Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–11.
ISSN-0037-8712
doi:10.5269/bspm.62215

On Inverse Problems for the Generalized Kawahara Equation With Integral Overdetermination *

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ABSTRACT: In the following paper we study the inverse problems for the generalized Kawahara equation and its linearized analog in a bounded domain with integral overdetermination. As the control function we consider either a special form of the right-hand side of the equation or the value of the derivative of the solution on one of the boundaries. In order to achieve the controllability of the generalized Kawahara equation we impose smallness conditions on either the time interval or the input data. Those restrictions are absent in linear case.

Key Words: Kawahara equation, inverse problems, integral overdetermination, initial-boundary value problem.

Contents

1. Introduction

In the following paper we consider an initial-boundary value problem for the generalized Kawahara equation:

$$u_t - u_{xxxxx} + bu_{xxx} + au_x + F'(u)u_x = f(t,x), \tag{1.1}$$

$u = u(t,x)$, $a, b \in \mathbb{R}$, posed on a rectangle $Q_T = (0,T) \times (0,R)$, where $T, R > 0$.

First introduced in Ref. [10] the Kawahara equation describes the propagation of long nonlinear waves in weakly dispersive media. It is a modification of the well-known Korteweg–de Vries equation:

$$u_t + u_{xxx} + au_x + uu_x = 0. \tag{1.2}$$

We consider equation (1.1) with the initial condition:

$$u(0,x) = u_0(x), \quad x \in [0,R], \tag{1.3}$$

and boundary conditions:

$$\begin{gathered} u(t,0) = \mu(t), \quad u(t,R) = \nu(t), \\ u_x(t,0) = \theta(t), \quad u_x(t,R) = h(t), \\ u_{xx}(t,R) = \sigma(t), \quad t \in [0,T]. \end{gathered} \tag{1.4}$$

The function $F(u) \in C^1(\mathbb{R})$ satisfies the following inequality:

$$| F(u) | \leq c | u |^q, \tag{1.5}$$

where $c > 0$ and $1 < q < 6$.

* The work was supported by the Ministry of Science and Higher Education of Russian Federation: agreement no. 075-03-2020-223/3 (FSSF-2020-0018).
2010 *Mathematics Subject Classification:* 35Q93.
Submitted January 24, 2022. Published December 04, 2025

For the overdetermination condition we take the integral condition

$$\int_0^R u(t,x)\omega(x)dx = \varphi(t), \quad t \in [0,T], \tag{1.6}$$

where ω and φ are known functions. We will consider either the boundary function σ or the special type of the right-hand side of the equation f as a control.

According to Ref. [11], there are two types of the overdetermination in the theory of inverse problems: integral condition, such as (1.6), and terminal condition, that is *for the given* $T > 0$ *and function* u_T *to find the solution of the considered problem that satisfies*

$$u(T,x) = u_T.$$

Problems with terminal conditions are also known as controllability problems.

Through the years there was a wide range of investigations dedicated to the inverse problems with terminal overdetermination for Kawahara equation. The very interesting results were obtained in Ref. [9]. The authors studied Cauchy problem with boundary conditions for Kawahara equation, where the right-hand side of equation $F(u)$ was a cubic polynomial, and established local controllability to the trajectories.

Another research was conducted in Ref. [12,13], where the authors studied the Kawahara equation posed on a periodic domain with a distributed control, where the function h was considered as the control input (see (1.5) in Ref. [12]).

In Ref. [4] the Kawahara equation posed on a bounded interval with a distributed control was considered and a Carleman estimate for the Kawahara equation with internal observation was established. Using this estimate the author introduced the conditions when the Kawahara equation is null controllable.

Recently in Ref. [1] controllability in a weighted L_2 and regional controllability in L_2 Sobolev spaces have been obtained for the Kawahara equation. The authors also established the well-posedness of the problem in a weighted L_2.

According to Ref. [11], integral overdetermination conditions, have physical meaning, thus it's worth studying.

Our methods and results are very similar to those given in Ref. [5], where the author studied the inverse problem for Korteweg-de Vries equation. Absolutely the same methods were used in Ref. [2], where the authors investigated the solvability of inverse problems for the Kawahara equation with integral overdetermination. It is necessary to mention, that this paper resurfaced after our research was already completed. Nevertheless, in our work we studied a more general nonlinearity, while in Ref. [2] nonlinearity was given by the term $u_x u$.

The inverse problem for Korteweg-de Vries on unbounded domains with integral overdetermination was also studied in Ref. [7]. Using this technique the same results for Kawahara equation were obtained in Ref. [3].

Moreover, recently the inverse problem for the multi-dimensional generalizations of Korteweg-de Vries equation (also known as Zakharov-Kuznetsov equation) was studied in Ref. [6].

We will consider an initial-boundary value problem in the functional space:

$$X(Q_T) = C([0,T]; L_2(0,R) \cap L_2(0,T; H^2(0,R))), \tag{1.7}$$

with a norm

$$\|u\|_{X(Q_T)} = \sup_{t\in[0,T]} \|u(t,\cdot)\|_{L_2(0,R)} + \|u_{xx}\|_{L_2(Q_T)}. \tag{1.8}$$

For any interval $I \subset \mathbb{R}$ define the fractional order Sobolev space $H^s(I), s \in \mathbb{R}$,

$$H^s(\mathbb{R}) = \{f : \mathfrak{F}^{-1}[(1+\xi^2)^{s/2}\hat{f}(\xi)] \in L_2(\mathbb{R})\}, \tag{1.9}$$

where $\hat{f} = \mathfrak{F}[f]$ is a direct Fourier transform, and $\hat{f} = \mathfrak{F}^{-1}[f]$- inverse, $H^0(I) = L_2(I)$, $H^k(I) = W_2^k(I)$, $k \in \mathbb{N}$. Impose the following conditions to the function ω:

$$\omega \in H^5(0,R), \quad \omega(0) = \omega(R) = \omega'(0) = \omega'(R) = \omega''(0) = 0. \tag{1.10}$$

The first inverse problem: for given $u_0, \mu, \nu, \theta, h, f$, it is required to find a function σ such that the solution of the problem (1.1) -(1.4) satisfies the condition (1.6).

