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# On $L_1$-biconservative Lorentzian hypersurfaces of Minkowski 5-space

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ABSTRACT: A Lorentzian hypersurface $x : M_1^4 \to \mathbb{L}^5$, isometrically immersed into the Lorentz-Minkowski 5-space $\mathbb{L}^5$, is said to be $L_1$-biconservative if the tangent component of vector field $L_1^2 x$ is identically zero, where $L_1$ is the linearized operator associated to the first variation of 2nd mean curvature vector field on $M_1^4$. Since $L_0 = \Delta$ is the well known Laplace operator, the concept of $L_1$-biconservative hypersurface is an extension of ordinary conservativity. The biconservativity is related to the physical concept of conservative stress-energy with respect to the bienergy functional. We discuss on Lorentzian hypersurfaces of $\mathbb{L}^5$ having at most two distinct principal curvatures. After illustrating some examples, we prove that every $L_1$-bicoservative Lorentzian hypersurface with constant ordinary mean curvature and at most two distinct principal curvatures in $\mathbb{L}^5$ has to be of constant 2nd mean curvature.

Key Words: Lorentzian hypersurface, $L_1$-biconservative, $L_1$-biharmonic.

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## 1. Introduction

From a geometric point of view, the variational problem associated to the bienergy functional on the set of Riemannian metrics on a domain has given rise to the biconservative stress-energy tensor. From physical points of view, the biconservative surfaces and, especially, the biharmonic surfaces play interesting roles in elastics and fluid mechanics. A differential geometric motivation of the subject of biharmonic maps is a well-known conjecture of Bang-Yen Chen which states that each biharmonic submanifold of an Euclidean space in minimal. Later on, Dimitrić proved that any biharmonic hypersurface in $\mathbb{E}^m$ with at most two distinct principal curvatures is minimal ([7]). An equivalent statement says that every biharmonic hypersurface in $\mathbb{E}^m$ with at most two distinct principal curvatures is harmonic, which means that there is no proper biharmonic hypersurface in $\mathbb{E}^m$ with at most two distinct principal curvatures. Remember that, a biharmonic hypersurface which is not harmonic is called *proper biharmonic*. In 1995, Hasanis and Vlachos affirmed Chen's conjecture on hypersurfaces in Euclidean 4-space ([10]). In 2013, Akutagawa and Maeta ([1]) have studied Chen's conjecture on biharmonic submanifolds in Euclidean $n$-space. As an extended case, a hypersurface $x : M_p^3 \to \mathbb{E}_s^4$, whose mean curvature vector field is an eigenvector of the Laplace operator $\Delta$, has been studied in [8,9] for the Euclidean case (where, $p = s = 0$), and in [3,4] for the Lorentz case (where $s = 1$ and $p = 0, 1$). On the other hand, Chen himself had found a nice relation between the finite type hypersurfaces and biharmonic ones. The theory of finite type hypersurfaces is a well-known subject interested by Chen (for instance, in [5,6]) and also L.J. Alias, S.M.B. Kashani and others. In [11], Kashani has studied the notion of $L_1$-finite type Euclidean hypersurfaces as an extension of finite type ones. One can see main results in Chapter 11 of Chen's book ([5]).

The linearized operator $L_1$ is an extension of the Laplace operator $L_0 = \Delta$, which stands for the linearized map of the first variation of the 2nd mean curvature of the hypersurface (see, for instance, [2,12,16,17,19]). This operator is defined by $L_1(f) = tr(P_1 \circ \nabla^2 f)$ for any $f \in C^\infty(M)$, where $P_1 = nHI - S$ denotes the first Newton transformation associated to the second fundamental from of the hypersurface and $\nabla^2 f$ is the Hessian of $f$. It is interesting to generalize the definition of biharmonic hypersurface by

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replacing $\Delta$ by $L_1$. Recently, in [14], we have studied the $L_1$-biharmonic spacelike hypersurfaces in 4-dimentional Minkowski space $\mathbb{E}^4_1$. In this paper, we study the $L_1$-biconservative Lorentzian hypersurfaces in the Minkowski 5-space $\mathbb{E}^5_1$. We show that, every $L_1$-biconservative Lorentzian hypersurface $x : M^4_1 \to \mathbb{E}^5_1$, with constant mean curvature and some additional conditions on principal curvatures, has constant second mean curvature.

We present the structure of paper. In section 2, we remember some preliminaries which will be used in paper. In section 3, we present several examples of $L_1$-biconservative hypersurfaces. Section 4 is allocated to $L_1$-biconservative Lorentzian hypersurfaces of $\mathbb{L}^5$ with at most two distinct principal curvatures. Considering four possible types $I$ to $IV$ of the shape operator of hypersurfaces, we prove several theorems separately in four cases. Three theorems is appropriated to $L_1$-biconservative Lorentzian hypersurfaces in the Lorentz-Minkowski 5-space which has diagonal shape operator with at most two distinct eigenvalue functions (theorems 4.1, 4.2 and 4.3). Then, we study $L_1$-biconservative Lorentzian hypersurfaces of type $II$, and we show that, if $M^4_1$ has at most two distinct principal curvatures and constant mean curvature, then it's 2nd mean curvature is constant (Theorem 4.4). When the shape operator of $M^4_1$ is of type $III$, by definition it has at most two distinct principal curvatures, so we show that if $M^4_1$ (of type $III$) has constant mean curvature, then it's 2nd mean curvature is constant (Theorem 4.5). Finally, we remember that a hypersurfaces of type $IV$ has two complex principal curvatures and two real ones. We show that if $M^4_1$ (of type $IV$) has at most two distinct principal curvatures, then it's 2nd mean curvature is constant (Theorem 4.6).

