0$, we obtain

$$\frac{2+q}{\lambda p}\delta\|u\|_{1,p}^p = \|u\|_{2+q,\partial\Omega}^{2+q} \leq C_*^{2+q}\|u\|_{1,p}^{2+q-p}\|u\|_{1,p}^p,$$

consequently, if $\|u\|_{1,p} \neq 0$, we get

$$\|u\|_{1,p} \geq \left(\frac{2+q}{\lambda p C_*^{2+q}}\delta\right)^{\frac{p}{2+q-p}},$$

which along with

$$E(u) = \frac{1-\delta}{p}\|u\|_{1,p}^p + E_\delta(u) = \frac{1-\delta}{p}\|u\|_{1,p}^p,$$

gives

$$\begin{aligned} E(u) &\geq \frac{1-\delta}{p}\left(\frac{2+q}{\lambda p C_*^{2+q}}\delta\right)^{\frac{p}{2+q-p}} \\ &= d(\delta). \end{aligned}$$

These give the conclusion of first assertion.
On the other hand, note that

$$\frac{1}{2+q}F(u) = \left(\frac{1}{2+q} - \frac{\delta}{p}\right)\|u\|_{1,p}^p + E_\delta(u).$$

From (ii) in Lemma (3.1) we have $E_{\delta_0}(u) = 0$ if and only if $F(u) = 0$. In view of Liu and Zhao [[14], Theorem 2.1], the depth of potential well given by (1.2) can be characterized as $d = \inf E(u)$ subject to the conditions $u \in W^{1,p}(\Omega)$, $\|u\|_{1,p} \neq 0$ and $F(u) = 0$. Hence, from the first conclusion of Proposition (3.2), we obtain $d = d(\delta_0)$. □

Lemma 3.3. *Assume that* $0 < E(u) < d$ *for some* $u \in W^{1,p}(\Omega)$, *and* $\delta_1 < \delta_2$ *are the two roots of equation* $d(\delta) = E(u)$. *Then the sign of* $E_\delta(u)$ *does not change for* $\delta_1 < \delta < \delta_2$.

Proof. Arguing by contradiction, we suppose that the sign of $E_\delta(u)$ is changeable for $\delta_1<\delta<\delta_2$, thus there exists a $\delta^*\in(\delta_1,\delta_2)$ such that $E_{\delta^*}(u)=0$. On the other hand, $E(u)>0$ implies $\|u\|_{1,p}\neq 0$. Combining Theorem (1.3) and Lemma (3.1) we obtain

$$E(u)\geq d\left(\delta^*\right)>d\left(\delta_1\right)=d\left(\delta_2\right),$$

which contradicts $E(u)=d\left(\delta_1\right)=d\left(\delta_2\right)$. □

Corollary 3.4. *Assume that $0<E(u)<d$ for some $u\in H^1(\Omega)$, and $\delta_1<\delta_2$ are the two roots of equation $d(\delta)=E(u)$. Then $E_\delta(u)>0$ (or <0) for all $\delta\in(\delta_1,\delta_2)$ if and only if there exists a $\bar{\delta}\in[\delta_1,\delta_2]$ such that $E_{\bar{\delta}}(u)>0$ (or <0).*

Proof of Theorcm 1.1. We start by constructing a sequence such that its limit equal to the solution in (1.1). Let $\left\{\varphi_j(x)\right\}_{j=1}^{\infty}$ be a system of base functions in $W^{1,p}(\Omega)$, define the approximate solution to (1.1) as follows:

$$u_m(x,t)=\sum_{j=1}^{m} f_{jm}(t)\varphi_j(x),\quad m=1,2,\ldots$$

satisfying

$$\langle u_{mt},\varphi_s\rangle+\langle|u_m|^{p-2}u_m,\varphi_s\rangle+\langle|\nabla u_m|^{p-2}\nabla u_m,\nabla\varphi_s\rangle=\lambda\langle|u_m|^{q}u_m,\varphi_s\rangle_0\ ,\quad 1\leq s\leq m, \tag{3.1}$$

$$u_m(0)=\sum_{j=1}^{m} f_{jm}(0)\varphi_j(x)\to u_0(x)\text{ in }W^{1,p}(\Omega). \tag{3.2}$$

Multiplying (3.1) by $f'_{sm}(t)$, summing for s and integrating with respect to t, we get

$$\int_0^t\left\|u_{m\tau}(\tau)\right\|^2\ \mathrm{d}\tau+E\left(u_m(t)\right)=E\left(u_m(0)\right),\quad \forall t\in[0,\infty). \tag{3.3}$$