Theorem 1.1 *Let $u_0 \in L_2(0,R)$, $\varphi \in W_p^1(0,T)$, $\mu,\nu \in (H^{2/5} \bigcap L_p)(0,T)$, h,$\theta \in (H^{1/5} \bigcap L_p)(0,T)$, $f \in L_p(0,T;L_2(0,R))$ for some $p \in [2,+\infty]$, inequality (1.5) be verified, the condition (1.10) is satisfied and, moreover, $\omega''(R) \neq 0$,*

$$\varphi(0) = \int_0^R u_0(x)\omega(x)dx. \tag{1.11}$$

Let

$$\begin{gathered} c_0 = \|u_0\|_{L_2(0,R)} + \|\mu\|_{H^{2/5}(0,T)} + \|\nu\|_{H^{2/5}(0,T)} + \\ \|h\|_{H^{1/5}(0,T)} + \|\theta\|_{H^{1/5}(0,T)} + \|f\|_{L_2(Q_T)} + \|\varphi'\|_{L_2(0,T)}. \end{gathered} \tag{1.12}$$

Then

1). For fixed c_0 there exists a $T_0 > 0$ such as if $T \in (0,T_0)$, then there exists a unique function $\sigma \in L_p(0,T)$ and the corresponding unique solution $u \in X(Q_T)$ of the problem (1.1)-(1.4) with condition (1.6).

2). For fixed T there exists a $\delta > 0$ such as if $c_0 \leq \delta$, then there exists a unique function $\sigma \in L_p(0,T)$ and the corresponding unique solution $u \in X(Q_T)$ of the problem (1.1)-(1.4) with condition (1.6).

Now let us assume, that the right-hand side of (1.1) is given by

$$f(t,x) \equiv f_0(t)g(t,x). \tag{1.13}$$

In the second inverse problem, for given functions $u_0, \mu, \nu, h, \theta, \sigma, g$, it is required to find a function f_0, such that the solution of the problem (1.1) -(1.4) satisfies the condition (1.6).

Theorem 1.2 *Let $u_0 \in L_2(0,R)$, $\varphi \in W_p^1(0,T)$, μ,$\nu \in (H^{2/5} \bigcap L_p)(0,T)$, h, $\theta \in (H^{1/5} \bigcap L_p)(0,T)$, $\sigma \in L_{max(2,p)}(0,T)$ for some $p \in [1,+\infty]$ the condition (1.10) is satisfied and there exists a positive constant g_0 such as for all $t \in [0,T]$*

$$g_0 \leq \left| \int_0^R g(t,x)\omega(x)dx \right| . \tag{1.14}$$

Let

$$\begin{gathered} c_0 = \|u_0\|_{L_2(0,R)} + \|\mu\|_{H^{2/5}(0,T)} + \|\nu\|_{H^{2/5}(0,T)} + \\ \|h\|_{H^{1/5}(0,T)} + \|\theta\|_{H^{1/5}(0,T)} + \|\sigma\|_{L_2(0,T)} + \|\varphi'\|_{L_1(0,T)}. \end{gathered} \tag{1.15}$$

Then

1). For fixed c_0 there exists a $T_0 > 0$ such as if $T \in (0,T_0)$, then there exists a unique function $f_0 \in L_p(0,T)$ and the corresponding unique solution $u \in X(Q_T)$ of the problem (1.1)-(1.4) with condition (1.6).

2). For fixed T there exists a $\delta > 0$ such as if $c_0 \leq \delta$, then there exists a unique function $f_0 \in L_p(0,T)$ and the corresponding unique solution $u \in X(Q_T)$ of the problem (1.1)-(1.4) with condition (1.6).

2. On solutions of initial-boundary value problem

In this selection we establish some results on solutions of initial-boundary value problem for Kawahara equation. Let us begin with giving the definition of weak solution of the problem (1.1)-(1.4). (1.1)-(1.4).

Definition 2.1 Let $u_0 \in L_2(0,R)$, $\mu, \nu, h, \theta, \sigma \in L_2(0,T)$, $f \equiv f_1 + f_{2x}$ where $f_1 \in L_1(0,T;L_2(0,R))$,$f_2 \in L_1(0,T;L_2(0,R))$. The function $u \in L_\infty(0,T;L_2(0,R))$ can be defined as a weak solution of problem (1.1)-(1.4), if for any $\phi \in L_2(0,T;H^5(0,R))$, such as $\phi_t \in L_2(Q_T)$, $\phi\big|_{t=T} = \phi\big|_{x=0} = \phi\big|_{x=R} =$

$\phi_x\big|_{x=0} = \phi_x\big|_{x=R} = \phi_{xx}\big|_{x=0} = 0$, the following relation is satisfied:

$$\begin{gathered}\int_0^R u_0\phi\big|_{t=0}dx+\\ \iint_{Q_T}(u\phi_t - u\phi_{xxxxx} + bu\phi_{xxx} + au\phi_x + F(u)\phi_x + f_1\phi - f_2\phi_x)dxdt+\\ \int_0^T(\sigma(t)\phi(R)_{xx} - h(t)\phi(R)_{xxx} + \theta(t)\phi_{xxx}(0)\\ +\nu(t)\phi_{xxxx}(R) - \mu\phi_{xxxx}(0) - b\nu(t)\phi_{xx}(R))dt = 0.\end{gathered} \tag{2.1}$$

