Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–16.
ISSN-0037-8712
doi:10.5269/bspm.62210

Existence and Multiplicity Results for a $(p(x), q(x))$-Laplacian Steklov Problem

Abdessamad Lakhdi, Belhadj Karim and Abdellah Zerouali

ABSTRACT: This paper investigates the existence and multiplicity of solutions for a Steklov problem involving the $(p(x), q(x))$-Laplacian operator. The discussion is based on Ricceri's three critical points theorem and different versions of mountain pass theorem.

Key Words: $(p(x), q(x))$-Laplacian, Steklov eigenvalue problem, Recceri's variational principle, Mountain Pass theorem.

Contents

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following generalized Steklov problem

$$\begin{cases} \triangle_{p(x)}u + \triangle_{q(x)}u = 0 & \text{in } \Omega, \\ \left(|\nabla u|^{p(x)-2} + |\nabla u|^{q(x)-2}\right)\frac{\partial u}{\partial \nu} + |u|^{p(x)-2}u + |u|^{q(x)-2}u = \lambda\left(|u|^{r(x)-2}u - \varepsilon|u|^{s(x)-2}u\right) & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda \in (0, +\infty)$, $\varepsilon \geq 0$, p, q, r, s are continuous functions on $\overline{\Omega}$ with $q(x) \leq p(x)$ for all $x \in \overline{\Omega}$ and ν is the outward unit normal vector on $\partial\Omega$.

Nonlinear eigenvalue problems for the $p(x)$-Laplacian with Dirichlet, Neumann, Robin or Steklov boundary conditions on a bounded domain have been studied extensively in recent years. Many interesting results have been obtained on this kind of problems, see for example [5,6,8,9,17,18,19] and references therein. This importance reflects directly into a various range of applications. There are applications concerning elastic materials, thermorheological and electrorheological fluids, image restoration and mathematical biology (See [2,4,11,20,22]), but these problems are very interesting from a purely mathematical point of view as well. Unlike the p-Laplacian ($p(x) = p$ constant), the $p(x)$-Laplacian ($p(x) \neq$ constant) possesses more complicated properties, for instance, it is inhomogeneous, so in the discussion, some special techniques will be needed. In the case where $p(x) = q(x)$ and $\varepsilon = 0$, A. Anane et al in [1], are considered the following Steklov problem

$$\begin{cases} \triangle_{p(x)}u = |u|^{p(x)-2}u & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = \lambda|u|^{r(x)-2}u & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

2010 *Mathematics Subject Classification:* 35J60, 58E30, 47J30, 58E05.
Submitted January 23, 2022. Published December 04, 2025

They have proved two different situations of the existence of solutions for the problem (1.2). In [14], B. Karim et al have studied the case $p(x) = q(x)$ and $\varepsilon = 1$, they proved in different cases the existence and multiplicity of a-harmonic solutions for the following elliptic problem

$$\begin{cases} div(a(x, \nabla u)) = 0 & \text{in } \Omega, \\ a(x, \nabla u).\nu + |u|^{p(x)-2}u = \lambda \left(|u|^{r(x)-2}u - |u|^{s(x)-2}u\right) & \text{on } \partial\Omega. \end{cases}$$

But these problems becomes $p(x)$-harmonic when $a(x, s) = |s|^{p(x)-2}s$. The purpose of this paper is to generalize the results of the $p(x)$-Laplacian in [1,14] to $(p(x), q(x))$-Laplacian.

Throughout this paper, we denote by :

$$h^+ := \max_{x\in\overline{\Omega}} h(x); \quad h^- := \min_{x\in\overline{\Omega}} h(x); \quad \text{for all } h \in C(\bar{\Omega}),$$

$$p^\partial(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \infty, & \text{if } p(x) \geq N, \end{cases}$$

and

$$C_+(\overline{\Omega}) := \left\{h \in C(\overline{\Omega});\ 1 < h^- < h^+ < \infty\right\}.$$

We assume that variable exponents p, q, r and s are in $C_+(\overline{\Omega})$.

The main results of this paper are Theorems 1.1-1.2 below. The first two results concern the cases of $\varepsilon = 0$ and the rest of results concern $\varepsilon = 1$. The first two results concern the case of $\varepsilon = 0$ and the rest of results address $\varepsilon = 1$.

Theorem 1.1 *For $\varepsilon = 0$:*

(i) If $q^+ \leq p^+ < r^- < r^+ < p^\partial(x)$ for all $x \in \overline{\Omega}$, where $p^\partial(x)$ is defined above. Then for any $\lambda > 0$ the problem (1.1) possesses a nontrivial weak solution.

(ii) If $r^+ < q^- < q^+ \leq p^-$. Then for any $\lambda > 0$ there exists a sequence (u_k) of nontrivial weak solutions for the problem (1.1). Moreover $u_k \to 0$, as $k \to \infty$.

Theorem 1.2 *For $\varepsilon = 1$:*

(i) If $s^+ \leq p^+ < r^- < r^+ < p^\partial(x)$ for all $x \in \overline{\Omega}$. Then for any $\lambda > 0$, the problem (1.1) possesses a non trivial weak solution.

(ii) If $N < p^-$ and $s^+ < r^- < r^+ < p^-$. Then there exist an open interval $\wedge \subset (0; +\infty)$ and a positive constant $\rho_0 > 0$ such that for any $\lambda \in \wedge$, the problem (1.1) has at least three weak solutions whose norms are less than ρ_0.

Remark 1.1 *It is easily seen that the case $\varepsilon > 0$ can be treated as the case $\varepsilon = 1$.*

Remark 1.2 *We find the same results if $q(x)$ plays the role of $p(x)$, it means if we take $p(x) \leq q(x)$.*

This paper consists of three sections. Section 1 contains an introduction and the main results. In Section 2, we state some necessary preliminary knowledge. The proofs of our main results are given in Section 3.

2. Preliminaries

We recall in what follows some definitions and basic properties of variable exponent Lebesgue and Sobolev spaces. We refer to [7,10,15] for the fundamental properties of these spaces. Here also, we collect the ingredients of our proofs.

We introduce the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) := \left\{u : \Omega \subset \mathbb{R}^N \to \mathbb{R} \text{ is measurable and } \int_\Omega |u|^{p(x)}dx < +\infty\right\},$$

and the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega)} := \inf\left\{\alpha > 0\ ;\ \int_\Omega \left|\frac{u(x)}{\alpha}\right|^{p(x)} dx \leq 1\right\}.$$

Equipped with this norm, $L^{p(x)}(\Omega)$ is a separable and reflexive Banach space [10].

Let us define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) := \left\{u \in L^{p(x)}(\Omega)\ /\ |\nabla u| \in L^{p(x)}(\Omega)\right\},$$

equipped with the norm

$$\|u\|_\Omega := \inf\left\{\alpha > 0\ ;\ \int_\Omega \left|\frac{u(x)}{\alpha}\right|^{p(x)} dx + \int_\Omega \left|\frac{\nabla u(x)}{\alpha}\right|^{p(x)} dx \leq 1\right\}; \quad \forall u \in W^{1,p(x)}(\Omega).$$

Proposition 2.1 *[6] Let $u \in W^{1,p(x)}(\Omega)$.*
Let $\|u\| := |\nabla u|_{L^{p(x)}(\Omega)} + |u|_{L^{p(x)}(\partial\Omega)}$. Then $\|u\|$ is a norm on $W^{1,p(x)}(\Omega)$ which is equivalent to $\|u\|_\Omega$.

The properties of $W^{1,p(x)}(\Omega)$ and the properties concerning the embedding results are given in the following proposition.

Proposition 2.2 *[8,15]*

(1) $W^{1,p(x)}(\Omega)$ is separable reflexive Banach space;

(2) If $q \in C_+(\overline{\Omega})$ and $q(x) < p^\partial(x)$ for any $x \in \overline{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\partial\Omega)$ is compact and continuous.

A big help in handling the generalized Lebesgue-Sobolev spaces is provided by the map $\rho_p : W^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_p(u) := \int_\Omega |\nabla u|^{p(x)} dx + \int_{\partial\Omega} |u|^{p(x)} d\sigma, \quad \forall u \in W^{1,p(x)}(\Omega).$$

The following proposition illuminates the close relation between the $\|.\|$ and the modular $\rho_p(u)$ (see for example [6]).