Next, if $0<E\left(u_0\right)<d$ and $E_{\delta_2}\left(u_0\right)>0$, then by Corollary (3.4) we have $E_\delta\left(u_0\right)>0$ and $E\left(u_0\right)<d(\delta)$ for all $\delta\in(\delta_1,\delta_2)$, consequently $u_0(x)\in X_\delta$ for all $\delta\in(\delta_1,\delta_2)$. For any fixed $\delta\in(\delta_1,\delta_2)$, we get $u_m(0)\in X_\delta$ for sufficiently large m.
Next, we prove that

$$u_m(t)\in X_\delta \tag{3.4}$$

for sufficiently large m and $t\in[0,\infty)$.
Arguing by contradiction, we assume that there exist a $t_0>0$ such that $u_m\left(t_0\right)\in\partial X_\delta$, i.e., $E_\delta\left(u_m\left(t_0\right)\right)=0$ and $\left\|u_m\left(t_0\right)\right\|_{1,p}\neq 0$ or $E\left(u_m\left(t_0\right)\right)=d(\delta)$. By (3.3) we obtain

$$E\left(u_m(t)\right)\leq E\left(u_m(0)\right)<d(\delta),\quad \forall t\in[0,\infty). \tag{3.5}$$

From (3.5) we can see that $E\left(u_m\left(t_0\right)\right)\neq d(\delta)$. If $E_\delta\left(u_m\left(t_0\right)\right)=0$ and $\left\|u_m\left(t_0\right)\right\|_{1,p}\neq 0$, then it follows from Proposition (3.2) that $E_\delta\left(u_m\left(t_0\right)\right)\geq d(\delta)$, which contradicts (3.5). Thus assertion (3.4) follows as desired.
From (3.3), (3.4) and

$$E\left(u_m(t)\right)=\frac{1-\delta}{p}\left\|u_m(t)\right\|_{1,p}^{p}+E_\delta\left(u_m(t)\right)$$

we see that

$$\int_0^t\left\|u_{m\tau}\right\|^2\ \mathrm{d}\tau<d(\delta),$$

and

$$\left\|u_m(t)\right\|_{1,p}<\left(\frac{2+q}{\lambda p C_*^{2+q}}\delta\right)^{\frac{1}{2+q-p}}.$$

Then

$$\| \, |u_m(t)|^{p-2}u_m(t) \, \|_s^s = \|u_m(t)\|_p^p < \left(\frac{2+q}{\lambda p C_*^{2+q}}\delta\right)^{\frac{p}{2+q-p}}, \qquad s = \frac{p}{p-1}, \; 0 \le t < \infty.$$

Moreover, from (1.4) we deduce

$$\|u_m(t)\|_{q+2,\partial\Omega} \le C_* \|u_m(t)\|_{1,p} < \left(\frac{2+q}{\lambda p C_*^{p}}\delta\right)^{\frac{1}{2+q-p}},$$

thus

$$\| \, |u_m(t)|^{q}u_m(t) \, \|_{r,\partial\Omega}^r = \|u_m(t)\|_{q+2,\partial\Omega}^{q+2} < \left(\frac{2+q}{\lambda p C_*^{p}}\delta\right)^{\frac{q+2}{2+q-p}}, \qquad r = \frac{q+2}{q+1}, \; 0 \le t < \infty,$$

for sufficiently large m and $t \in [0, \infty)$.
Then, there exist a u and subsequence $\{u_v\}$ of $\{u_m\}$ such that as $v \to \infty$,

$$\begin{aligned}
&u_v \to u \;\text{ weakly star in } L^\infty\left(0,\infty; W^{1,p}(\Omega)\right), \\
&u_{vt} \to u_t \;\text{ weakly in } L^2\left(0,\infty; L^2(\Omega)\right), \\
&|u_v|^{p-2}u_v \to |u|^{p-2}u \;\text{ weakly star in } L^\infty\left(0,\infty; L^s(\Omega)\right), \\
&|u_v|^{q}u_v \to |u|^{q}u \;\text{ weakly star in } L^\infty\left(0,\infty; L^r(\Omega) \times L^r(\partial\Omega, \rho)\right).
\end{aligned}$$