The solution existing by Theorem 2.1 of the linear problem (1.1)-(1.4) will be denoted by $u = S(u_0, \mu, \nu, \theta, h, \sigma, f_1, f_2)$. We set $W = (u_0, \mu, \nu, \theta, h)$ and intrudes the following notations for special case of the operator S:

$$\begin{gathered}S_0W = S(u_0, \mu, \nu, \theta, h, 0, 0, 0), \quad S_0 : L_2(0,R)\times(H^{2/5})^2\times(H^{1/5})^2 \to X(Q_T),\\ S_1f_1 = S(0,0,0,0,0,0,f_1,0), \quad S_1 : L_1(0,T;L_2(0,R)) \to X(Q_T),\\ S_2f_2 = S(0,0,0,0,0,0,0,f_2), \quad S_2 : L_{4/3}(0,T;L_2(0,R)) \to X(Q_T),\\ S_3\sigma = S(0,0,0,0,0,\sigma,0,0), \quad S_3 : L_2(0,T) \to X(Q_T).\end{gathered}$$

In this paper we will heavily rely on the following theorem.

Theorem 2.1 *Let $F(u)\equiv 0$, $u_0 \in L_2(0,R)$, $\mu,\nu \in H^{2/5}(0,T)$,$h,\theta \in H^{1/5}(0,T)$, $\sigma \in L_2(0,T)$,$f_1 \in L_1(0,T;L_2(0,R))$, $f_2 \in L_{4/3}(0,T;L_2(0,R))$, then there exists a unique solution $u \in X(Q_T)$ of the problem (1.1)-(1.4). Moreover, the operator $S : (u_0,\mu,\nu,\theta,h,\sigma,f_1,f_2) \to u$ is continuous in the corresponding norms and its norm does not decrease with the growth of T.*

Proof: Operator S can be represented as the sum of operators S_0, S_1, S_2, S_3. This result was obtained for all those operators, but S_2 in [8].

Consider $u = S_2f_2$. If $f_1 \in C_0^\infty(Q_T)$, then there exists a smooth solution of the problem (1.1)-(1.4) [8]. Multiply (1.1) by $(1+x)2u$ and integrate:

$$\begin{gathered}\frac{d}{dt}\int_0^R(1+x)u^2(t,x)dx + u_{xx}^2(t,0) + \int_0^R(3u_{xx}^2 + 3bu_x^2 - au^2)dx\\ +\int_0^R 2u_xf_2(1+x)dx + \int_0^R f_2udx = 0\end{gathered} \tag{2.2}$$

We will use interpolating inequality

$$|\int_I Y'^2dx\int| \le (\int_I Y^2dx)^{1/2}(\int_I Y''^2dx)^{1/2} \tag{2.3}$$

for the smooth function Y, that is equal zero on the boundaries of the interval I.

Apply (2.3) to (2.2), and, after simple manipulations, we obtain

$$\begin{gathered}\frac{d}{dt}\int_0^R(1+x)u^2(t,x)dx + u_{xx}^2(t,0) + C_1\int_0^R u_{xx}^2dx\\ \le C_2\int_0^R u^2dx + C_3\|f_2(t,\cdot)\|_{L_2(0,R)}^{4/3}(\int_0^R u^2dx)^{1/3}\end{gathered} \tag{2.4}$$

where C_i is a constant.

From inequality

$$\frac{d}{dt}\int_0^R(1+x)u^2(t,x)dx \le C_2\int_0^R(1+x)u^2dx + C_3\|f_2(t,\cdot)\|_{L_2(0,R)}^{4/3}(\int_0^R(1+x)u^2dx)^{1/3},$$

we obtain that

$$\|u\|_{C([0,T];L_2(0,R))} \le C_4(t)\|f_2(t,\cdot)\|_{L_{4/3}(0,T;L_2(0,R))} \tag{2.5}$$

Next, integrate (2.4)

$$\|u_{xx}\|^2_{L_2(Q_T)} \le C(T)\sup_{t\in[0,T]}\|u\|^2_{L_2(0,R)} + C(T)\sup_{t\in[0,T]}\|u\|^{2/3}_{L_2(0,R)}\int_0^T \|f_2(t,\cdot)\|^{4/3}_{L_2(0,R)}dt \tag{2.6}$$

Now, apply (2.5) to (2.6), we have

$$\|u\|_{X(Q_T)} \le c(T)\|f_2(t,\cdot)\|_{L_{4/3}(0,T;L_2(0,R))}.$$

In the general case the desired estimate is obtained via closure. □

Let $\widetilde{W}^1_p(0,T) = \{\varphi \in W^1_p(0,T) : \varphi(0) = 0\}$ for $1 \le p \le +\infty$. Introduce the liner operator Q on the space of functions $u(t,x)$ that belong to the space $L_1(0,R)$ $\forall t \in [0,T]$ by the formula $(Qu)(t) = q(t)$ where

$$q(t) \equiv \int_0^R u(t,x)\omega(x)dx, \quad t \in [0,T]. \tag{2.7}$$

Lemma 2.1 *Let $F(u) \equiv 0$ and conditions of theorem 2.1 are satisfied. Moreover, let $\mu,\nu,\theta,h,\sigma \in L_p(0,T)$, $f_1 \in L_p(0,T;L_2(0,R))$, $f_2 \in L_p(0,T;L_1(0,R))$ for some $p \in [1,+\infty]$. If $u = S(u_0,\mu,\nu,\theta,h,\sigma, f_1,f_2)$ and equality (1.10) is verified then the corresponding function $q = Qu$ belongs to the space $W^1_p(0,T)$. And $\forall t \in (0,T)$ the following equality is verified*

$$\begin{gathered} q'(t) = \sigma(t)\omega''(R) - h(t)\omega'''(R) + \theta(t)\omega'''(0) + \nu(t)\omega''''(R) \\ -\mu(t)\omega''''(0) - b\nu(t)\omega''(R) + \int_0^R u[-\omega''''' + b\omega''' + a\omega' + f_1\omega - f_2\omega']dx. \end{gathered} \tag{2.8}$$