## 2. Preliminaries

In this section, we recall preliminaries from [2,12,13] and [15]-[18]. The 5-dimensional Lorentz-Minkowski space, $\mathbb{L}^5$, is the real vector space $\mathbb{R}^5$ endowed with the scalar product defined by

$$\langle x, y\rangle := -x_1y_1 + \Sigma^5_{i=2}x_iy_i,$$

for every $x, y \in \mathbb{R}^5$. Throughout the paper, we study on every Lorentzian hypersurface of $\mathbb{L}^5$, defined by an isometric immersion $x : M^4_1 \to \mathbb{L}^5$. The symbols $\nabla$ and $\bar{\nabla}$ stand for the Levi-Civita connection on $M^4_1$ and $\mathbb{L}^5$, respectively. For every tangent vector fields $X$ and $Y$ on $M$, the Gauss formula is given by $\bar{\nabla}_XY = \tilde{\nabla}_XY + \langle AX, Y\rangle\mathbf{n}$, for every $X, Y \in \chi(M)$, where, $\mathbf{n}$ is a (locally) unit normal vector field on $M$ and $A$ is the shape operator of $M$ relative to $\mathbf{n}$. For each non-zero vector $X \in \mathbb{L}^5$, the real value $\langle X, X\rangle$ may be a negative, zero or positive number and then, the vector $X$ is said to be time-like, light-like or space-like, respectively.

**Definition 2.1** For a 4-dimensional Lorentzian vector space $V^4_1$, a basis $\mathcal{B} := \{e_1, \dots, e_4\}$ is said to be *orthonormal* if it satisfies $\langle e_i, e_j\rangle = \epsilon_i\delta^j_i$ for $i, j = 1, \dots, 4$, where $\epsilon_1 = -1$ and $\epsilon_i = 1$ for $i = 2, 3, 4$. As usual, $\delta^j_i$ stands for the Kronecker delta. $\mathcal{B}$ is called *pseudo-orthonormal* if it satisfies $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 0$, $\langle e_1, e_2\rangle = -1$ and $\langle e_i, e_j\rangle = \delta^j_i$, for $i = 1, 2, 3, 4$ and $j = 3, 4$.

As well-known, the shape operator $A$ of the Lorentzian hypersurface $M^4_1$ in $\mathbb{L}^5$, as a self-adjoint linear map on the tangent bundle of $M^4_1$, locally can be put into one of four possible canonical matrix forms, usually denoted by $I$, $II$, $III$ and $IV$. Where, in cases $I$ and $IV$, with respect to an orthonormal basis of the tangent space of $M^4_1$, the matrix representation of the induced metric on $M^4_1$ is

$$G_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the shape operator of $M^4_1$ can be put into matrix forms

$$B_1 = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \quad \text{and} \quad B_4 = \begin{pmatrix} \kappa & \lambda & 0 & 0 \\ -\lambda & \kappa & 0 & 0 \\ 0 & 0 & \eta_1 & 0 \\ 0 & 0 & 0 & \eta_2 \end{pmatrix}, \quad (\lambda \neq 0)$$

respectively. For cases $II$ and $III$, using a pseudo-orthonormal basis of the tangent space of $M^4_1$, the induced metric on which has matrix form

$$G_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the shape operator of $M_1^4$ can be put into matrix forms

$$B_2 = \begin{pmatrix} \kappa & 0 & 0 & 0 \\ 1 & \kappa & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} \qquad \text{and} \qquad B_3 = \begin{pmatrix} \kappa & 0 & 0 & 0 \\ 0 & \kappa & 1 & 0 \\ -1 & 0 & \kappa & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix},$$

respectively. In case $IV$, the matrix $B_4$ has two conjugate complex eigenvalues $\kappa \pm i\lambda$, but in other cases the eigenvalues of the shape operator are real numbers.

**Remark 2.1** In two cases $II$ and $III$, one can substitute the pseudo-orthonormal basis $\mathcal{B} := \{e_1, e_2, e_3, e_4\}$ by a new orthonormal basis $\tilde{\mathcal{B}} := \{\tilde{e_1}, \tilde{e_2}, e_3, e_4\}$ where $\tilde{e_1} := \frac{1}{2}(e_1 + e_2)$ and $\tilde{e_2} := \frac{1}{2}(e_1 - e_2)$. Therefore, we obtain new matrix representations $\tilde{B_2}$ and $\tilde{B_3}$ (instead of $B_2$ and $B_3$, respectively) as

$$\tilde{B_2} = \begin{pmatrix} \kappa + \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \kappa - \frac{1}{2} & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} \qquad \text{and} \qquad \tilde{B_3} = \begin{pmatrix} \kappa & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & \kappa & -\frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \kappa & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

After this changes, to unify the notations we denote the orthonormal basis by $\mathcal{B}$ in all cases.

**Notation:** According to four possible matrix representations of the shape operator of $M_1^4$, we define its principal curvatures, denoted by unified notations $\kappa_i$ for $i = 1, \ldots, 4$, as follow.

In case $I$, we put $\kappa_i := \lambda_i$, for $i = 1, \ldots, 4$, where $\lambda_i$'s are the eigenvalues of $B_1$.

In case $II$, where the matrix representation of $A$ is $\tilde{B_2}$, we take $\kappa_i := \kappa$ for $i = 1, 2$, and $\kappa_i := \lambda_{i-2}$, for $i = 3, 4$.

In case $III$, where the shape operator has matrix representation $\tilde{B_3}$, we take $\kappa_i := \kappa$ for $i = 1, 2, 3$, and $\kappa_4 := \lambda$.

Finally, in the case $IV$, where the shape operator has matrix representation $\tilde{B_4}$, we put $\kappa_1 = \kappa + i\lambda$, $\kappa_2 = \kappa - i\lambda$, and $\kappa_i := \eta_{i-2}$, for $i = 3, 4$.