Proposition 2.3 *If $u, u_k \in W^{1,p(x)}(\Omega)$; $\quad k = 1, 2, \ldots,$ then*

1. $||u|| \geq 1 \Rightarrow ||u||^{p^-} \leq \rho_p(u) \leq ||u||^{p^+}$;
2. $||u|| \leq 1 \Rightarrow ||u||^{p^+} \leq \rho_p(u) \leq ||u||^{p^-}$;
3. $||u_k|| \to 0 (\to \infty) \Leftrightarrow \rho_p(u_k) \to 0 (\to \infty)$.

Remark 2.1 *If $N < p^- \leq p(x)$ for any $x \in \overline{\Omega}$, by Theorem 2.2 in [10] and Remark 1 in [17], we have $W^{1,p(x)}(\Omega)$ is compactly embedded in $C(\overline{\Omega})$. Defining $\|u\|_\infty = \sup_{x \in \overline{\Omega}} |u(x)|$, we find that there exists a positive constant c such that $\|u\|_\infty \leq c\|u\|$ for all $u \in W^{1,p(x)}(\Omega)$.*

Definition 2.1 (Palais-Smale condition) *Let X be a Banach space and $\Phi : X \to \mathbb{R}$ a C^1-functional. We say that Φ satisfies the Palais-Smale condition on X ((PS)-condition for short), if every sequence $u_n \in X$ such that $\Phi(u_n)$ is bounded and $\Phi'(u_n) \to 0$, has a converging subsequence in X.*

To prove (i) of Theorem 1.1 and (i) of Theorem 1.2, we shall give a variant of the mountain pass theorem of Ambrosetti and Rabinowitz as follows.

Theorem 2.1 *[12] Let E endowed with the norm $\|.\|_E$, be a Banach space. Assume that $\phi \in C^1(E;\mathbb{R})$ satisfies the Palais $-$ Smale (PS) condition. Also, assume that ϕ has a mountain pass geometry, that is,*

1. *there exists two constants $\eta > 0$ and $\rho \in \mathbb{R}$ such that $\phi(u) \geq \rho$ if $\|u\|_E = \eta$;*
2. *$\phi(0) < \rho$ and there exists $e \in E$ such that $\|e\|_E > \eta$ and $\phi(e) < \rho$.*

Then ϕ has a critical point $u_0 \in E$ such that $u_0 \neq 0$ and $u_0 \neq e$ with critical value

$$\phi(u_0) = \inf_{\gamma \in \mathcal{P}} \sup_{u \in \gamma} \phi(u) \geq \rho > 0.$$

Where $\mathcal{P}$ denotes the class of the paths $\gamma \in C([0,1];E)$ joining 0 to e.

The proof of (ii) of Theorem 1.1 will be based on the following version of the symmetric mountain pass theorem (see for example [13]).

Theorem 2.2 *Let E be an infinite dimensional Banach space and $I \in C^1(E,\mathbb{R})$ satisfy the following two assumptions:*

(A_1). *$I(u)$ is even, bounded from below, $I(0) = 0$ and $I(u)$ satisfies the Palais-Smale condition (PS);*

(A_2). *For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u \in A_k} I(u) < 0$, where Γ_k denote the family of symmetric subsets A of E such that $0 \notin A$ and $\gamma(A) \geq k$ with*

$$\gamma(A) := \inf\{k \in \mathbb{N}; \exists h : A \to \mathbb{R}^k \backslash \{0\} \text{ such that } h \text{ is continuous and odd}\}$$

is the genus of A.

Then $I(u)$ admits a sequence of critical points u_k such that $I(u_k) < 0$; $u_k \neq 0$ and $u_k \to 0$ as $k \to \infty$.

Finally, we remind the version of Recciri theorem (see [3, Theorem 1]) that will be used in the proof of (ii) of Theorem 1.2.

Theorem 2.3 *Let E be a separable and reflexive real Banach space; $\Phi : E \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on E^*; $\Psi : E \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*

(1) $\lim_{\|u\| \to +\infty} (\Phi(u) + \lambda \Psi(u)) = +\infty$ *for all $\lambda > 0$; and that are $\alpha \in \mathbb{R}$ and $u_0, u_1 \in E$ such that*

(2) $\Phi(u_0) < \alpha < \Phi(u_1)$;

(3) $\inf_{u \in \Phi^{-1}((-\infty,\alpha])} \Psi(u) > \dfrac{(\Phi(u_1) - \alpha)\Psi(u_0) + (\alpha - \Phi(u_0))\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)}$.

Then there exist an open interval $\wedge \subset (0,+\infty)$ and a positive real number ρ_0 such that for each $\lambda \in \wedge$ the equation $\Phi'(u) + \lambda\Psi'(u) = 0$ has at least three solutions in E whose norms are less than ρ_0.

3. Proof of main results

The Sobolev space $W^{1,p(x)}(\Omega)$ is the space where we will try to find weak solutions for problem (1.1). Since we will rely in different situations on the critical point theory, we formulate the definition of weak solution.

Definition 3.1 *We say that $u \in W^{1,p(x)}(\Omega)$ is a nontrivial weak solution of (1.1) if and only if :*

$$\int_\Omega \left(|\nabla u|^{p(x)-2} + |\nabla u|^{q(x)-2}\right) \nabla u \nabla v dx + \int_{\partial\Omega} \left(|u|^{p(x)-2} + |u|^{q(x)-2}\right) uv d\sigma$$
$$-\lambda \int_{\partial\Omega} \left(|u|^{r(x)-2} - \varepsilon |u|^{s(x)-2}\right) uv d\sigma = 0, \quad \text{for any } v \in W^{1,p(x)}(\Omega).$$

The Euler-Lagrange functional associated to (1.1) is defined on $W^{1,p(x)}(\Omega)$, by

$$\begin{aligned}\Phi_{\lambda,\varepsilon}(u) = &\int_{\Omega} \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \frac{|\nabla u|^{q(x)}}{q(x)} \right) dx + \int_{\partial\Omega} \left(\frac{|u|^{p(x)}}{p(x)} + \frac{|u|^{q(x)}}{q(x)} \right) d\sigma \\ &- \lambda \int_{\partial\Omega} \left(\frac{|u|^{r(x)}}{r(x)} - \varepsilon \frac{|u|^{s(x)}}{s(x)} \right) d\sigma,\end{aligned} \tag{3.1}$$

where $d\sigma$ is the $N-1$ dimensional Hausdorff measure restricted to the boundary $\partial\Omega$.

It is clear that $\Phi_{\lambda,\varepsilon} \in C^1(W^{1,p(x)}(\Omega); \mathbb{R})$ and we have

$$\begin{aligned}\langle \Phi'_{\lambda,\varepsilon}(u), v \rangle = &\int_{\Omega} \left(|\nabla u|^{p(x)-2} + |\nabla u|^{q(x)-2} \right) \nabla u \nabla v dx + \int_{\partial\Omega} \left(|u|^{p(x)-2} + |u|^{q(x)-2} \right) uv d\sigma \\ &- \lambda \int_{\partial\Omega} \left(|u|^{r(x)-2} - \varepsilon |u|^{s(x)-2} \right) uv d\sigma, \quad \text{for any } u, v \in W^{1,p(x)}(\Omega).\end{aligned} \tag{3.2}$$

Hence it is easy to see that the weak solutions of problem (1.1) given by (3.2) are critical points of $\Phi_{\lambda,\varepsilon}$ defined by (3.1).

3.1. Proof of Theorem 1.1

In this subsection and next subsection we provide existence and multiplicity results for problem (1.1). Therefore each subsection will be divided into two sub-subsections. Here we focus on the case when $\varepsilon = 0$.

3.1.1. Proof of case (i). The energy functional associated to problem (1.1) in the case where $\varepsilon = 0$ is given by

$$\Phi_{\lambda,0}(u) = \int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx + \int_{\Omega} \frac{|\nabla u|^{q(x)}}{q(x)} dx + \int_{\partial\Omega} \frac{|u|^{p(x)}}{p(x)} d\sigma + \int_{\partial\Omega} \frac{|u|^{q(x)}}{q(x)} d\sigma - \lambda \int_{\partial\Omega} \frac{|u|^{r(x)}}{r(x)} d\sigma,$$

and we have

$$\begin{aligned}\langle \Phi'_{\lambda,0}(u), v \rangle = &\int_{\Omega} \left(|\nabla u|^{p(x)-2} + |\nabla u|^{q(x)-2} \right) \nabla u \nabla v dx + \int_{\partial\Omega} \left(|u|^{p(x)-2} + |u|^{q(x)-2} \right) uv d\sigma \\ &- \lambda \int_{\partial\Omega} |u|^{r(x)-2} uv d\sigma, \quad \text{for any } u, v \in W^{1,p(x)}(\Omega).\end{aligned}$$

We show now that the Mountain Pass Theorem of Ambrosetti and Rabinowitz (Theorem 2.1) can be applied. To this aim, we prove three auxiliary lemmas.