Moreover,

$$\begin{gathered} \|q'\|_{L_p(0,T)} \le \|u_0\|_{L_2(0,R)} + \|\mu\|_{L_p(0,T)} + \|\nu\|_{L_p(0,T)} \\ +\|\theta\|_{L_p(0,T)} + \|h\|_{L_p(0,T)} + \|\sigma\|_{L_p(0,T)} + \|f_1\|_{L_p(0,T;L_2(0,R))} + \|f_2\|_{L_p(0,T;L_1(0,R))} \\ +\|\mu\|_{H^{2/5}(0,T)} + \|\nu\|_{H^{2/5}(0,T)} + \|\theta\|_{H^{1/5}(0,T)} + \|h\|_{H^{1/5}(0,T)} + \|\sigma\|_{L_2(0,T)} + \|f_2\|_{L_{4/3}(0,T;L_2)}. \end{gathered} \tag{2.9}$$

Proof: Let $\phi(t.x) \equiv \psi(t)\omega(x)$, $\psi \in C_0^\infty(0,T)$. Equality (2.1) yields that

$$\begin{gathered} \int_0^T \psi'(t)q(t)d\tau = -\int_0^T \psi(t)[\sigma(t)\omega''(R) - h(t)\omega'''(R) \\ +\theta(t)\omega'''(0) + \nu(t)\omega''''(R) - \mu(t)\omega''''(0) - b\nu(t)\omega''(R) \\ +\int_0^R u(-\omega''''' + b\omega''' + a\omega' + f_1\omega - f_2\omega')dx]dt \equiv -\int_0^T \psi(t)r(t)dt. \end{gathered} \tag{2.10}$$

Since $r \in L_p(0,T)$, then by definition the last equality implies the existence of the generalized Sobolev derivative $q'(t) = r(t) \in L_p(0,T)$. Moreover, the following inequality is verified

$$\|u\|_{L_p(0,T;L_2(0,R))} \le T^{1/p}\|u\|_{C([0,T];L_2(0,R))}.$$

Now apply theorem 2.1 and obtain the desired result. □

Lemma 2.2 *Let $\sigma \in L_2(0,T), f_1 \in L_1(0,T;L_2(0,R)), u = S_3\sigma + S_1 f_1$, then for $t \in [0,T]$ the following inequality is verified*

$$\int_0^R u^2(t,x)dx \le \int_0^t \sigma^2(\tau)d\tau + 2\int_0^t\int_0^R f_1(\tau,x)u(\tau,x)dxd\tau. \tag{2.11}$$

Proof: If $\sigma \in C_0^\infty(0,T), f_1 \in C_0^\infty(Q_T)$, then there exists a smooth solution of the problem (1.1)-(1.4) [8]. Multiply (1.1) by $2u(t,x)$ and integrate. Thus, we get (2.11). The general case is obtained via closure due to the continuity of operator S. □

3. Control by boundary function

In this selection we will establish the proof of theorem 1.1.

Without loss of generality we will assume that $\omega''(R) = 1$ which can be achieved by scaling of functions ω and φ.

Lemma 3.1 *Let $F(u) \equiv 0, u_0 = 0$, the condition (1.10) is verified and $\omega''(R) = 1, \mu = \nu = h = \theta \equiv 0, f \equiv 0$ and $\varphi \in \widetilde{W}_p^1(0,T)$ for some $p \in [2, +\infty]$, then there exists the unique function $\sigma = \Gamma\varphi \in L_p(0,T)$ for which the corresponding function $u = S_3\sigma$ satisfies the condition (1.6), and the linear operator $\Gamma : \widetilde{W}_p^1(0,T) \to L_p(0,T)$ is bounded and its norm does not decrease with increasing T.*

Proof: Introduce $\Lambda = Q \circ S_3$ a linear operator on the space $L_p(0,T)$. According to Lemma 2.1 and the continuity of the operator $S_3 : L_2(0,T) \to X(Q_T)$, operator $\Lambda : L_p(0,T) \to \widetilde{W}_p^1(0,T)$ is bounded.

Equality $\varphi = \Lambda\sigma$ for $\sigma \in L_p(0,T)$ yields that function σ gives the desired solution of the considered control problem. Introduce an operator $A : L_p(0,T) \to L_p(0,T)$:

$$(A\sigma)(t) = \varphi'(t) - \int_0^R u(t,x)(-\omega''''' + b\omega''' + a\omega')dx, \quad u = S_3\sigma. \tag{3.1}$$

Next, we will show, that $\varphi = \Lambda\sigma$ if and only if $\sigma = A\sigma$. If $\varphi = \Lambda\sigma$, then equality $q'(t) = \varphi(t)$ is verified for the function $q(t) = (\Lambda\sigma)(t)$, and according to Lemma 2.1 we obtain

$$(A\sigma)(t) = q'(t) - \int_0^R u(t,x)(-\omega''''' + b\omega''' + a\omega')dx = \sigma(t). \tag{3.2}$$

And vice versa, if $\sigma = A\sigma$, then

$$\sigma(t) = \varphi'(t) - \int_0^R u(t,x)(-\omega''''' + b\omega''' + a\omega')dx. \tag{3.3}$$

And according to lemma 2.1 then equality $q'(t) = \varphi'(t)$ is verified for the function $q(t) = (\Lambda\sigma)(t)$. Note that $q(t) = \varphi(t)$ because $q(0) = \varphi(0) = 0$

Now we show that the operator A is a contraction when choosing an equivalent norm in the space $L_p(0,T)$.