The characteristic polynomial of $A$ on $M_1^4$ is of the form $Q(t) = \prod_{i=1}^4 (t - \kappa_i) = \sum_{j=0}^4 (-1)^j s_j t^{4-j}$, where, $s_0 := 1$, $s_i := \sum_{1 \le j_1 < \cdots < j_i \le 4} \kappa_{j_1} \cdots \kappa_{j_i}$ for $i = 1, \ldots, 4$.

For $j = 1, \ldots, 4$, the *jth mean curvature* $H_j$ of $M_1^4$ is defined by $H_j = \frac{1}{\binom{4}{j}} s_j$. When $H_j$ is identically null, $M_1^4$ is said to be $(j-1)$-*minimal*.

**Definition 2.2** $(i)$ A timelike hypersurface $x : M_1^4 \to \mathbb{L}^5$, with diagonalizable shape operator, is said to be *isoparametric* if all of it's principal curvatures are constant.

$(ii)$ A timelike hypersurface $x : M_1^4 \to \mathbb{L}^5$, with non-diagonalizable shape operator, is said to be *isoparametric* if the minimal polynomial of it's shape operator is constant.

**Remark 2.2** Here we remember Theorem 4.10 from [13], which assures us that there is no isoparametric timelike hypersurface of $\mathbb{L}^5$ with complex principal curvatures.

The well-known Newton transformation $P_j : \chi(M) \to \chi(M)$ on $M_1^4$, is defined by

$$P_0 = I, \quad P_j = s_j I - A \circ P_{j-1}, \qquad (j = 1, 2, 3, 4), \tag{2.1}$$

where, $I$ is the identity map. Using its explicit formula, $P_j = \sum_{i=0}^j (-1)^i s_{j-i} A^i$ (where $A^0 = I$) and by the Cayley-Hamilton theorem (stating that any operator is annihilated by its characteristic polynomial) we get $P_4 = 0$. It can be seen that, $P_j$ is self-adjoint and commutative with $A$ (see [2,16]).

Now, we define a notation as

$$\mu_{i;k} = \sum_{1 \le j_1 < \cdots < j_k \le 4;\, j_l \neq i} \kappa_{j_1} \cdots \kappa_{j_k}, \qquad (i = 1, 2, 3, 4; \quad 1 \le k \le 3). \tag{2.2}$$

Corresponding to four possible forms $\tilde{B}_i$ (for $1 \le i \le 4$) of $A$, the Newton transformation $P_j$ has different representations. In the case $I$, where $A = \tilde{B}_1$, we have $P_j = diag[\mu_{1;j}, \cdots, \mu_{4;j}]$, for $j = 1, 2, 3$.

When $A = B_2$ (in the case $II$), we have

$$P_1 = \begin{pmatrix} \lambda_1 + \lambda_2 + \kappa - \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \lambda_1 + \lambda_2 + \kappa + \frac{1}{2} & 0 & 0 \\ 0 & 0 & 2\kappa + \lambda_2 & 0 \\ 0 & 0 & 0 & 2\kappa + \lambda_1 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} \lambda_1\lambda_2 + (\kappa - \frac{1}{2})(\lambda_1 + \lambda_2) & -\frac{1}{2}(\lambda_1 + \lambda_2) & 0 & 0 \\ \frac{1}{2}(\lambda_1 + \lambda_2) & \lambda_1\lambda_2 + (\kappa + \frac{1}{2})(\lambda_1 + \lambda_2) & 0 & 0 \\ 0 & 0 & \kappa(\kappa + 2\lambda_2) & 0 \\ 0 & 0 & 0 & \kappa(\kappa + 2\lambda_1) \end{pmatrix}.$$

In the case $III$, we have $A = B_3$, and

$$P_1 = \begin{pmatrix} 2\kappa + \lambda & 0 & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 2\kappa + \lambda & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 2\kappa + \lambda & 0 \\ 0 & 0 & 0 & 3\kappa \end{pmatrix}, \quad P_2 = \begin{pmatrix} 2\kappa\lambda + \kappa^2 - \frac{1}{2} & -\frac{1}{2} & -\frac{\sqrt{2}}{2}(\kappa + \lambda) & 0 \\ \frac{1}{2} & 2\kappa\lambda + \kappa^2 + \frac{1}{2} & \frac{\sqrt{2}}{2}(\kappa + \lambda) & 0 \\ \frac{\sqrt{2}}{2}(\kappa + \lambda) & \frac{\sqrt{2}}{2}(\kappa + \lambda) & 2\kappa\lambda + \kappa^2 & 0 \\ 0 & 0 & 0 & 3\kappa^2 \end{pmatrix}.$$

In the case $IV$, $A = B_4$,

$$P_1 = \begin{pmatrix} \kappa + \eta_1 + \eta_2 & -\lambda & 0 & 0 \\ \lambda & \kappa + \eta_1 + \eta_2 & 0 & 0 \\ 0 & 0 & 2\kappa + \eta_2 & 0 \\ 0 & 0 & 0 & 2\kappa + \eta_1 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} \kappa(\eta_1 + \eta_2) + \eta_1\eta_2 & -\lambda(\eta_1 + \eta_2) & 0 & 0 \\ \lambda(\eta_1 + \eta_2) & \kappa(\eta_1 + \eta_2) + \eta_1\eta_2 & 0 & 0 \\ 0 & 0 & \kappa^2 + \lambda^2 + 2\kappa\eta_2 & 0 \\ 0 & 0 & 0 & \kappa^2 + \lambda^2 + 2\kappa\eta_1 \end{pmatrix}.$$

Fortunately, in all cases we have the following important identities, similar to those in [2,16].