We give the first one.

Lemma 3.1 *For $\varepsilon = 0$, assume that $q^+ \leq p^+ < r^- < r^+ < p^\partial(x)$ for all $x \in \overline{\Omega}$. Then there exists $\eta, b > 0$ such that : $\Phi_{\lambda,0}(u) \geq b$ for $u \in W^{1,p(x)}(\Omega)$ with $\|u\| = \eta$.*

Proof: According to the fact that

$$|u(x)|^{r^+} + |u(x)|^{r^-} \geq |u(x)|^{r(x)}; \quad \forall x \in \overline{\Omega},$$

it may be concluded that for all $u \in W^{1,p(x)}(\Omega)$,

$$\begin{aligned}\Phi_{\lambda,0}(u) &= \int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx + \int_{\Omega} \frac{|\nabla u|^{q(x)}}{q(x)} dx + \int_{\partial\Omega} \frac{|u|^{p(x)}}{p(x)} d\sigma + \int_{\partial\Omega} \frac{|u|^{q(x)}}{q(x)} d\sigma - \lambda \int_{\partial\Omega} \frac{|u|^{r(x)}}{r(x)} d\sigma \\ &\geq \frac{1}{p^+} \rho_p(u) + \frac{1}{q^+} \rho_q(u) - \frac{\lambda}{r^-} \left(\int_{\partial\Omega} |u|^{r^+} d\sigma + \int_{\partial\Omega} |u|^{r^-} d\sigma \right) \\ &\geq \frac{1}{p^+} \rho_p(u) - \frac{\lambda}{r^-} \left(\int_{\partial\Omega} |u|^{r^+} d\sigma + \int_{\partial\Omega} |u|^{r^-} d\sigma \right).\end{aligned}$$

As $r^- \leq r^+ < p^\partial(x)$ for any $x \in \overline{\Omega}$, Proposition 2.2 shows that $W^{1,p(x)}(\Omega)$ is continuously embedded in $L^{r^+}(\partial\Omega)$ and in $L^{r^-}(\partial\Omega)$. we deduce that there exists two positive constants C_1 and C_2 such that

$$\int_{\partial\Omega} |u|^{r^+} d\sigma \leq C_1 \|u\|^{r^+}, \quad \int_{\partial\Omega} |u|^{r^-} d\sigma \leq C_2 \|u\|^{r^-}, \quad \forall u \in W^{1,p(x)}(\Omega).$$

Then we get

$$\Phi_{\lambda,0}(u) \geq \frac{1}{p^+}\rho_p(u) - \frac{\lambda}{r^-}(C_1\|u\|^{r^+} + C_2\|u\|^{r^-}).$$

By Proposition 2.3, we obtain

$$\Phi_{\lambda,0}(u) \geq \frac{1}{p^+}\|u\|^{p^-} - \frac{\lambda}{r^-}\Big(C_1\|u\|^{r^+} + C_2\|u\|^{r^-}\Big), \text{if} \quad \|u\| \geq 1;$$

and

$$\Phi_{\lambda,0}(u) \geq \frac{1}{p^+}\|u\|^{p^+} - \frac{\lambda}{r^-}\Big(C_1\|u\|^{r^+} + C_2\|u\|^{r^-}\Big), \text{if} \quad \|u\| \leq 1.$$

Therefore

$$\Phi_{\lambda,0}(u) \geq \|u\|^{p^-}\Big(\frac{1}{p^+} - \frac{\lambda}{r^-}(C_1\|u\|^{r^+-p^-} + C_2\|u\|^{r^--p^-})\Big), \text{if} \quad \|u\| \geq 1;$$

and

$$\Phi_{\lambda,0}(u) \geq \|u\|^{p^+}\Big(\frac{1}{p^+} - \frac{\lambda}{r^-}(C_1\|u\|^{r^+-p^+} + C_2\|u\|^{r^--p^+})\Big), \text{if} \quad \|u\| \leq 1.$$

Since $p^+ < r^- < r^+$, it follows that the functional $h : [0;1] \to \mathbb{R}$ defined by

$$h(t) = \frac{1}{p^+} - \frac{\lambda C_1}{r^-}t^{r^+-p^+} - \frac{\lambda C_2}{r^-}t^{r^--p^+},$$

is positive on neighborhood of the origin, and the lemma follows. □

We carry on to the next lemma.

Lemma 3.2 *For $\varepsilon = 0$, assume that $q^+ \leq p^+ < r^- < r^+ < p^\partial(x)$ for all $x \in \bar{\Omega}$. Then there exists $e \in W^{1,p(x)}(\Omega)$ with $\|e\| > \eta$ such that $\Phi_{\lambda,0}(e) < 0$; where η is given in Lemma 3.1.*

Proof: Let $\varphi \in C_0^\infty(\Omega)$ be such that $\varphi \geq 0$ and $\varphi \not\equiv 0$. For $t > 1$, we have

$$\begin{aligned}\Phi_{\lambda,0}(t\varphi) &= \int_\Omega \frac{t^{p(x)}}{p(x)}|\nabla\varphi|^{p(x)}dx + \int_\Omega \frac{t^{q(x)}}{q(x)}|\nabla\varphi|^{q(x)}dx + \int_{\partial\Omega} \frac{t^{p(x)}}{p(x)}|\varphi|^{p(x)}d\sigma \\ &+ \int_{\partial\Omega} \frac{t^{q(x)}}{q(x)}|\varphi|^{q(x)}d\sigma - \lambda\int_{\partial\Omega} \frac{t^{r(x)}}{r(x)}|\varphi|^{r(x)}d\sigma \\ &\leq \frac{t^{p^+}}{p^-}\rho_p(\varphi) + \frac{t^{q^+}}{q^-}\rho_q(\varphi) - \frac{\lambda t^{r^-}}{r^+}\int_{\partial\Omega}|\varphi|^{r(x)}d\sigma.\end{aligned}$$

Since $q^+ \leq p^+ < r^-$, we conclude that $\lim_{t\to+\infty} \Phi_{\lambda,0}(t\varphi) = -\infty$. Consequently, for all $\epsilon > 0$ there exists $\alpha > 0$ such that $|t| > \alpha$, $\Phi_{\lambda,0}(t\varphi) < -\epsilon < 0$. This proves the lemma. □

Finally, we give the third auxiliary lemma.

Lemma 3.3 *For $\varepsilon = 0$, assume that $q^+ \leq p^+ < r^- < r^+ < p^\partial(x)$ for all $x \in \overline{\Omega}$. Then the functional $\Phi_{\lambda,0}$ satisfies the Palais-Smale (PS) condition.*

Proof: Let $(u_k) \subset W^{1,p(x)}(\Omega)$ be such that $C = \sup_{k\in\mathbb{N}^*} \Phi_{\lambda,0}(u_k)$ and $\Phi'_{\lambda,0}(u_k) \to 0$.
Let us first show that (u_k) is bounded. To see this, we argue by contradiction and we assume that $\|u_k\| \to \infty$ as $k \to \infty$. Then, there exists $k_0 \in \mathbb{N}^*$ such that $\|u_k\| > 1$ for any $k > k_0$. hence