Let $\sigma_1, \sigma_2 \in L_p(0,T), u_j = S_3\sigma_j$, then

$$A\sigma_1 - A\sigma_2 = -\int_0^R (u_1 - u_2)(-\omega''''' + b\omega''' + a\omega')dx. \tag{3.4}$$

According to lemma 2.2 when $t \in [0,T]$ the following inequality is verified

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L_2(0,R)} \le \|\sigma_1 - \sigma_2\|_{L_2(0,t)}. \tag{3.5}$$

Let $\eta > 0$, then for $p < +\infty$ we obtain

$$\begin{aligned} &\|e^{-\eta t}(A\sigma_1 - A\sigma_2)\|_{L_p(0,T)} \\ &\le (\|\omega'''''\|_{L_2(0,R))} + |b|\|\omega'''\|_{L_2(0,R))} + |a|\|\omega'\|_{L_2(0,R))}) \\ &\times (\int_0^T e^{-p\eta t}\|u_1 - u_2\|^p_{L_2(0,R)}dt)^{1/p} \\ &\le c\big[(\int_0^T e^{-p\eta t}(\int_0^t (\sigma_1(\tau) - \sigma_2(\tau))^2 d\tau)^{p/2} dt\big]^{1/p} \\ &\le c_1\big[\int_0^T e^{-p\eta t}\int_0^t \mid \sigma_1(\tau) - \sigma_2(\tau) \mid^p d\tau dt\big]^{1/p} \\ &= c_1\big[(\int_0^T e^{-p\eta\tau} \mid \sigma_1(\tau) - \sigma_2(\tau) \mid^p \int_\tau^T e^{-p\eta(\tau - t)} dt d\tau\big]^{1/p} \\ &\le \frac{c_1}{(p\eta)^{1/p}}\|e^{-\eta t}(\sigma_1(\tau) - \sigma_2(\tau))\|_{L_p(0,T)}, \end{aligned} \tag{3.6}$$

where $c_1 = c(T,p,a,b,\|\omega'''''\|_{L_2(0,R)},\|\omega'''\|_{L_2(0,R)},\|\omega'\|_{L_2(0,R)})$. Thus, for instance, we can choose $\eta = \frac{(2c_1)^p}{p}$. If $p=+\infty$, then

$$\begin{aligned} &\sup_{t\in[0,T]} e^{-\eta t}\mid (A\sigma_1 - A\sigma_2)\mid \\ &\le (\|\omega'''''\|_{L_2(0,R))} + \mid b\mid \|\omega'''\|_{L_2(0,R))} + \mid a\mid \|\omega'\|_{L_2(0,R))}) \\ &\quad\times \sup_{t\in[0,T]} e^{-\eta t}\,\|\, u_1 - u_2 \,\|_{L_2(0,R)} \\ &\le c \sup_{t\in[0,T]} e^{-\eta t}\,\|\,\sigma_1(\tau)-\sigma_2(\tau)\,\|_{L_2(0,t))} \\ &\le c \operatorname*{ess\,sup}_{\tau,t\in[0,T]} e^{-\eta\tau}\mid \sigma_1(\tau)-\sigma_2(\tau)\mid (\int_0^t e^{2\eta(\tau-t)}d\tau)^{1/2} \\ &\le \frac{c}{(2\eta)^{1/2}} \operatorname*{ess\,sup}_{\tau\in[0,T]} e^{-\eta\tau}\mid \sigma_1(\tau)-\sigma_2(\tau)\mid, \end{aligned} \tag{3.7}$$

where $c = c(T,p,a,b,\|\omega'''''\|_{L_2(0,R)},\|\omega'''\|_{L_2(0,R)},\|\omega'\|_{L_2(0,R)})$. Thus, for instance, we can choose $\eta = 2c^2$.

So, we derive that for any $\varphi \in \widetilde{W}^1_p(0,T)$ there exists a unique function $\sigma \in L_p(0,T)$ such as $\sigma = A\sigma$ or $\varphi = \Lambda\sigma$. And according to the Banach theorem inverse operator $\Gamma = \Lambda^{-1} : \widetilde{W}^1_p(0,T) \to L_p(0,T)$ is continuous. Moreover,

$$\|\Gamma\varphi\|_{L_p(0,T)} \le c(T)\|\varphi'\|_{L_p(0,T))}. \tag{3.8}$$

For arbitrary $T_1 > T$, we extend the function φ by continuity by the constant $\varphi(T)$ on the interval (T,T_1). Then an analogue of inequality (3.8) on the interval $(0,T_1)$ for such a function is valid. Obviously, $c(T) \le c(T_1)$ means that the norm of the operator Γ does not decrease with increasing T. □

Now let us introduce a result on controllability for the linear problem.