$$\mu_{i,1} = 4H_1 - \lambda_i, \quad \mu_{i,2} = 6H_2 - \lambda_i\mu_{i,1} = 6H_2 - 4\lambda_i H_1 + \lambda_i^2, \ \ (1 \le i \le 4), \tag{2.3}$$

$$tr(P_1) = 12H_1, \quad tr(P_2) = 12H_2, \quad tr(P_1 \circ A) = 12H_2, \quad tr(P_2 \circ A) = 12H_3, \tag{2.4}$$

$$trA^2 = 4(4H_1^2 - 3H_2), \quad tr(P_1 \circ A^2) = 12(2H_1H_2 - H_3), \quad tr(P_2 \circ A^2) = 4(4H_1H_3 - H_4). \tag{2.5}$$

The *linearized operator* of the $(j+1)$th mean curvature of $M$, $L_j : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is defined by the formula $L_j(f) := tr(P_j \circ \nabla^2 f)$, where, $\langle \nabla^2 f(X), Y\rangle = \langle \nabla_X \nabla f, Y\rangle$ for every $X, Y \in \chi(M)$.

Associated to the orthonormal frame $\{e_1, \cdots, e_4\}$ of tangent space on a local coordinate system in the hypersurface $x : M_1^4 \to \mathbb{L}^5$ , $L_1(f)$ has an explicit expression as

$$L_1(f) = \sum_{i=1}^{4} \epsilon_i \mu_{i,1}(e_ie_if - \nabla_{e_i}e_if). \tag{2.6}$$

On a Lorentzian hypersurface $x : M_1^4 \to \mathbb{L}^5$, with a chosen (local) unit normal vector field $\mathbf{n}$, for an arbitrary vector $\mathbf{a} \in \mathbb{L}^5$ we use the decomposition $\mathbf{a} = \mathbf{a}^T + \mathbf{a}^N$ where $\mathbf{a}^T \in TM$ is the tangential component of $\mathbf{a}$, $\mathbf{a}^N \perp TM$, and we have the following formulae from [2,16].

$$\nabla\langle x, \mathbf{a}\rangle = \mathbf{a}^T, \quad \nabla\langle \mathbf{n}, \mathbf{a}\rangle = -S\mathbf{a}^T. \tag{2.7}$$

$$L_1x = 12H_2\mathbf{n}, \quad L_1\mathbf{n} = -6\nabla(H_2) - 12[2H_1H_2 - H_3]\mathbf{n}, \tag{2.8}$$

$$L_1^2x = 12L_1(H_2\mathbf{n}) = 24[P_2\nabla H_2 - 9H_2\nabla H_2] + 12[L_1H_2 - 12H_2(2H_1H_2 - H_3)]\mathbf{n}. \tag{2.9}$$

Assume that a hypersurface $x : M_1^4 \to \mathbb{L}^5$ satisfies the condition $L_1^2x = 0$, then it is said to be $L_1$-*biharmonic*. By equalities (2.8) and (2.9), from the condition $L_1(H_2\mathbf{n}) = 0$ (which is equivalent to $L_1$-biharmonicity) we obtain simpler conditions on $M_1^4$ to be a $L_1$-biharmonic hypersurface in $\mathbb{E}_1^5$, as:

$$(i)\ L_1H_2 = 12H_2(2H_1H_2 - H_3) = H_2 tr(P_1 \circ A^2), \qquad (ii)\ P_2\nabla H_2 = 9H_2\nabla H_2. \tag{2.10}$$

A Lorentzian hypersurface $x : M_1^4 \to \mathbb{E}_1^5$ is said to be $L_1$-*biconservative*, if its 2nd mean curvature satisfies the condition (2.10)$(ii)$.

The well-known structure equations on $\mathbb{E}_1^5$ are given by $d\omega_i = \sum_{j=1}^{5} \omega_{ij} \wedge \omega_j$, $\omega_{ij} + \omega_{ji} = 0$ and $d\omega_{ij} = \sum_{l=1}^{5} \omega_{il} \wedge \omega_{lj}$. Restricted on $M$, we have $\omega_5 = 0$ and then, $0 = d\omega_5 = \sum_{i=1}^{4} \omega_{5,i} \wedge \omega_i$. So, by Cartan's lemma, there exist functions $h_{ij}$ such that $\omega_{5,i} = \sum_{j=1}^{4} h_{ij}\omega_j$ and $h_{ij} = h_{ji}$ which give the second fundamental form of $M$, as $B = \sum_{i,j} h_{ij}\omega_i\omega_j e_5$. The mean curvature $H$ is given by $H = \frac{1}{4}\sum_{i=1}^{4} h_{ii}$. Therefore, we obtain the structure equations on $M$ as follow.

$$d\omega_i = \sum_{j=1}^{4} \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0, \tag{2.11}$$

$$d\omega_{ij} = \sum_{k=1}^{4} \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l=1}^{4} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.12}$$

for $i, j = 1, 2, 3$, and the Gauss equations $R_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk})$, where $R_{ijkl}$ denotes the components of the Riemannian curvature tensor of $M$. Denoting the covariant derivative of $h_{ij}$ by $h_{ijk}$, we have

$$dh_{ij} = \sum_{k=1}^{4} h_{ijk}\omega_k + \sum_{k=1}^{4} h_{kj}\omega_{ik} + \sum_{k=1}^{4} h_{ik}\omega_{jk}, \tag{2.13}$$

and by the Codazzi equation we get

$$h_{ijk} = h_{ikj}. \tag{2.14}$$

Finally, we recall the definition of an $L_1$-finite type hypersurface from [11], which is the basic notion of the paper.