$$\begin{aligned}
C + \|u_k\| \geq & \Phi_{\lambda,0}(u_k) - \frac{1}{r^-}\langle \Phi'_{\lambda,0}(u_k), u_k\rangle \\
\geq & \int_\Omega \frac{|\nabla u_k|^{p(x)}}{p(x)}dx + \int_\Omega \frac{|\nabla u_k|^{q(x)}}{q(x)}dx + \int_{\partial\Omega} \frac{|u_k|^{p(x)}}{p(x)}d\sigma + \int_{\partial\Omega} \frac{|u_k|^{q(x)}}{q(x)}d\sigma \\
& - \lambda \int_{\partial\Omega} \frac{|u_k|^{r(x)}}{r(x)}d\sigma - \frac{1}{r^-}\left[\int_\Omega |\nabla u_k|^{p(x)}dx + \int_\Omega |\nabla u_k|^{q(x)}dx \right.\\
& \left. + \int_{\partial\Omega} \left(|u_k|^{p(x)} + |u_k|^{q(x)}\right) d\sigma - \lambda \int_{\partial\Omega} |u_k|^{r(x)}d\sigma\right] \\
\geq & \frac{1}{p^+}\rho_p(u_k) + \frac{1}{q^+}\rho_q(u_k) - \lambda \int_{\partial\Omega} \frac{1}{r(x)}|u_k|^{r(x)}d\sigma + \frac{\lambda}{r^-}\int_{\partial\Omega} |u_k|^{r(x)}d\sigma \\
& - \frac{1}{r^-}\left(\rho_p(u_k) + \rho_q(u_k)\right) \\
\geq & \left(\frac{1}{p^+} - \frac{1}{r^-}\right)\rho_p(u_k) + \left(\frac{1}{q^+} - \frac{1}{r^-}\right)\rho_q(u_k) + \lambda \int_{\partial\Omega} \left(\frac{1}{r^-} - \frac{1}{r(x)}\right)|u_k|^{r(x)}d\sigma \\
\geq & \left(\frac{1}{p^+} - \frac{1}{r^-}\right)\rho_p(u_k) \\
\geq & \left(\frac{1}{p^+} - \frac{1}{r^-}\right)\|u_k\|^{p^-}.
\end{aligned}$$

As $p^+ < r^-$, we obtain a contradiction with the fact that $p^- > 1$. Therefore (u_k) is bounded in $W^{1,p(x)}(\Omega)$. Taking into account the fact that $W^{1,p(x)}(\Omega)$ is a reflexive space (Proposition 2.2), we infer that, up to a subsequence still denoted (u_k), we obtain $(u_k)_k$ converges weakly to u in $W^{1,p(x)}(\Omega)$ and strongly in $L^{p(x)}(\partial\Omega)$, $L^{q(x)}(\partial\Omega)$ and $L^{r(x)}(\partial\Omega)$ respectively (Proposition 2.2).
This leads to

$$\langle \Phi'_{\lambda,0}(u_k), u_k - u\rangle \to 0,$$

and

$$\int_{\partial\Omega} |u_k|^{p(x)-2}u_k(u_k - u)d\sigma + \int_{\partial\Omega} |u_k|^{q(x)-2}u_k(u_k - u)d\sigma - \int_{\partial\Omega} |u_k|^{r(x)-2}u_k(u_k - u)dx \to 0.$$

Thus

$$\int_\Omega |\nabla u_k|^{p(x)-2}\nabla u_k(\nabla u_k - \nabla u)dx + \int_\Omega |\nabla u_k|^{q(x)-2}\nabla u_k(\nabla u_k - \nabla u)dx \to 0.$$

Since

$$\int_\Omega |\nabla u_k|^{p(x)-2}\nabla u_k(\nabla u_k - \nabla u)dx$$

and

$$\int_\Omega |\nabla u_k|^{q(x)-2}\nabla u_k(\nabla u_k - \nabla u)dx$$

have the same sign, then $\int_\Omega |\nabla u_k|^{p(x)-2}\nabla u_k(\nabla u_k - \nabla u)dx \to 0$. Using (S_+) property (see, [9]), we deduce that $u_k \to u$ strongly in $W^{1,p(x)}(\Omega)$, and the lemma follows. □

Proof: [Proof of case (i) of Theorem 1.1] Using Lemmas 3.1 and 3.2, we obtain

$$\max(\Phi_{\lambda,0}(0), \Phi_{\lambda,0}(e)) = \Phi_{\lambda,0}(0) < \inf_{\|u\|=\eta} \Phi_{\lambda,0}(u) =: \beta.$$

Taking into consideration Lemma 3.3 and Theorem 2.1, we deduce that there exits a nontrivial critical point of $\Phi_{\lambda,0}$ associated of the critical value given by

$$\inf_{\gamma\in\mathcal{P}}\sup_{t\in[0;1]}\Phi_{\lambda,0}(\gamma(t))\geq\beta,$$

Where

$$\mathcal{P}=\left\{\gamma\in C\left([0;1],W^{1,p(x)}(\Omega)\right);\quad \gamma(0)=0\quad\text{and}\quad\gamma(1)=e\right\}.$$

Therefore, problem (1.1) in this case, admits at least one nontrivial weak solution in $W^{1,p(x)}(\Omega)$. □

3.1.2. Proof of case (ii). For the proof of (ii) of Theorem 1.1, we will use the version of the symmetric mountain pass theorem (see Theorem 2.2). We organize this proof as follows:

Lemma 3.4 *For $\varepsilon=0$, assume that $r^+<q^-<q^+\leq p^+$. Then the functional $\Phi_{\lambda,0}$ is even, bounded from below, $\Phi_{\lambda,0}(0)=0$ and satisfies the Palais-Smale (PS) condition.*

Proof: It is easy to check that $\Phi_{\lambda,0}$ is even and $\Phi_{\lambda,0}(0)=0$.
Since $r^+<p^{\partial}(x)$ for all $x\in\overline{\Omega}$, the same reasoning applies in the proof of Lemma 3.1 give the following inequalities :

$$\Phi_{\lambda,0}(u)\geq\tfrac{1}{p^+}\|u\|^{p^-}-\tfrac{\lambda}{r^-}(C_1\|u\|^{r^+}+C_2\|u\|^{r^-}),\quad\text{if}\quad\|u\|\geq1;$$
$$\Phi_{\lambda,0}(u)\geq\tfrac{1}{p^+}\|u\|^{p^+}-\tfrac{\lambda}{r^-}(C_1\|u\|^{r^+}+C_2\|u\|^{r^-}),\quad\text{if}\quad\|u\|\leq1.$$

Since $r^+<p^-$, we see that $\Phi_{\lambda,0}$ is bounded from below and coercive. We are left with the task of showing that $\Phi_{\lambda,0}$ satisfies the *Palais-Smale* condition.

Let $(u_k)\subset W^{1,p(x)}(\Omega)$ be a sequence such that $(\Phi_{\lambda,0}(u_k))$ is bounded and $\Phi'_{\lambda,0}(u_k)\to0$. By the coercivity of $\Phi_{\lambda,0}$, the sequence (u_k) is bounded and similar arguments to those in the proof of Lemma 3.3 shows that $u_k\to u$ strongly in $W^{1,p(x)}(\Omega)$, and the lemma is proved. □

Lemma 3.5 *For $\varepsilon=0$, assume that $r^+<q^-<q^+\leq p^+$. Then for each $k\in\mathbb{N}^*$, there exists an $H_k\in\Gamma_k$ such that : $\sup_{u\in H_k}\Phi_{\lambda,0}(u)<0$.*

Proof: Let $v_1,v_2,\cdots,v_k\in C^\infty(\mathbb{R}^N)$ such that

$$\overline{\{x\in\partial\Omega;\ v_i(x)\neq0\}}\cap\overline{\{x\in\partial\Omega;\ v_j(x)\neq0\}}=\emptyset\ \text{ if } i\neq j$$

and

$$|\{x\in\partial\Omega;\ v_i(x)\neq0\}|>0,\ \forall i,j\in\{1,2,\cdots,k\}.$$

Take $F_k=span\{v_1,v_2,\cdots,v_k\}$, we have $\dim F_k=k$. Denote

$$S=\{v\in W^{1,p(x)}(\Omega);\ \|v\|=1\}$$

and for $0<t\leq1$, $H_k(t)=t(F_k\cap S)$. For all $t\in]0,1]$, we have $\gamma(H_k(t))=k$.

The task is now to prove that for any $k\in\mathbb{N}^*$, there exists $t_k\in]0,1]$ such that $\sup_{u\in H_k(t_k)}\Phi_{\lambda,0}(u)<0$.