Hence, for fixed s, taking $m = v \to \infty$ in (3.1), we obtain

$$\langle u_t, \varphi_s \rangle + \langle |u|^{p-2}u, \varphi_s \rangle + \langle |\nabla u|^{p-2}\nabla u, \nabla \varphi_s \rangle = \lambda \langle |u|^q u, \varphi_s \rangle_0.$$

Furthermore, by (3.2) we get $u(x,0) = u_0(x)$ in $W^{1,p}(\Omega)$. Then, problem (1.1) admits a global solution $u(t) \in L^\infty\left(0,\infty; W^{1,p}(\Omega)\right) \cap C\left(0,\infty; L^p(\Omega) \times L^p(\partial\Omega, \rho)\right)$ with $u_t(t) \in L^2\left(0,\infty; L^2(\Omega)\right)$ and $u(t) \in X_\delta$ for all $t \in [0,\infty)$. Since δ is arbitrary, then $u(t) \in X_\delta$ for all $\delta \in (\delta_1, \delta_2)$ and $t \in [0, \infty)$.

3.2. Proof of Theorem 1.2

To derive the Theorem (1.2) we need the following results.

Lemma 3.5. *If $E(u) \le d(\delta)$, then*

1. *$E_\delta(u) > 0$ if and only if*

$$0 < \|u\|_{1,p} < \left(\frac{2+q}{\lambda p C_*^{2+q}}\delta\right)^{\frac{1}{2+q-p}}; \tag{3.6}$$

2. *$E_\delta(u) < 0$ if and only if*

$$\|u\|_{1,p} > \left(\frac{2+q}{\lambda p C_*^{2+q}}\delta\right)^{\frac{1}{2+q-p}}. \tag{3.7}$$

Proof. 1. If (3.6) holds, then we have

$$\|u\|_{2+q,\partial\Omega}^{2+q} \le C_*^{2+q}\|u\|_{1,p}^{2+q} = C_*^{2+q}\|u\|_{1,p}^{2+q-p}\|u\|_{1,p}^{p} < \frac{2+q}{\lambda p}\delta \|u\|_{1,p}^p.$$

Consequently, $E_\delta(u) > 0$.
If $E_\delta(u) > 0$, then $\|u\|_{1,p} > 0$. Thus, from

$$E(u) = \frac{1-\delta}{p}\|u\|_{1,p}^p + E_\delta(u) \le d(\delta) \tag{3.8}$$

we get (3.6).

2. It is easy to see $\|u\|_{1,p} \neq 0$ from $E_\delta(u) < 0$. Hence, by

$$\frac{2+q}{\lambda p}\delta\|u\|_{1,p}^{p} < \|u\|_{2+q,\partial\Omega}^{2+q} \leq C_*^{2+q}\|u\|_{1,p}^{2+q-p}\|u\|_{1,p}^{p}$$

we obtain (3.7).
Combining (3.7) and (3.8) we obtain $E_\delta(u) < 0$.

□

Proposition 3.6. *If $E(u) \leq d(\delta)$, then $X_\delta \subset B_\delta$ and $Y_\delta \subset B_\delta^c$.*

Proof. This Theorem follows from Lemma (3.5). □

Corollary 3.7. *Assume that $E(u) \leq d(\delta)$. Then,*

1. *$u \in X_\delta$ if and only if $u \in B_\delta$;*
2. *$u \in Y_\delta$ if and only if $u \in B_\delta^c$.*

Proof. The conclusions of Corollary (3.7) can be derived by a combination of Proposition (3.6) and Lemma (3.5). □

Lemma 3.8. *Let p, q satisfy (H), then the solutions given in Theorem (1.1) satisfy*

$$\int_0^t \left\|u_\tau(\tau)\right\|^2 \,\mathrm{d}\tau + E(u(t)) \leq E\left(u_0\right), \quad \forall t \in [0, \infty). \tag{3.9}$$

Proof. The proof of this lemma follows from the arguments similar to the proof of [[14], Lemma 2.8]. □

Corollary 3.9. *Let $u_0(x) \in W^{1,p}(\Omega)$, p and q satisfy (H). Assume that $0 < e < d$ and $\delta_1 < \delta_2$ are the two roots of equation $d(\delta) = e$. Then,*

1. *Solutions of problem (1.1) with $0 < E\left(u_0\right) \leq e$ belong to $\overline{X}_{\delta_1}$, provided $F\left(u_0\right) > 0$;*
2. *Solutions of problem (1.1) with $0 < E\left(u_0\right) \leq e$ belong to $\overline{Y}_{\delta_2}$, provided $F\left(u_0\right) < 0$.*