**Definition 2.3** An isometrically immersed hypersurface $x : M_1^4 \to \mathbb{L}^5$ is said to be of $L_1$-finite type if $x$ has a finite decomposition $x = \sum_{i=0}^{m} x_i$, for some positive integer $m$, satisfying the condition $L_1x_i = \tau_i x_i$, where , $\tau_i \in \mathbb{R}$ and $x_i : M_1^4 \to \mathbb{L}^5$ is smooth maps, for $i = 1, 2, \cdots, m$, and $x_0$ is constant. If all $\tau_i$'s are mutually different, $M_1^4$ is said to be of $L_1$-$m$-type. An $L_1$-$m$-type hypersurface is said to be null if for at least one $i$ $(1 \le i \le m)$ we have $\tau_i = 0$.

## 3. Examples

For instance, we give five examples of $L_1$-biconservative Lorentzian hypersurfaces in $\mathbb{L}^5$.

**Example 3.1** Assume that $\mathcal{M}_1(r)$ be the product $\mathbb{S}_1^3(r) \times \mathbb{R} \subset \mathbb{L}^5$ where $r > 0$. It has another representation as

$$\mathcal{M}_1(r) = \{(y_1, ..., y_5) \in \mathbb{L}^5 | - y_1^2 + y_2^2 + y_3^2 + y_4^2 = r^2\},$$

having the spacelike vector field $\mathbf{n}(y) = -\frac{1}{r}(y_1, y_2, y_3, y_4, 0)$ as the Gauss map. Clearly, it has two distinct principal curvatures $\kappa_1 = \kappa_2 = \kappa_3 = \frac{1}{r}$, $\kappa_4 = 0$, and the constant higher order mean curvatures $H_1 = \frac{3}{4}r^{-1}$, $H_2 = \frac{1}{2}r^{-2}$ and $H_3 = 0$.

**Example 3.2** Assume that $\mathcal{M}_2(r)$ be the product $\mathbb{S}^2_1(r)\times\mathbb{R}^2\subset\mathbb{L}^5$ where $r>0$. It has another representation as

$$\mathcal{M}_2(r)=\{(y_1,...,y_5)\in\mathbb{L}^5|-y_1^2+y_2^2+y_3^2=r^2\},$$

having the spacelike vector field $\mathbf{n}(y)=-\frac{1}{r}(y_1,y_2,y_3,0,0)$ as the Gauss map. Clearly, it has two distinct principal curvatures $\kappa_1=\kappa_2=\frac{1}{r}$, $\kappa_3=\kappa_4=0$, and the constant higher order mean curvatures $H_1=\frac{1}{2}r^{-1}$, $H_2=\frac{1}{6}r^{-2}$ and $H_3=0$.

**Example 3.3** Assume that $\mathcal{M}_3(r)$ be the product $\mathbb{S}^1_1(r)\times\mathbb{R}^3\subset\mathbb{L}^5$ where $r>0$. It has another representation as

$$\mathcal{M}_3(r)=\{(y_1,...,y_5)\in\mathbb{L}^5|-y_1^2+y_2^2=r^2\},$$

having the spacelike vector field $\mathbf{n}(y)=-\frac{1}{r}(y_1,y_2,0,0,0)$ as the Gauss map. Clearly, it has two distinct principal curvatures $\kappa_1=\frac{1}{r}$, $\kappa_2=\kappa_3=\kappa_4=0$, and the constant higher order mean curvatures $H_1=\frac{1}{4}r^{-1}$, and $H_2=H_3=0$.

**Example 3.4** Let $\mathcal{M}_4(r)$ be the product $\mathbb{L}^3\times\mathbb{S}^1(r)\subset\mathbb{L}^5$ where $r>0$. It can be represented as

$$\mathcal{M}_4(r)=\{(y_1,...,y_5)\in\mathbb{L}^5|y_4^2+y_5^2=r^2\},$$

with the Gauss map $\mathbf{n}(y)=-\frac{1}{r}(0,0,0,y_4,y_5)$. it has two distinct principal curvatures $\kappa_1=\kappa_2=\kappa_3=0$, $\kappa_4=\frac{1}{r}$, and the constant higher order mean curvatures $H_1=\frac{1}{4r}$, and $H_k=0$ for $k=2,3,4$.

**Example 3.5** Let $\mathcal{M}_5(r)$ be the product $\mathbb{L}^2\times\mathbb{S}^2(r)\subset\mathbb{L}^5$ where $r>0$. It can be represented as

$$\mathcal{M}_5(r)=\{(y_1,...,y_5)\in\mathbb{L}^5|y_3^2+y_4^2+y_5^2=r^2\},$$

with the Gauss map $\mathbf{n}(y)=-\frac{1}{r}(0,0,y_3,y_4,y_5)$. it has two distinct principal curvatures $\kappa_1=\kappa_2=0$, $\kappa_3=\kappa_4=\frac{1}{r}$, and the constant higher order mean curvatures $H_1=\frac{1}{2r}$, $H_2=\frac{1}{6}r^{-2}$ and $H_k=0$ for $k=3,4$.

## 4. Main results

The next lemma can be proved by the same manner of similar one in [19].

**Lemma 4.1** Let $M^4_1$ be a timelike hypersurface in $\mathbb{L}^5$ of type $I$ with real principal curvatures of constant multiplicities. Then the distribution of the space of principal directions corresponding to the principal curvatures is completely integrable. In addition, if a principal curvature be of multiplicity greater than one, then it will be constant on each integral submanifold of the corresponding distribution.

**Theorem 4.1** Every $L_1$-biconservative timelike hypersurface of $\mathbb{L}^5$, having diagonalizable shape operator with exactly one eigenvalue function of multiplicity four, has constant 2nd mean curvature.