Let $k\in\mathbb{N}^*$. For $0<t\leq1$; we have

$$\begin{aligned}
\sup_{u\in H_k(t)}\Phi_{\lambda,0}(u)&\leq\sup_{v\in F_k\cap S}\Phi_{\lambda,0}(tv)\\
&\leq\sup_{v\in F_k\cap S}\left\{\int_\Omega\frac{t^{p(x)}}{p(x)}|\nabla v|^{p(x)}dx+\int_\Omega\frac{t^{q(x)}}{q(x)}|\nabla v|^{q(x)}dx+\int_{\partial\Omega}\frac{t^{p(x)}}{p(x)}|v|^{p(x)}d\sigma\right.\\
&\quad\left.+\int_{\partial\Omega}\frac{t^{q(x)}}{q(x)}|v|^{q(x)}d\sigma-\lambda\int_{\partial\Omega}\frac{t^{r(x)}}{r(x)}|v|^{r(x)}d\sigma\right\}\\
&\leq\sup_{v\in F_k\cap S}\left\{\frac{t^{p^-}}{p^-}\rho_p(v)+\frac{t^{q^-}}{q^-}\rho_q(v)-\lambda\frac{t^{r^+}}{r^+}\int_{\partial\Omega}|v|^{r(x)}d\sigma\right\}\\
&\leq\sup_{v\in F_k\cap S}\left\{\frac{t^{p^-}}{p^-}+\frac{t^{q^-}}{q^-}\rho_q(v)-\lambda\frac{t^{r^+}}{r^+}\int_{\partial\Omega}|v|^{r(x)}d\sigma\right\}.
\end{aligned}$$

As $W^{1,p(x)}(\Omega)$ is continuously embedded in $W^{1,q(x)}(\Omega)$, there exists a positive constant c_0 such that $\|v\|_{q(x)} \le c_0\|v\|$, for all $v \in W^{1,p(x)}(\Omega)$ and since $v \in S$, we conclude that $\|v\| = 1$ which means that $\|v\|_{q(x)} \le c_0$. Thus $\left\|\frac{v}{c_0}\right\|_{q(x)} \le 1$. We get then by Proposition 2.3

$$\rho_q\left(\frac{v}{c_0}\right) = \int_\Omega \frac{|\nabla v|^{q(x)}}{c_0^{q(x)}}dx + \int_{\partial\Omega} \frac{|v|^{q(x)}}{c_0^{q(x)}}d\sigma \le 1. \tag{3.3}$$

We know that $c_0^{q(x)} \le \max(c_0^{q^-}, c_0^{q^+}) := M$, which implies that

$$\rho_q\left(\frac{v}{c_0}\right) \ge \frac{\rho_q(v)}{M}. \tag{3.4}$$

From (3.3) and (3.4), we deduce that

$$\rho_q(v) \le M.$$

Let $C_3 = \min_{v\in F_k\cap S} \int_{\partial\Omega} |v|^{r(x)}d\sigma > 0$. Since $1 < q^- \le p^-$ and $0 < t \le 1$, an easy computation shows that

$$\begin{aligned}\sup_{u\in H_k(t)} \Phi_{\lambda,0}(u) \le& \frac{t^{q^-}}{q^-} + \frac{t^{q^-}M}{q^-} - \frac{\lambda C_3 t^{r^+}}{r^+}\\ \le& t^{q^-}\left(\frac{1+M}{q^-} - \frac{\lambda C_3}{r^+ t^{q^- - r^+}}\right).\end{aligned}$$

Since $r^+ < q^-$, we may choose $t_k \in]0;1]$ small enough such that

$$\frac{1+M}{q^-} - \frac{\lambda C_3}{r^+ t_k^{q^- - r^+}} < 0.$$

It follows immediately that

$$\sup_{u\in H_k(t_k)} \Phi_{\lambda,0}(u) < 0.$$

which is the desired conclusion. □

Proof: [Proof of case (ii) of Theorem 1.1] Taking into consideration Lemmas 3.4, 3.5 and Theorem 2.2, we deduce that the problem (1.1) admits a sequence of non trivial weak solutions (u_k) , such that $\Phi_{\lambda,0}(u_k) < 0$ and $\lim_{k\to\infty} u_k = 0$. □

3.2. Proof of Theorem 1.2

The rest of our paper is dedicated to the study of problem 1.1 in the case when $\varepsilon = 1$.

3.2.1. Proof of case (i). We give the energy functional associated to problem (1.1) in this case as follows

$$\begin{aligned}\Phi_{\lambda,1}(u) =& \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx + \int_\Omega \frac{|\nabla u|^{q(x)}}{q(x)}dx + \int_{\partial\Omega} \frac{|u|^{p(x)}}{p(x)}d\sigma + \int_{\partial\Omega} \frac{|u|^{q(x)}}{q(x)}d\sigma\\ &- \lambda\int_{\partial\Omega}\left(\frac{|u|^{r(x)}}{r(x)} - \frac{|u|^{s(x)}}{s(x)}\right)d\sigma,\end{aligned}$$

and we have

$$\begin{aligned}\langle\Phi'_{\lambda,1}(u), v\rangle =& \int_\Omega \left(|\nabla u|^{p(x)-2} + |\nabla u|^{q(x)-2}\right)\nabla u\nabla v dx + \int_{\partial\Omega}\left(|u|^{p(x)-2} + |u|^{q(x)-2}\right)uv d\sigma\\ &- \lambda\int_{\partial\Omega}\left(|u|^{r(x)-2} - |u|^{s(x)-2}\right)uv d\sigma, \qquad \text{for any } u, v \in W^{1,p(x)}(\Omega).\end{aligned}$$

As in the proof of (i) of Theorem 1.1, this proof is based on the Theorem 2.1. For deducing that our functional $\Phi_{\lambda,1}$ satisfies the Palais-Smale condition and it has a mountain pass geometry, we need to show the following Lemmas.

Lemma 3.6 *For $\varepsilon = 1$, assume that $s^+ \leq p^+ < r^- < r^+ < p^\partial(x)$ for all $x \in \bar{\Omega}$. Then there exists $\eta, b > 0$ such that $\Phi_{\lambda,1}(u) \geq b$ for $u \in W^{1,p(x)}(\Omega)$ with $\|u\| = \eta$.*

Proof: Since $r^+ < p^\partial(x)$ for all $x \in \overline{\Omega}$, by Proposition 2.2 and 2.3, we have

$$\Phi_{\lambda,1}(u) \geq \frac{1}{p^+}\|u\|^{p^+} - \frac{\lambda}{r^-}\left(C_1\|u\|^{r^+} + C_2\|u\|^{r^-}\right) \quad \text{if} \quad \|u\| \leq 1.$$

Consequently if $\|u\| \leq 1$, then

$$\Phi_{\lambda,1}(u) \geq \|u\|^{p^+}\Big(\frac{1}{p^+} - \frac{\lambda}{r^-}(C_1\|u\|^{r^+-p^+} + C_2\|u\|^{r^--p^+})\Big).$$

Since $p^+ < r^- \leq r^+$, we conclude that the functional $h : [0,1] \to \mathbb{R}$ defined as follows

$$h(t) = \frac{1}{p^+} - \frac{\lambda C_1}{r^-}t^{r^+-p^+} - \frac{\lambda C_2}{r^-}t^{r^--p^+},$$

is positive on neighborhood of the origin and the proof of this lemma is complete. □

Lemma 3.7 *For $\varepsilon = 1$, assume that $s^+ \leq p^+ < r^- < r^+ < p^\partial(x)$ for all $x \in \overline{\Omega}$. Then there exists $e \in W^{1,p(x)}(\Omega)$ with $\|e\| > \eta$ such that $\Phi_{\lambda,1}(e) < 0$; where η is given in Lemma 3.6.*

Proof: Let φ be a positive function in $C_0^\infty(\overline{\Omega})$ such that $\varphi \geq 0$ and $\varphi \not\equiv 0$, on $\partial\Omega$. For $t > 1$, we have

$$\begin{aligned}
\Phi_{\lambda,1}(t\varphi) &= \int_\Omega \frac{t^{p(x)}}{p(x)}|\nabla\varphi|^{p(x)}dx + \int_\Omega \frac{t^{q(x)}}{q(x)}|\nabla\varphi|^{q(x)}dx + \int_{\partial\Omega} \frac{t^{p(x)}}{p(x)}|\varphi|^{p(x)}d\sigma + \int_{\partial\Omega} \frac{t^{q(x)}}{q(x)}|\varphi|^{q(x)}d\sigma \\
&\quad -\lambda\int_{\partial\Omega}\left(\frac{t^{r(x)}}{r(x)}|\varphi|^{r(x)} - \frac{t^{s(x)}}{s(x)}|\varphi|^{s(x)}\right)d\sigma \\
&\leq \frac{t^{p^+}}{p^-}\rho_p(\varphi) + \frac{t^{q^+}}{q^-}\rho_q(\varphi) - \frac{\lambda t^{r^-}}{r^+}\int_{\partial\Omega}|\varphi|^{r(x)}d\sigma + \frac{\lambda t^{s^+}}{s^-}\int_{\partial\Omega}|\varphi|^{s(x)}d\sigma.
\end{aligned}$$