Proof. Let $u(t)$ be any solution of problem (1.1) with $0 < E\left(u_0\right) \leq e$, and T be the maximum existence time of $u(t)$. By (3.9) we get $E(u) \leq d\left(\delta_1\right) = d\left(\delta_2\right)$. For fixed $t \in [0, T)$, taking $\delta \to \delta_1\,(\delta \to \delta_2)$ in $E_\delta(u) > 0\,(E_\delta(u) < 0)$, we obtain $E_{\delta_1}(u) \geq 0\,(E_{\delta_2}(u) \leq 0)$ for all $t \in [0, T)$. This shows the conclusions of the corollary (3.9). □

Proof of Theorcm 1.2.

1. If $u_0\left(x\right) \in B_{\delta_0}$, then

$$\|u_0\| < \left(\frac{2+q}{\lambda p C_*^{2+q}}\delta_0\right)^{\frac{1}{2+q-p}},$$

it follows from Lemma (3.5) that $E_{\delta_0}\left(u_0\right) > 0$, thus $F\left(u_0\right) > 0$. Finally, from Corollaries (3.9) and (3.7) we deduce that $u \in \overline{B}_{\delta_1}$.

2. By a similar argument as above, we can prove that if $u_0(x) \in B_{\delta_0}^c$, then the solutions of problem (1.1) with initial condition $0 < E\left(u_0\right) \leq$ e lie in $B_{\delta_2}^c \cup \partial B_{\delta_2}$.

3.3. Proof of Theorem 1.3

1. Let $u(t)$ be any solution of problem (1.1) and T be the maximum existence time of $u(t)$. Next we schow $T<\infty$. Arguing by contradiction, we suppose that $T=\infty$.
Set
$$H(t)=\frac{1}{2}\int_0^t\|u\|^2\ \mathrm{d}\tau.$$
Then
$$H'(t)=\frac{1}{2}\|u\|^2,$$
and
$$H''(t)=\langle u,u_t\rangle=-F(u). \tag{3.10}$$
By (3.9) and
$$E\left(u\right)=\frac{2+q-p}{p\left(2+q\right)}\|u\|_{1,p}^p+\frac{1}{2+q}F\left(u\right), \tag{3.11}$$
we obtain
$$F(u)\le-\frac{2+q-p}{p}\|u\|_{1,p}^p+(2+q)E\left(u_0\right)-(2+q)\int_0^t\left\|u_\tau\right\|^2\ \mathrm{d}\tau.$$
Now, from (3.10), we can write
$$H''(t)\ge\frac{2+q-p}{p}\|u\|_{1,p}^p-(2+q)E\left(u_0\right)+(2+q)\int_0^t\left\|u_\tau\right\|^2\ \mathrm{d}\tau. \tag{3.12}$$
Next, we show that
$$H''(t)\ge(2+q)\int_0^t\left\|u_\tau\right\|^2\ \mathrm{d}\tau. \tag{3.13}$$
To see this, we consider the following two cases.
Case 1. The case $E\left(u_0\right)\le 0$.
Assertion (3.13) follows directly from (3.12).
Case 2. The case $0<E\left(u_0\right)<d$.
By $E_{\delta_1}\left(u_0\right)<0$ and Corollary (3.4) we get $E_{\delta_0}\left(u_0\right)<0$. Note that $E(u)\le E\left(u_0\right)<d$. Hence, by recaling the definition of Y_δ, we obtain $u\in Y_{\delta_0}$. Consequently, from (ii) in Corollary (3.7), we obtain $u\in B_{\delta_0}^c$, i.e.,
$$\|u\|_{1,p}>\left(\frac{2+q}{\lambda pC_*^{2+q}}\delta_0\right)^{\frac{1}{2+q-p}}.$$
Which together with (ii) in Lemma (3.1) and Proposition (3.2), we can deduce
$$\|u\|_{1,p}^p>C_*^{-\frac{(2+q)p}{2+q-p}}=\frac{(2+q)p}{2+q-p}d>\frac{(2+q)p}{2+q-p}E\left(u_0\right).$$
Combining this with (3.12), thus assertion (3.13) follows as desired.
Next, from (3.13), there exists a $t^*>0$ such that $H'(t)\ge H'\left(t^*\right)>0$ and $H(t)\ge H'\left(t^*\right)\left(t-t^*\right)+H\left(t^*\right)$ for all $t\in\left[t^*,\infty\right)$. Consequently
$$\lim_{t\to\infty}H(t)=\infty. \tag{3.14}$$
Combining (3.13) and the Cauchy-Schwarz inequality, we have
$$\begin{aligned}H(t)H''(t)&\ge\frac{2+q}{p}\int_0^t\|u\|^2\ \mathrm{d}\tau\int_0^t\left\|u_\tau\right\|^2\ \mathrm{d}\tau\\&\ge\frac{2+q}{p}\left(\int_0^t\langle u,u_\tau\rangle\mathrm{d}\tau\right)^2\\&=\frac{2+q}{p}\left(H'(t)-H'(0)\right)^2.\end{aligned}$$