Theorem 3.1 *Let $F(u)\equiv 0$ and functions u_0,μ,ν,θ satisfy the conditions of theorem 1.1, $f = f_1 + f_{2x}$, where $f_1 \in L_p(0,T;L_2(0,R))$, $f_2 \in L_p(0,T;L_1(0,R)\cap L_{4/3}(0,T;L_2(0,R)))$, conditions (1.10) and (1.11) are verified (where $\omega''(R)=1$ and $\varphi \in W^1_p(0,T)$). Then there exists a unique function $\sigma \in L_p(0,T)$ such as that the condition (1.6) is satisfied for $u = S(u_0,\mu,\nu,\theta,h,\sigma,f_1,f_2)$*

Proof: Let $\widetilde{\varphi} = \varphi - Q(S_0W + S_1f_1 + S_2f_2)$. Then lemma 2.1 and theorem 1.1 yield, that $\widetilde{\varphi} \in \widetilde{W}^1_p(0,T)$. Thus, according to lemma 3.1 it turns out that $\sigma \in \Gamma\widetilde{\varphi}$ is the desired function. In particular, if $u = S(u_0,\mu,\nu,\theta,h,\sigma,f_1,f_2)$ then

$$u = S_0W + S_1f_1 + S_2f_2 + (S_3\circ\Gamma)(\varphi - Q(S_0W + S_1f_1 + S_2f_2)). \tag{3.9}$$

The uniqueness of σ also follows from the previous lemma. □

Proof: [Proof of Theorem 1.1] In the condition of Theorem 3.1, let us assume that $f_1 \equiv f$, $f_2 \equiv F(v)$, $|F(v)| \le c|v|^q$, where $v \in X(Q_T)$. It follows from

$$\sup_{x\in(0,R)} g^2(x) \le c(R)(\|g''\|^{1/2}_{L_2(0,R)}\|g\|^{3/2}_{L_2(0,R)} + \|g\|^2_{L_2(0,R)}), \tag{3.10}$$

that $F(v) \in L_{4/3}(0,T;L_2(0,R))$ and

$$\begin{aligned} &\|F(v)\|_{L_{4/3}(0,T;L_2(0,R))} \le c\||v|^q\|_{L_{4/3}(0,T;L_2(0,R))} \\ &\le c(\int_0^T \sup_{x\in(0,R)} |v|^{\frac{4q-4}{3}}(\int_0^R v^2dx)^{2/3}dt)^{3/4} \\ &\le c(T)\sup_{t\in[0,T]}\|v\|^{\frac{3(q-1)+4}{4}}_{L_2(0,R)}\|v_{xx}\|^{\frac{q-1}{4}}_{L_2(Q_T)}T^{\frac{7-q}{8}} + c(T)\sup_{t\in[0,T]}\|v\|^q_{L_2(0,R)}T^{3/4} \\ &\le C(T)(T^{\frac{7-q}{8}} + T^{3/4})\|v\|^q_{X(Q_T)}. \end{aligned} \tag{3.11}$$

Also note that

$$\begin{aligned} \|F(v)\|_{L_2(0,T;L_1(0,R))} &\le c\||v|^q\|_{L_2(0,T;L_1(0,R))} \\ &\le (\int_0^T \sup_{x\in(0,R)} |v|^{2(q-2)}(\int_0^R v^2dx)^2dt)^{1/2} \\ \le c(T)\sup_{t\in[0,T]}\|v\|^{\frac{3q+2}{4}}_{L_2(0,R)}\|v_{xx}\|^{\frac{q-2}{4}}_{L_2(Q_T)}T^{\frac{6-q}{8}} &+ c(T)\sup_{t\in[0,T]}\|v\|^q_{L_2(0,R)}T^{1/2} \\ &\le C(T)(T^{\frac{6-q}{8}}+T^{1/2})\|v\|^q_{X(Q_T)}. \end{aligned} \tag{3.12}$$

In $X(Q_T)$ consider the mapping

$$u = \Theta v \equiv S_0W + S_1f - S_2(F(v)) + (S_3\circ\Gamma)(\varphi - Q(S_0W + S_1f - S_2(F(v)))). \tag{3.13}$$

If $\tilde{q}(t) \equiv Q(S_0W + S_1f - S_2(F(v)))(t)$ then according to lemma 2.1 we obtain

$$\begin{aligned} \|\tilde{q}'\|_{L_p(0,T)} \le c(T)(\|u_0\|_{L_2(0,R)} &+ \|\mu\|_{(H^{2/5}\bigcap L_p)(0,T)} + \|\nu\|_{(H^{2/5}\bigcap L_p)(0,T)} \\ +\|\theta\|_{(H^{1/5}\bigcap L_p)(0,T)} &+ \|h\|_{(H^{1/5}\bigcap L_p)(0,T)} + \|f\|_{L_p(0,T;L_2(0,R))} \\ +\|F(v)\|_{L_p(0,T;L_1(0,R))} &+ \|F(v)\|_{L_{4/3}(0,T;L_2(0,R))}). \end{aligned} \tag{3.14}$$

Let $p=2$. Using (3.11) and (3.12) we obtain:

$$\|\Theta v\|_{X(Q_T)} \le c(T)(c_0 + (T^{\frac{6-q}{8}} + T^{1/2} + T^{\frac{7-q}{8}} + T^{3/4}))\|v\|^q_{X(Q_T)}, \tag{3.15}$$

where c_0 is defined by formula (1.12). Next

$$\begin{aligned} \|\Theta v_1 - \Theta v_2\|_{X(Q_T)} \le C(T)(T^{\frac{6-q}{8}} + T^{1/2} + T^{\frac{7-q}{8}} + T^{3/4}) \\ (\|v_1\|^{q-1}_{X(Q_T)} + \|v_2\|^{q-1}_{X(Q_T)})\|v_1 - v_2\|_{X(Q_T)}. \end{aligned} \tag{3.16}$$

The constant $c(T)$ does not decrease with the growth of T.