As $s^+ \leq p^+ < r^-$ and $q^+ \leq p^+$, it follows that $\lim\limits_{t\to+\infty}\Phi_{\lambda,1}(t\varphi) = -\infty$. So, for all $\epsilon > 0$ there exists $\alpha > 0$ such that for $t > \alpha$, $\Phi_{\lambda,1}(t\varphi) < -\epsilon < 0$, thus the proof is complete. □

Lemma 3.8 *For $\varepsilon = 1$, assume that $s^+ \leq p^+ < r^- < r^+ < p^\partial(x)$ for all $x \in \overline{\Omega}$. Then the functional $\Phi_{\lambda,1}$ satisfies the Palais-Smale condition.*

Proof: Let $(u_k) \subset W^{1,p(x)}(\Omega)$ be such that $C = \sup\limits_{k\in\mathbb{N}^*}\Phi_{\lambda,1}(u_k)$ and $\Phi'_{\lambda,1}(u_k) \to 0$. Assume by contradiction that $\|u_k\| \to \infty$, then there exists $k_0 \in \mathbb{N}$ such that $\|u_k\| > 1$ for any $k > k_0$. We have

$$\begin{aligned}
C + \|u_k\| \geq & \Phi_{\lambda,1}(u_k) - \frac{1}{r^-}\langle \Phi'_{\lambda,1}(u_k), u_k\rangle \\
\geq & \int_\Omega \frac{|\nabla u_k|^{p(x)}}{p(x)}dx + \int_\Omega \frac{|\nabla u_k|^{q(x)}}{q(x)}dx + \int_{\partial\Omega} \frac{|u_k|^{p(x)}}{p(x)}d\sigma + \int_{\partial\Omega} \frac{|u_k|^{q(x)}}{q(x)}d\sigma \\
& - \lambda \int_{\partial\Omega} \left(\frac{|u_k|^{r(x)}}{r(x)} - \frac{|u_k|^{s(x)}}{s(x)} \right) d\sigma - \frac{1}{r^-}\int_\Omega \left(|\nabla u_k|^{p(x)} + |\nabla u_k|^{q(x)} \right) dx \\
& - \frac{1}{r^-}\int_{\partial\Omega} (|u_k|^{p(x)} + |u_k|^{q(x)}) d\sigma + \frac{\lambda}{r^-}\int_{\partial\Omega} \left(|u_k|^{r(x)} - |u_k|^{s(x)} \right) d\sigma \\
\geq & \left(\frac{1}{p^+} - \frac{1}{r^-} \right) \rho_p(u_k) + \left(\frac{1}{q^+} - \frac{1}{r^-} \right) \rho_q(u_k) + \lambda \int_{\partial\Omega} \left(\frac{1}{r^-} - \frac{1}{r(x)} \right) |u_k|^{r(x)} d\sigma \\
& + \lambda \int_{\partial\Omega} \left(\frac{1}{s(x)} - \frac{1}{r^-} \right) |u_k|^{s(x)} d\sigma \\
\geq & \left(\frac{1}{p^+} - \frac{1}{r^-} \right) \rho_p(u_k) \\
\geq & \left(\frac{1}{p^+} - \frac{1}{r^-} \right) \|u_k\|^{p^-}.
\end{aligned}$$

Letting $n \to \infty$ and dividing by $\|u_k\|^{p^-}$ in the above inequality, we obtain a contradiction. Therefore (u_k) is bounded in $W^{1,p(x)}(\Omega)$. From the fact that $W^{1,p(x)}(\Omega)$ is a reflexive space (Proposition 2.2), we deduce that, for a subsequence still denoted (u_k), we have $u_k \rightharpoonup u$ weakly in $W^{1,p(x)}(\Omega)$ and $u_k \to u$ strongly in $L^{p(x)}(\partial\Omega)$, $L^{q(x)}(\partial\Omega)$, $L^{r(x)}(\partial\Omega)$ and in $L^{s(x)}(\partial\Omega)$ respectively (Proposition 2.2).

In consequence

$$\langle \Phi'_{\lambda,1}(u_k), u_k - u\rangle \to 0,$$

and

$$\int_{\partial\Omega} \left(|u_k|^{p(x)-2} + |u_k|^{q(x)-2} \right) u_k(u_k - u) d\sigma - \lambda \int_{\partial\Omega} \left(|u_k|^{r(x)-2} - |u_k|^{s(x)-2} \right) u_k(u_k - u) d\sigma \to 0.$$

Thus

$$\int_\Omega |\nabla u_k|^{p(x)-2}\nabla u_k(\nabla u_k - \nabla u)dx + \int_\Omega |\nabla u_k|^{q(x)-2}\nabla u_k(\nabla u_k - \nabla u)dx \to 0.$$

The same approach used in the proof of Lemma 3.3, give us $u_k \to u$ strongly in $W^{1,p(x)}(\Omega)$, Which completes the proof. □

Proof: [Proof of case (i) of Theorem 1.2] From Lemmas 3.6-3.8 and Theorem 2.1, we deduce that there exists a nontrivial critical point for functional $\Phi_{\lambda,1}$, thus, the problem (1.1) has at least one nontrivial weak solution $u_0 \in W^{1,p(x)}(\Omega)$, with

$$\Phi_{\lambda,1}(u_0) = \inf_{\gamma \in \mathcal{P}} \sup_{t\in[0,1]} \Phi_{\lambda,1}(\gamma(t)).$$

□

3.2.2. Proof of case (ii). Our proof is based on the version of Recciri Theorem (see Theorem 2.3).

Along this proof for brevity, X denotes the generalized Sobolev space $W^{1,p(x)}(\Omega)$. In order to apply Recciri's result we define the functionals $\Phi, \Psi : X \to \mathbb{R}$ by

$$\Phi(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx + \int_\Omega \frac{|\nabla u|^{q(x)}}{q(x)}dx + \int_{\partial\Omega} \frac{|u|^{p(x)}}{p(x)}d\sigma + \int_{\partial\Omega} \frac{|u|^{q(x)}}{q(x)}d\sigma,$$

and

$$\Psi(u) = -\int_{\partial\Omega}\left(\frac{|u|^{r(x)}}{r(x)} - \frac{|u|^{s(x)}}{s(x)}\right)d\sigma.$$

It's clear that $\Phi, \Psi \in C^1(X,\mathbb{R})$ and

$$\langle \Phi'(u), v\rangle = \int_\Omega \left(|\nabla u|^{p(x)-2} + |\nabla u|^{q(x)-2}\right)\nabla u \nabla v dx + \int_{\partial\Omega}\left(|u|^{p(x)-2} + |u|^{q(x)-2}\right)uvd\sigma,$$

$$\langle \Psi'(u), v\rangle = -\int_{\partial\Omega}\left(|u|^{r(x)-2} - |u|^{s(x)-2}\right)uvd\sigma.$$

Hence it is easy to see that the critical points of $\Phi + \lambda\Psi$ are weak solutions to problem (1.1) when $\varepsilon = 1$ and we use this fact in the search concerning weak solutions from the next paragraphs.

Our proof starts with proving the following theorem, which gives some properties of Φ'.

Theorem 3.1 *The following statements holds :*

1. *Φ' is continuous, bounded and strictly monotone;*
2. *Φ' is of type (S_+);*
3. *Φ' is an homeomorphism.*

Proof:

1. Since Φ' is the Fréchet derivative of Φ, it follows that Φ' is continuous.

   Using the elementary inequalities (see, [21])

$$\begin{array}{ll} |x-y|^\gamma \le 2^\gamma\left(|x|^{\gamma-2}x - |y|^{\gamma-2}y\right)(x-y) & ,\text{if} \quad \gamma \ge 2, \\ |x-y|^2 \le \frac{1}{(\gamma-1)}(|x|+|y|)^{2-\gamma}\left(|x|^{\gamma-2}x - |y|^{\gamma-2}y\right)(x-y) & ,\text{if} \quad 1<\gamma<2, \end{array}$$

   for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$. We obtain for all $u, v \in X$ such that $u \ne v$

$$\begin{aligned} \langle \Phi'(u) - \Phi'(v), u-v\rangle &= \int_\Omega (|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)(\nabla u - \nabla v)dx \\ &+ \int_\Omega (|\nabla u|^{q(x)-2}\nabla u - |\nabla v|^{q(x)-2}\nabla v)(\nabla u - \nabla v)dx \\ &+ \int_{\partial\Omega} (|u|^{p(x)-2}u - |v|^{p(x)-2}v)(u-v)d\sigma \\ &+ \int_{\partial\Omega} (|u|^{q(x)-2}u - |v|^{q(x)-2}v)(u-v)d\sigma > 0, \end{aligned}$$

   which means that Φ' is strictly monotone.