**Proof:** Let $x:M^4_1\to\mathbb{L}^5$ satisfies the assumed conditions. Taking the open subset $\mathcal{U}$ of $M$ as $\mathcal{U}:=\{p\in M^4_1:\nabla H_2^2(p)\neq0\}$, we prove that $\mathcal{U}$ is empty. Assuming $\mathcal{U}\neq\varnothing$, we consider $\{e_1,e_2,e_3,e_4\}$ as a local orthonormal frame of principal directions of $A$ on $\mathcal{U}$ such that for $i=1,2,3,4$ we have $Se_i=\lambda e_i$ and

$$\mu_{i,2}=3\lambda^2,\quad H_2=\lambda^2. \tag{4.1}$$

By condition (2.10)$(ii)$, we have $P_2(\nabla H_2)=9H_2\nabla H_2$ which, using the polar decomposition $\nabla H_2=\sum_{i=1}^4\epsilon_i\langle\nabla H_2,e_i\rangle e_i$, gives $\epsilon_i\langle\nabla H_2,e_i\rangle(\mu_{i,2}-9H_2)=0$ on $\mathcal{U}$ for $i=1,2,3$. Hence, if for some $i$ we have $\langle\nabla H_2,e_i\rangle\neq0$ on $\mathcal{U}$, then we get

$$\mu_{i,2}=9H_2, \tag{4.2}$$

which, using equalities (4.1), gives $\lambda^2=0$ and then $H_2=0$ on $\mathcal{U}$, which is a contradiction. Hence $\mathcal{U}$ is empty and $H_2$ is constant on $M$. □

**Theorem 4.2** Every $L_1$-biconservative timelike hypersurface of $\mathbb{L}^5$, having diagonalizable shape operator, constant ordinary mean curvature and exactly two distinct principal curvature functions $\lambda$ and $\mu$ respectively of multiplicities 3 and 1, has constant second mean curvature.

**Proof:** Let $x : M_1^4 \to \mathbb{L}^5$ satisfies the assumed conditions. Taking the open subset $\mathcal{U}$ of $M$ as $\mathcal{U} := \{p \in M_1^4 : \nabla H_2^2(p) \neq 0\}$, we prove that $\mathcal{U}$ is empty. Assuming $\mathcal{U} \neq \varnothing$, we consider $\{e_1, e_2, e_3, e_4\}$ as a local orthonormal frame of principal directions of $A$ on $\mathcal{U}$ such that $Se_i = \lambda e_i$ for $i = 1, 2, 3$ and $Se_4 = \mu e_4$. Therefore, we obtain

$$\begin{aligned}&\mu_{1,2} = \mu_{2,2} = \mu_{3,2} = \lambda^2 + 2\lambda\mu, \quad \mu_{4,2} = 3\lambda^2,\\ &4H_1 = 3\lambda + \mu, \quad 6H_2 = 3\lambda(\lambda + \mu), \quad 4H_3 = \lambda^2(\lambda + 3\mu), \quad H_4 = \lambda^3\mu.\end{aligned} \tag{4.3}$$

By condition (2.10)$(ii)$, we have

$$P_2(\nabla H_2) = 9H_2\nabla H_2. \tag{4.4}$$

Then, using the polar decomposition

$$\nabla H_2 = \sum_{i=1}^{4} \epsilon_i \langle \nabla H_2, e_i\rangle e_i, \tag{4.5}$$

we see that (4.4) is equivalent to

$$\epsilon_i \langle \nabla H_2, e_i\rangle(\mu_{i,2} - 9H_2) = 0$$

on $\mathcal{U}$ for $i = 1, 2, 3$. Hence, for some $i$ such that $\langle \nabla H_2, e_i\rangle \neq 0$ on $\mathcal{U}$, we get

$$\mu_{i,2} = 9H_2. \tag{4.6}$$

By definition, we have $\nabla H_2 \neq 0$ on $\mathcal{U}$, which gives one or both of the following states.

**State 1.** $\langle \nabla H_2, e_i\rangle \neq 0$, for some $i \in \{1, 2, 3\}$. Using equalities (4.3), from (4.6) we obtain

$$\lambda(5\mu + 7\lambda) = 0. \tag{4.7}$$

If $\lambda = 0$ then $H_2 = 0$. Otherwise, we get $\lambda = \frac{5}{2}H$, $H_2 = -\frac{15}{2}H^2$.

**State 2.** $\langle \nabla H_2, e_i\rangle = 0$, for $i \in \{1, 2, 3\}$ and $\langle \nabla H_2, e_4\rangle \neq 0$. By equalities (4.3) and (4.6), we obtain $3\lambda^2 = \frac{9}{2}(\lambda\mu + \lambda^2)$, which gives

$$\lambda(3\mu + \lambda) = 0. \tag{4.8}$$

If $\lambda = 0$ then $H_2 = 0$. Otherwise, we have $\lambda = \frac{3}{2}H$, $H_2 = \frac{3}{4}H^2$.

Hence, $H_2$ is constant on $M$. □

**Theorem 4.3** Every $L_1$-biconservative timelike hypersurface of $\mathbb{L}^5$, having diagonalizable shape operator, constant ordinary mean curvature and exactly two distinct principal curvature functions $\lambda$ and $\mu$ both of multiplicities 2, has constant 2nd mean curvature.