Then there exists a $\alpha > 0$ such that

$$H(t)H''(t) \geq (1+\alpha)H'(t)^2.$$

For all $t \in [t^*, \infty)$, consequently

$$\left(H^{-\alpha}(t)\right)' = -\frac{\alpha H'(t)}{H^{\alpha+1}(t)} < 0,$$

and

$$\left(H^{-\alpha}(t)\right)'' = -\frac{\alpha}{H^{\alpha+2}(t)}\left[H(t)H''(t) - (\alpha+1)\left(H'(t)\right)^2\right] \leq 0.$$

Therefore, $H^{-\alpha}(t) > 0$ is decreasing and concave on $[t^*, \infty)$, which contradicts (3.14), then $T < \infty$. Hence the conclusion of (i) holds.

2. First, we show that

$$E_{\delta_0}(u) < 0, \quad \forall t \in [0, \infty). \tag{3.15}$$

Arguing by contradiction, we assume that there exist a first time $t_0 > 0$ such that $E_{\delta_0}(u(t_0)) = 0$ and $E_{\delta_0}(u) < 0$ for all $t \in [0, t_0)$. By (ii) in Lemmas (3.5) and (3.1), we can deduce

$$\|u\|_{1,p}^p > C_*^{-\frac{(2+q)p}{2+q-p}}, \quad \forall t \in [0, t_0)$$

which together with Proposition (3.2) gives

$$\|u\|_{1,p}^p > \frac{(2+q)p}{2+q-p}d, \quad \forall t \in [0, t_0),$$

consequently

$$\|u(t_0)\|_{1,p}^p \geq \frac{(2+q)p}{2+q-p}d.$$

Which together with (3.11), we obtain

$$E(u(t_0)) \geq d. \tag{3.16}$$

At the same time, by (3.10), we have $\langle u, u_t\rangle > 0$, which implies that $\int_0^t \|u_\tau\|^2 \, d\tau$ is increasing in time.
Consequently

$$\int_0^{t_0} \|u_\tau\|^2 \, d\tau > 0.$$

Combining this with (3.9) and $E(u_0) = d$, we get

$$E(u(t_0)) < d.$$

which contradicts (3.16), then assertion (3.15) holds.
For any $\tilde{t} > 0$, let

$$d_1 := d - \int_0^{\tilde{t}} \|u_t\|^2 \, dt,$$

Thus

$$0 < E(u) \leq d_1 < d \text{ for all } t \in [\tilde{t}, \infty),$$

which together with assertion (3.15) and (ii) in Corollary (3.9) gives

$$u \in \overline{Y}_{\tilde{\delta}_2} \text{ for all } t \in [\tilde{t}, \infty),$$

where $\tilde{\delta}_1 < \tilde{\delta}_2$ are two roots of equation $d(\delta) = d_1$.
Consequently

$$E_{\tilde{\delta}_2}(u) \leq 0 \text{ for all } t \in [\tilde{t}, \infty).$$

We also obtain

$$E_{\tilde{\delta}_1}(u) < 0 \text{ for all } t \in [\tilde{t}, \infty).$$

From these and the argument in the proof for the case Case 2 in (i), we can obtain (ii).

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Anass Lamaizi,
LaMAO Laboratory,
Faculty of sciences, Oujda,
Morocco.
E-mail address: lamaizi.anass@ump.ac.ma

and

Abdellah Zerouali,
Department of Mathematics,
CRMEF, Oujda,
Morocco.
E-mail address: abdellahzerouali@yahoo.fr

and

Omar Chakrone,
LaMAO Laboratory,
Faculty of sciences, Oujda,
Morocco.
E-mail address: chakrone@yahoo.fr

and

Karim Belhadj,
Department of Mathematics,
Sciences And Technologies Faculty, Errachidia,
Morocco.
E-mail address: karembelf@gmail.com