2. Let $(u_n)_n$ be a sequence of X such that $u_n \rightharpoonup u$ weakly in X as $n \to +\infty$ and $\limsup_{n\to+\infty}\langle \Phi'(u_n), u_n - u\rangle \le 0$.

   Since $q(x) \le p(x) < p^\partial(x)$ for any $x \in \bar{\Omega}$, then by Proposition 2.2 the embeddings from $W^{1,p(x)}(\Omega)$ to $L^{p(x)}(\partial\Omega)$ and to $L^{q(x)}(\partial\Omega)$ are compact. So we have

$$\int_{\partial\Omega} |u_n|^{p(x)-2}u_n(u_n - u)d\sigma \to 0,$$

   and

$$\int_{\partial\Omega} |u_n|^{q(x)-2}u_n(u_n - u)d\sigma \to 0.$$

   Thus

$$\int_\Omega |\nabla u_k|^{p(x)-2}\nabla u_k(\nabla u_k - \nabla u)dx + \int_\Omega |\nabla u_k|^{q(x)-2}\nabla u_k(\nabla u_k - \nabla u)dx \to 0.$$

   It follows by the same method as in Lemma 3.3 that $u_k \to u$ strongly in X.

3. Finally, show that Φ' is an homeomorphism : We know that the strict monotonicity of Φ' implies its injectivity. To show that Φ' is a surjection, it suffices to show that it is coercive. Indeed Φ' is a coercive operator, using Proposition 2.3 and since $p^- - 1 > 0$, we obtain then for each $u \in X$ such that $\|u\| \geq 1$

$$\frac{\langle \Phi'(u), u\rangle}{\|u\|} = \frac{\int_\Omega |\nabla u|^{p(x)}dx + \int_\Omega |\nabla u|^{q(x)}dx + \int_{\partial\Omega} |u|^{p(x)}d\sigma + \int_{\partial\Omega} |u|^{q(x)}d\sigma}{\|u\|}$$
$$= \frac{\rho_p(u) + \rho_q(u)}{\|u\|} \geq \frac{\rho_p(u)}{\|u\|} \geq \frac{\|u\|^{p^-}}{\|u\|} \geq \|u\|^{p^- - 1} \to \infty \quad \text{as} \quad \|u\| \to \infty.$$

Consequently, the operator Φ' is a bijection, thus Φ' admits an inverse mapping. It remains to show that Φ'^{-1} is continuous. For that, let $(f_n)_n$ be a sequence of X' such that $f_n \to f$ in X' as $n \to +\infty$. Let u_n and u in X such that

$$\Phi'^{-1}(f_n) = u_n \quad \text{and} \quad \Phi'^{-1}(f) = u.$$

By the coercivity of Φ', we deduct that the sequence (u_n) is bounded in X. As X is reflexive (Proposition 2.2), for a subsequence still denoted (u_n), we have $u_n \rightharpoonup \hat{u}$ weakly in X as $n \to +\infty$, which implies

$$\lim_{n\to+\infty} \langle \Phi'(u_n) - \Phi'(u), u_n - \hat{u}\rangle = \lim_{n\to+\infty} \langle f_n - f, u_n - \hat{u}\rangle = 0.$$

It follows by the property (S_+) and by the continuity of Φ' that

$$u_n \to \hat{u} \quad \text{in } X \quad \text{and} \quad \Phi'(u_n) \to \Phi'(\hat{u}) = \Phi'(u) \quad \text{in } X' \quad \text{as } n \to +\infty.$$

Moreover, since Φ' is an injection, we conclude that $\hat{u} = u$. This completes the proof.

□

Now are now in a position to give the proof of our fourth main result.

Proof: [Proof of (ii) of Theorem 1.2] By Theorem 3.1, we have Φ is a continuously Gâteaux differentiable and sequentially weakly semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X'. Furthermore, Ψ is continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. It is clear that Φ is bounded on every bounded subset of X.

For $\|u\| \geq 1$, by Proposition 2.3, we have

$$\begin{aligned}
\Phi(u) &= \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx + \int_\Omega \frac{|\nabla u|^{q(x)}}{q(x)}dx + \int_{\partial\Omega} \frac{|u|^{p(x)}}{p(x)}d\sigma + \int_{\partial\Omega} \frac{|u|^{q(x)}}{q(x)}d\sigma \\
&\geq \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx + \int_{\partial\Omega} \frac{|u|^{p(x)}}{p(x)}d\sigma \\
&\geq \frac{1}{p^+}\rho_p(u) \\
&\geq \frac{1}{p^+}\|u\|^{p^-},
\end{aligned}$$

moreover, for all $u \in X$ and for each $\lambda > 0$,

$$\begin{aligned}
\lambda\Psi(u) &= -\lambda \int_{\partial\Omega} \left(\frac{|u|^{r(x)}}{r(x)} - \frac{|u|^{s(x)}}{s(x)} \right) d\sigma \\
&\geq -\lambda \int_{\partial\Omega} \frac{|u|^{r(x)}}{r(x)}d\sigma \\
&\geq -\frac{\lambda}{r^-}(C_1\|u\|^{r+} + C_2\|u\|^{r-}).
\end{aligned}$$

We conclude from the two inequalities above that

$$\Phi(u)+\lambda\Psi(u)\geq\frac{1}{p^+}\|u\|^{p^-}-\frac{\lambda}{r^-}(C_1\|u\|^{r+}+C_2\|u\|^{r-}).$$

Since $r^+<p^-$, it follows that

$$\lim_{\|u\|\to+\infty}(\Phi(u)+\lambda\Psi(u))=+\infty\quad\forall\lambda\in[0,+\infty),$$

and finally that assumption (1) of Theorem 2.3 is verified.

we proceed to show that assumption (2) is also satisfied. For that purpose, we introduce the function $F:\overline{\Omega}\times[0,+\infty[\to\mathbb{R}$ defined by $F(x,t)=\frac{t^{r(x)}}{r(x)}-\frac{t^{s(x)}}{s(x)}$.
We see at once that $F(x,.)\in C^1(]0,+\infty[,\mathbb{R})$ for all $x\in\overline{\Omega}$. It's derivative with respect to t is the function $F_t:\bar{\Omega}\times(0,+\infty)\to\mathbb{R}$ defined by $F_t(x;t)=t^{s(x)-1}(t^{r(x)-s(x)}-1)$.
Therefore, for all $x\in\overline{\Omega}$, we have :

$$F_t(x,t)\geq 0,\quad\text{if } t\geq 1\qquad\text{and}\qquad F_t(x,t)\leq 0,\quad\text{if } 0<t\leq 1.$$

We conclude that for any $x\in\overline{\Omega}$, $F(x,.)$ is increasing on $(1,+\infty)$ and decreasing on $(0,1)$. Since $s^+<r^-$, it follows that $\lim_{t\to+\infty}F(x,t)=+\infty$ for all $x\in\overline{\Omega}$. Moreover, for all $x\in\overline{\Omega}$, $F(x,t)=\frac{s(x)t^{r(x)}-r(x)t^{s(x)}}{r(x).s(x)}=0$ if and only if $t=t_0=0$ or $t=t_x=\left(\frac{r(x)}{s(x)}\right)^{\frac{1}{r(x)-s(x)}}$. From what has been proved, we deduce that for all $x\in\overline{\Omega}$, $F(x,t)\leq 0$ if $0\leq t\leq t_x$ and $F(x,t)>0$ if $t>t_x$.

Let a,b be two real numbers such that $0<a<\min(1,c)$, with c given in Remark 2.1 and $b>\max\left((\frac{r^+}{s^-})^{\frac{1}{r^--s^+}},(\frac{1}{|\partial\Omega|})^{\frac{1}{p^-}}\right)$.