**Proof:** Let $x : M_1^4 \to \mathbb{L}^5$ be a $L_1$-biharmonic timelike hypersurface of $\mathbb{L}^5$, which has diagonal shape operator with two eigenvalue functions, each of them is of multiplicity two. Defining the open subset $\mathcal{U}$ of $M$ as $\mathcal{U} := \{p \in M_1^4 : \nabla H_2^2(p) \neq 0\}$, we prove that $\mathcal{U}$ is empty. Assuming $\mathcal{U} \neq \varnothing$, we consider $\{e_1, e_2, e_3, e_4\}$ as a local orthonormal frame of principal directions of $A$ on $\mathcal{U}$ such that $Ae_i = \lambda e_i$ for $i = 1, 2$, and $Ae_i = \mu e_i$ for $i = 3, 4$. Therefore, we obtain

$$\begin{aligned}&\mu_{1,1} = \mu_{2,1} = \lambda + 2\mu, \quad \mu_{3,1} = \mu_{4,1} = 2\lambda + \mu,\\ &\mu_{1,2} = \mu_{2,2} = \mu^2 + 2\lambda\mu, \quad \mu_{3,2} = \mu_{4,2} = \lambda^2 + 2\lambda\mu, \quad 2H_1 = \lambda + \mu,\\ &6H_2 = \lambda^2 + 4\lambda\mu + \mu^2, \qquad 4H_3 = 2\lambda\mu(\lambda + \mu), \qquad H_4 = \lambda^2\mu^2.\end{aligned} \tag{4.9}$$

By condition (2.10)$(ii)$, we have $P_2(\nabla H_2) = 9H_2\nabla H_2$, which, using the polar decomposition $\nabla H_2 = \sum_{i=1}^4 \epsilon_i\langle\nabla H_2, e_i\rangle e_i$, gives $\epsilon_i\langle\nabla H_2, e_i\rangle(\mu_{i,2} - 9H_2) = 0$ on $\mathcal{U}$ for $i = 1, 2, 3, 4$. Hence, for some $i$ such that $\langle\nabla H_2, e_i\rangle \neq 0$ on $\mathcal{U}$, we get

$$\mu_{i,2} = 9H_2. \tag{4.10}$$

By definition, we have $\nabla H_2 \neq 0$ on $\mathcal{U}$, which gives one or both of the following states.

**State 1.** $\langle\nabla H_2, e_i\rangle \neq 0$, for some $i \in \{1, 2\}$. Using equalities (4.9), from (4.10) we obtain $\mu = c_0\lambda$, where $c_0 := -4 \pm \sqrt{13}$. Then, $H_2 = \frac{4(c_0^2+4c_0+1)}{(1+c_0)^2}H^2$.

**State 2.** $\langle\nabla H_2, e_i\rangle = 0$ for $i \in \{1, 2\}$, and $\langle\nabla H_2, e_j\rangle \neq 0$ for some $j \in \{3, 4\}$. By equalities (4.9) and (4.10), we obtain $3\lambda^2 = \frac{9}{2}(\lambda\mu + \lambda^2)$, which gives $\lambda = c_0\mu$, then, $H_2 = \frac{4(c_0^2+4c_0+1)}{(1+c_0)^2}H^2$. Hence, $H_2$ is constant on $M$. □

**Theorem 4.4** Every $L_1$-biconservative timelike hypersurface in $\mathbb{L}^5$ with shape operator of type $II$, having constant ordinary mean curvature and at most two distinct principal curvatures, has constant 2nd mean curvature.

**Proof:** Assuming an isometrical immersion $x : M_1^4 \to \mathbb{L}^5$ to be a $L_1$-biharmonic timelike hypersurface with shape operator of type $II$ in $\mathbb{L}^5$, which has constant ordinary mean curvature and at most two distinct principal curvatures, as the first stage, we show that $H_2$ (i.e. the 2th mean curvature of $M_1^4$) is constant on $M_1^4$. We suppose that, $H_2$ is non-constant and considering the open subset $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$, we try to show $\mathcal{U} = \emptyset$. By the assumption, with respect to a suitable (local) orthonormal tangent frame $\{e_1, \cdots, e_4\}$ on $M$, the shape operator $A$ has the matrix form $\tilde{B}_2$, such that $Ae_1 = (\kappa + \frac{1}{2})e_1 - \frac{1}{2}e_2$, $Ae_2 = \frac{1}{2}e_1 + (\kappa - \frac{1}{2})e_2$, $Ae_3 = \lambda e_3$ and $Ae_4 = \lambda e_4$, and then, we have $P_2e_1 = [\lambda^2 + 2(\kappa - \frac{1}{2})\lambda]e_1 + 2\lambda e_2$, $P_2e_2 = -\lambda e_1 + [\lambda^2 + 2(\kappa + \frac{1}{2})\lambda]e_2$, and $P_2e_3 = (\kappa^2 + 2\kappa\lambda)e_3$ and $P_2e_4 = (\kappa^2 + 2\kappa\lambda)e_4$.

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{4} \epsilon_i e_i(H_2)e_i$, from condition (2.10)$(ii)$ we have $P_2\nabla H_2 = 9H_2\nabla H_2$, which gives

$$\begin{aligned}
&(i) \quad [\lambda^2 + 2(\kappa - \frac{1}{2})\lambda) - 9H_2]\epsilon_1 e_1(H_2) = \lambda\epsilon_2 e_2(H_2)\\
&(ii) \quad [\lambda^2 + 2(\kappa + \frac{1}{2})\lambda - 9H_2]\epsilon_2 e_2(H_2) = -\lambda\epsilon_1 e_1(H_2),\\
&(iii) \quad (\kappa^2 + 2\kappa\lambda - 9H_2)\epsilon_3 e_3(H_2) = 0,\\
&(iv) \quad (\kappa^2 + 2\kappa\lambda - 9H_2)\epsilon_3 e_4(H_2) = 0.
\end{aligned} \tag{4.11}$$

Now, we prove some simple claims.

*Claim*: $e_1(H_2) = e_2(H_2) = e_3(H_2) = e_4(H_2) = 0$.