Further, let $M(T) = T^{\frac{6-q}{8}} + T^{1/2} + T^{\frac{7-q}{8}} + T^{3/4}$. Then choose $T_0 > 0$ for fixed c_0 so that $2c(T_0)c_0(4c(T_0)M(T_0))^{\frac{1}{q-1}} \le 1$. Then for any $T \in (0, T_0]$ take an arbitrary $r \in [2c(T)c_0, (4c(T)M(T))^{\frac{1}{q-1}}]$. For fixed T we take $r = (4c(T)M(T))^{\frac{1}{q-1}}$ and $c_0 \le \delta = (4c(T)M(T))^{\frac{1}{1-q}}/(2c(T))$. In both cases

$$c(T)c_0 \le r/2, \quad 2c(T)c_0 \le \frac{1}{(4c(T)M(T))^{\frac{1}{q-1}}}. \tag{3.17}$$

and then the mapping Θ is a contraction on the ball of radius r (centered at zero) in the space $X(Q_T)$. The only fixed point of the mapping $u = \Theta u \in X(Q_T)$ satisfies (1.1)-(1.4) and (1.6) for $\sigma = \Gamma(\varphi - Q(S_0W + S_1f - S_2F(v))) \in L_p(0,T)$.

The uniqueness follows from the fact, that the solution of the problem (1.1)-(1.6) $u \in X(Q_{T_0})$ for a sufficiently small T_0 and and a sufficiently large r is a fixed point of the mapping Θ that is a contraction in the ball of radius r in the space $X(Q_{T_0})$. □

4. Controllability by the right side of the equations

In this selection we will establish the proof of theorem 1.2.

Lemma 4.1 *Let $F(u) \equiv 0, u_0 \equiv 0$, $\mu = \nu = h = \theta = \sigma \equiv 0, f \equiv 0$, $g \in C([0,T]; L_2(0,R))$, conditions (1.14) and (1.10) are satisfied, $\varphi \in \widetilde{W}^1_p(0,T)$ for some $p \in [1,+\infty]$, then there exists a unique function $f_0 = \Gamma\varphi \in L_p(0,T)$ for which the corresponding function $u = S_1(f_0g)$ satisfies the condition (1.6), and the linear operator $\Gamma : \widetilde{W}^1_p(0,T) \to L_p(0,T)$ is bounded and its norm does not decrease with increasing T.*

Proof: Let $Gf_0 \equiv f_0 g$ for any f_0 on $(0,T)$. Introduce a linear operator $\Lambda = Q \circ S_1 \circ G$. According to Lemma 2.1 and the continuity of the operator $S_1 : L_1(0,T; L_2(0,R)) \to C([0,T]; L_2(0,R))$, operator $\Lambda : L_p(0,T) \to \widetilde{W}_p^1(0,T)$ and it is bounded.

Let

$$g_1(t) \equiv \int_0^R g(t,x)\omega(x)dx, g_0 \leq\mid g_1(t) \mid . \tag{4.1}$$

Introduce the linear operator $A : L_p(0,T) \to L_p(0,T)$

$$(Af_0)(t) \equiv \frac{\varphi'(t)}{g_1(t)} - \frac{1}{g_1(t)} \int_0^R u(t,x)(-\omega''''' + b\omega''' + a\omega')dx, \quad u = (S_1 \circ G)f_0. \tag{4.2}$$

Now we proof that $\varphi = \Lambda f_0$, then and only then $f_0 = Af_0$. If $\varphi = \Lambda f_0$, then the equality $q'(t) \equiv \varphi'(t)$ is verified for $q(t) \equiv (\Lambda f_0)(t)$. Thus, according to (2.8) we obtain

$$(Af_0)(t) = \frac{q'(t)}{g_1(t)} - \frac{1}{g_1(t)} \int_0^R u(t,x)(-\omega''''' + b\omega''' + a\omega')dx = f_0(t). \tag{4.3}$$

And vice versa, if $f_0 = Af_0$, then

$$f_0(t) = \frac{\varphi'(t)}{g_1(t)} - \frac{1}{g_1(t)} \int_0^R u(t,x)(-\omega''''' + b\omega''' + a\omega')dx. \tag{4.4}$$

And according to (2.8) the equality $q'(t) \equiv \varphi(t)$ is verified for function $q(t) \equiv (\Lambda f_0)(t)$. Note that $q(t) \equiv \varphi(t)$ because $q(0) = \varphi(0)$.

Next, we show that the operator A is a contraction when choosing an equivalent norm in the space $L_p(0,T)$.

Let $f_{01}, f_{02} \in L_p(0,T), u_j = (S_1 \circ G)f_{0j}$ then

$$Af_{01} - Af_{02} = -\frac{1}{g_1} \int_0^R (u_1 - u_2)(-\omega''''' + b\omega''' + a\omega')dx. \tag{4.5}$$

According to (2.11) with $t \in [0,T]$ the following inequality is verified

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L_2(0,R)} \leq \sqrt{2}\|g\|_{C([0,T];L_2(0,R))}\|f_{01} - f_{02}\|_{L_1(0,t)}. \tag{4.6}$$