Consider $u_0,u_1\in X$, $u_0(x)=0$, $u_1(x)=b$, for any $x\in\Omega$.
Remark 2.1 gives $u_0(x)=0$ and $u_1(x)=b$, for any $x\in\overline{\Omega}$. We thus get

$$\int_{\partial\Omega}\sup_{0\leq t\leq a}F(x,t)d\sigma\leq 0<\int_{\partial\Omega}F(x,b)d\sigma.$$

We set $\alpha=\frac{1}{p^+}(\frac{a}{c})^{p^+}$, we have $\alpha\in(0,1)$ and $\Phi(u_0)=-\Psi(u_0)=0$.
$\Phi(u_1)=\int_{\partial\Omega}\frac{1}{p(x)}b^{p(x)}d\sigma+\int_{\partial\Omega}\frac{1}{q(x)}b^{q(x)}d\sigma\geq\int_{\partial\Omega}\frac{1}{p(x)}b^{p(x)}d\sigma\geq\frac{1}{p^+}b^{p^-}|\partial\Omega|>\frac{1}{p^+}(\frac{a}{c})^{p^+}=\alpha$,
$\Psi(u_1)=-\int_{\partial\Omega}F(x,b)d\sigma$.
Thus we deduce that $\Phi(u_0)<\alpha<\Phi(u_1)$, which is the condition (2) of Theorem 2.3.

What is left is to show that assumption (3) is satisfied. We have

$$-\frac{(\Phi(u_1)-\alpha)\Psi(u_0)+(\alpha-\Phi(u_0))\Psi(u_1)}{\Phi(u_1)-\Phi(u_0)}=-\alpha\frac{\Psi(u_1)}{\Phi(u1)}=\alpha\frac{\int_{\partial\Omega}F(x,b)d\sigma}{\int_{\partial\Omega}\frac{1}{p(x)}b^{p(x)}d\sigma+\int_{\partial\Omega}\frac{1}{q(x)}b^{q(x)}d\sigma}>0.$$

Let $u\in X$ with $\Phi(u)\leq\alpha<1$. By Proposition 2.3, we have

$$\frac{1}{p^+}\|u\|^{p^+}\leq\frac{1}{p^+}\rho_p(u)\leq\Phi(u)\leq\alpha=\frac{1}{p^+}(\frac{a}{c})^{p^+}<1.$$

Remark 2.1 shows that for every $u\in X$ with $\Phi(u)\leq\alpha$,

$$|u(x)|\leq c\|u\|\leq c(p^+\alpha)^{\frac{1}{p^+}}=a,\quad\forall x\in\bar{\Omega}.$$

Which leads to

$$-\inf_{u\in\Phi^{-1}((-\infty,\alpha])}\Psi(u)=\sup_{u\in\Phi^{-1}((-\infty,\alpha])}-\Psi(u)\leq\int_{\partial\Omega}\sup_{0\leq t\leq a}F(x,t)d\sigma\leq 0.$$

hence

$$-\inf_{u\in\Phi^{-1}((-\infty,\alpha])}\Psi(u) < \alpha\frac{\int_{\partial\Omega}F(x,b)d\sigma}{\int_{\partial\Omega}\frac{1}{p(x)}b^{p(x)}d\sigma+\int_{\partial\Omega}\frac{1}{q(x)}b^{q(x)}d\sigma},$$

it means that

$$\inf_{u\in\Phi^{-1}((-\infty,\alpha])}\Psi(u) > \frac{(\Phi(u_1)-\alpha)\Psi(u_0)+(\alpha-\Phi(u_0))\Psi(u_1)}{\Phi(u_1)-\Phi(u_0)},$$

and hence condition (3) is verified.

Applying Theorem 2.3, we conclude that there exists an open interval $\wedge \subset (0,+\infty)$ and a positive constant $\rho_0 > 0$ such that for any $\lambda \in \wedge$ the equation $\Phi'(u) + \lambda\Psi'(u) = 0$ has at least three solutions in X whose norms are less than ρ_0. Which completes the proof of (ii) of Theorem 1.2. □

Acknowledgments

We would like to thank the referee for carefully reading our manuscript and for giving such constructive comments which substantially helped improving the quality of the paper. In this revised version of the manuscript we have tried to consider all the points that were raised.

References

1. A. Anane, O. Chakrone, A. Zerouali, Belhadj Karim, *Existence of solutions for a Steklov problem ivolving the $p(x)$-Laplacian.* Bol. Soc. Paran. Mat. (3s)v. 32 1, 207-215, (2014).
2. S. N. Antontsev, J. F. Rodrigues, *On stationary thermorheological viscous flows.* Ann. Univ. Ferrara Sez. VII Sci. Mat. 52, 19-36, (2006).
3. G. Bonanno, P. Candito, *Three solutions to a Neumann problem for elliptic equations involving the p-Laplacian.* Arch. Math. (Basel) 80, 424-429, (2003).
4. Y. Chen, S. Levine, R. Ran, *Variable exponent, linear growth functionals in image restoration.* Ann. Math. Sinica. 17A(5), 557-564, (1996).
5. S. G. Deng, *Eigenvalues of the $p(x)$-Laplacian Steklov problem.* J. Math. Anal. Appl. 339, 925-937, (2008).
6. S. G. Deng, *Positive solutions for Robin problem involving the $p(x)$-Laplacian.* J. Math. Anal. Appl. 360, 548-560, (2009).
7. D. Edmunds, J. Răkosnic, *Sobolev embeddings with variable exponent.* Studia Math. 143, 267-293, (2000).
8. X. L. Fan, X. Y. Han, *Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$.* Nonlinear Anal. 59, 173-188, (2004).
9. X. L. Fan, Q. H. Zhang, *Existence of solutions for $p(x)$-Laplacian Dirichlet problems.* Nonlinear Anal. 52, 1843–1852, (2003).
10. X. L. Fan, D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$.* J. Math. Anal. App. 263, 424-446, (2001).
11. G. Fragnelli, *Positive periodic solutions for a system of anisotropic parabolic equations.* J. Math. Anal. Appl. 73. 110-121, (2010).
12. Y. Jabri, *The mountain Pass Theorem. Variants, Generalizations and Some Application.* Cambridge University Press, 2003.
13. R. Kajikia, *A critical point theorem related to the symmetric mountain pass lemma and its applications to elliptic equations.* J. Funct. Anal. 225, 352-370, (2005).
14. B. Karim, A. Zerouali, O. Chakrone, *Existence and multiplicity of a-harmonic solutions for a Steklov problem with variable exponents.* Bol. Soc. Paran. Mat. (3s)v. 36 2, 125-136, (2018).
15. O. Kovácik, J. Rákosnik, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$.* Czechoslovak Math. J. 41, 592-618, (1991).
16. G. Li, G. Zhang, *Multiple solutions for the (p,q)-Laplacian problem with critical exponent.* Acta Mathematica Scientia. 29B , No.4, 903-918, (2009).
17. M. Mihailescu, *Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$-Laplace operator.* Nonlinear Anal. 67, 1419-1425, (2007).
18. M. Mihailescu, V. Radulescu, *On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent.* Proc. Amer Math. Soc. 135, 2929-2937, (2007).
19. D. Motreanu, M. Tanaka, *Generalized eigenvalue problems of nonhomogeneous elliptic operators and their application.* Pacific J. Math. 265, 151-184, (2013).

20. M. Ruzicka, *Electrorheological Fluids: Modeling and Mathematical Theory*. Springer-verlag, Berlin, 2002.

21. J. Simon, *Régularité de la solution d'une équation non linéaire dans* $\mathbb{R}^N$. vol. 665 of Lecture Notes in Math., Springer, Berlin, 205-227, (1978).

22. V. V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*. Izv. Akad. Nauk SSSR Ser. Mat. 50, 675-710, (1986).

Abdessamad Lakhdi

University Moulay Ismail, Faculty of Sciences and Technics, Errachidia, Morocco.

E-mail address: lakhdi.abdessamad@gmail.com

and

Belhadj Karim

University Moulay Ismail, Faculty of Sciences and Technics, Errachidia, Morocco.

E-mail address: karembelf@gmail.com

and

Abdellah ZEROUALI

Regional Centre of Trades Education and Training, Oujda, Morocco.

E-mail address: abdellahzerouali@yahoo.fr