Let $\eta > 0$, then for $p < +\infty$ we obtain

$$\begin{aligned} &\|e^{-\eta t}(Af_{01} - Af_{02})\|_{L_p(0,T)} \\ &\leq \frac{1}{g_0}(\|\omega'''''\|_{L_2(0,R)} + \mid b \mid \|\omega'''\|_{L_2(0,R)} + \mid a \mid \|\omega'\|_{L_2(0,R)}) \\ &\times \Big(\int_0^T e^{-p\eta t}\|u_1 - u_2\|^p_{L_2(0,R)} dx\Big)^{1/p} \\ &\leq c\Big[\int_0^T e^{-p\eta t}\Big(\int_0^t \mid f_{01}(\tau) - f_{02}(\tau) \mid d\tau\Big)^p dt\Big]^{1/p} \\ &\leq c_1\Big[\int_0^T e^{-p\eta t} \int_0^t \mid f_{01}(\tau) - f_{02}(\tau) \mid^p d\tau dt\Big]^{1/p} \\ &\leq \frac{c_1}{(p\eta)^{1/p}}\|e^{-\eta t}(f_{01} - f_{02})\|_{L_{p(0,T)}}, \end{aligned} \tag{4.7}$$

where $c_1 = c(T, p, a, b, \|\omega'''''\|_{L_2(0,R)}, \|\omega'''\|_{L_2(0,R)}, \|\omega'\|_{L_2(0,R)}, \|g\|_{C([0,T];L_2(0,R))}, g_0)$.

If $p=+\infty$, then

$$\begin{aligned} &\sup_{t\in[0,T]} e^{-\eta t} \mid Af_{01}-Af_{02}\mid \\ &\leq \frac{1}{g_0}(\|\omega'''''\|_{L_2(0,R)}+\mid b\mid \|\omega'''\|_{L_2(0,R)}+\mid a\mid \|\omega'\|_{L_2(0,R)}) \\ &\times \sup_{t\in[0,T]} e^{-\eta t}\|u_1-u_2\|_{L_2(0,R)} \\ &\leq c \sup_{t\in[0,T]} e^{-\eta t}\|f_{01}-f_{02}\|_{L_1(0,t)} \\ &\leq c \operatorname*{ess\,sup}_{t,\tau\in(0,T)} e^{-\eta t}\mid f_{01}(\tau)-f_{02}(\tau)\mid \int_0^t e^{\eta(\tau-t)}d\tau \\ &\leq \frac{c}{\eta}\operatorname*{ess\,sup}_{\tau\in(0,T)} e^{-\eta t}\mid f_{01}-f_{02}\mid . \end{aligned} \tag{4.8}$$

where $c=c(a,b,\|\omega'''''\|_{L_2(0,R)},\|\omega'''\|_{L_2(0,R)},\|\omega\|_{L_2(0,R)},\|g\|_{C([0,T];L_2(0,R))},g_0)$.

For any function $\varphi\in\widetilde{W}_p^1(0,T)$, there exists a unique function $f_0\in L_p(0,T)$, such as $f_0=Af_0$ or $\varphi=\Lambda f_0$. And according to the Banach theorem the inverse operator $\Gamma=\Lambda^{-1}:\widetilde{W}_p^1(0,T)\to L_p(0,T)$ is continuous. Moreover,

$$\|\Gamma\varphi\|_{L_p(0,T)}\leq c(T)\|\varphi'\|_{L_p(0,T)}. \tag{4.9}$$

The rest of the proof is similar to lemma 3.1. □

Now let us introduce a result on controllability for the linear problem.

Theorem 4.1 *Let $F(u)\equiv 0$ and functions u_0,μ,ν,θ satisfy the conditions of theorem 1.2, $f_2\in L_p(0,T;L_1(0,R))\bigcap L_{4/3}(0,T;L_2(0,R))$, conditions (1.10)$\varphi\in W_p^1(0,T)$ and (1.11) are verified. Then there exists a unique function$f_0\in L_p(0,T)$, such as function $u=S(u_0,\mu,\nu,\theta,h,\sigma,f_0g,f_2)$ satisfies the condition (1.6).*

Proof: Let $\widetilde{\varphi}\equiv\varphi-Q(S_0W+S_2f_2+S_3\sigma)$ then lemma 2.1 and theorem 1.1 yields, that $\widetilde{\varphi}\in\widetilde{W}_p^1(0,T)$. Thus, according to lemma 4.1 function $f_0=\Gamma\widetilde{\varphi}$ is the desired function. In particular, if $u=S(u_0,\mu,\nu,\theta,h,\sigma,f_0g,f_2)$, then

$$u=S_0W+(S_1\circ G\circ\Gamma)(\varphi-Q(S_0W+S_2f_2+S_3\sigma))+S_2f_2+S_3\sigma. \tag{4.10}$$

The uniqueness of f_0 also follows from the previous lemma. □

Proof: [Proof of Theorem 1.2] In the condition of Theorem 3.1, let us assume that $f_1\equiv f$, $f_2\equiv F(v)$, $|F(v)|\leq c|v|^q$, where $v\in X(Q_T)$. In $X(Q_T)$ consider the mapping:

$$\begin{gathered} u=\Theta v\equiv S_0W+(S_1\circ G\circ\Gamma) \\ (\varphi-Q(S_0W-S_2(F(v))+S_3\sigma))-S_2(F(v))+S_3\sigma. \end{gathered} \tag{4.11}$$

Apply the results of Theorem 2.1, Lemma 2.1, and Theorem 3.1 with $p=1$ (similar to (3.14)-(3.16) (in an analog of (3.14) swap $\|f\|_{L_p(0,T;L_2(0,R))}$ with $\|h\|_{(H^{1/5}\bigcap L_p)(0,T)})$). Thus, we obtain the inequalities (3.15) and (3.16), where the quantity c_0 is defined by equality (1.15), and the constant $c(T)$ does not decrease with increasing T. The rest of the proof is similar to the proof of the Theorem 1.1. □

Acknowledgments

The author thanks professor A.V. Faminskii for his guidance and suggestions.

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