If $e_1(H_2) \neq 0$, then by dividing both sides of equalities (4.11)$(i, ii)$ by $\epsilon_1 e_1(H_2)$ we get

$$\begin{aligned}
&(i) \quad \lambda^2 + 2(\kappa - \frac{1}{2})\lambda - 9H_2 = \lambda u,\\
&(ii) \quad [\lambda^2 + 2(\kappa + \frac{1}{2})\lambda - 9H_2]u = -\lambda,
\end{aligned} \tag{4.12}$$

where $u := \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$. By substituting $(i)$ in $(ii)$, we obtain $\lambda(1 + u)^2 = 0$, then $\lambda = 0$ or $u = -1$. If $\lambda = 0$, then, from (4.12)$(i)$ we obtain $H_2 = 0$, which means $H_2$ is constant. Also, by assumption $\lambda \neq 0$ and $u = -1$, which gives $\lambda^2 + 2\kappa\lambda = 9H_2$, then $3\kappa^2 + \lambda^2 + 8\kappa\lambda = 0$. Since $2H_1 = \kappa + \lambda$ is assumed to be constant on $M$, by instituting which in the last equality, we get $\lambda^2 - H_1\lambda - 3H_1^2 = 0$, which means $\lambda$, $\kappa$ and the $k$th mean curvatures (for $k = 2, 3, 4$) are constant on $M$. So, we got a contradiction and therefore, the first part of the claim is proved.

By a similar manner, each of assumptions $e_i(H_2) \neq 0$ for $i = 2, 3, 4$, gives the equality $\lambda^2 + 2\kappa\lambda = 9H_2$, which implies the contradiction that $H_2$ is constant on $M$. So, the claim is confirmed. □

**Theorem 4.5** Every $L_1$-biconservative timelike hypersurface in $\mathbb{L}^5$ with shape operator of type $III$, having constant ordinary mean curvature, has constant 2nd mean curvature.

**Proof:** Similar to proof of Theorem 4.4, we assume that $H_2$ is non-constant and considering the open subset $\mathcal{U}$, we prove that $\mathcal{U} = \emptyset$. Similarly, we get the conditions

$$\begin{aligned}
&(i) \quad (\kappa^2 + 2\kappa\lambda - \frac{1}{2} - 9H_2)\epsilon_1 e_1(H_2) - \frac{1}{2}\epsilon_2 e_2(H_2) - \frac{\sqrt{2}}{2}(\kappa+\lambda)\epsilon_3 e_3(H_2) = 0 \\
&(ii) \quad \frac{1}{2}\epsilon_1 e_1(H_2) + (\kappa^2 + 2\kappa\lambda + \frac{1}{2} - 9H_2)\epsilon_2 e_2(H_2) + \frac{\sqrt{2}}{2}(\kappa+\lambda)\epsilon_3 e_3(H_2) = 0 \\
&(iii) \quad \frac{\sqrt{2}}{2}(\kappa+\lambda)(\epsilon_1 e_1(H_2) + \epsilon_2 e_2(H_2)) + (\kappa^2 + 2\kappa\lambda - 9H_2)\epsilon_3 e_3(H_2) = 0, \\
&(iv) \quad (3\kappa^2 - 9H_2)\epsilon_4 e_4(H_2) = 0.
\end{aligned} \tag{4.13}$$

Now, we prove that $H_2$ is constant .

*Claim*: $e_1(H_2) = e_2(H_2) = e_3(H_2) = e_4(H_2) = 0$.
If $e_1(H_2) \neq 0$, then by dividing both sides of equalities (4.13)$(i, ii, iii)$ by $\epsilon_1 e_1(H_2)$, and using the identity $2H_2 = \kappa^2 + \kappa\lambda$ in type $III$, putting $u_1 := \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$ and $u_2 := \frac{\epsilon_3 e_3(H_2)}{\epsilon_1 e_1(H_2)}$, we get

$$\begin{aligned}
&(i) \quad -\frac{1}{2} - \frac{7}{2}\kappa^2 - \frac{5}{2}\kappa\lambda - \frac{1}{2}u_1 - \frac{\sqrt{2}}{2}(\kappa+\lambda)u_2 = 0 \\
&(ii) \quad \frac{1}{2} + (\frac{1}{2} - \frac{7}{2}\kappa^2 - \frac{5}{2}\kappa\lambda)u_1 + \frac{\sqrt{2}}{2}(\kappa+\lambda)u_2 = 0 \\
&(iii) \quad \frac{-\sqrt{2}}{2}(\kappa+\lambda)(1+u_1) - (\frac{7}{2}\kappa^2 + \frac{5}{2}\kappa\lambda)u_2 = 0,
\end{aligned} \tag{4.14}$$

which, by comparing $(i)$ and $(ii)$, gives $\frac{-1}{2}\kappa(7\kappa + 5\lambda)(1+u_1) = 0$. If $\kappa = 0$, then $H_2 = 0$. Assuming $\kappa \neq 0$, we get $u_1 = -1$ or $\lambda = -\frac{7}{5}\kappa$. If $u_1 \neq -1$ then $\lambda = -\frac{7}{5}\kappa$, then by (4.14)$(iii)$ we obtain $u_1 = -1$, which is a contradiction. Hence we have $u_1 = -1$, which by (4.14)$(i, iii)$ implies $u_2 = 0$.

Now we discuss on two cases $\lambda = -\frac{7}{5}\kappa$ or $\lambda \neq -\frac{7}{5}\kappa$. If $\lambda = -\frac{7}{5}\kappa$, then, $\kappa = \frac{5}{2}H_1$, $H_2 = \frac{-1}{5}\kappa^2$, $H_3 = \frac{-4}{5}\kappa^3$ and $H_4 = \frac{-7}{5}\kappa^4$ are all constants on $\mathcal{U}$. Also, the case $\lambda \neq -\frac{7}{5}\kappa$ is in contradiction with (4.14)$(ii)$.

Hence, the first claim $e_1(H_2) \equiv 0$ is proved. The second one (i.e. $e_2(H_2) = 0$) can be proved in same manner. Applying the results $e_1(H_2) = e_2(H_2) = 0$, from (4.14)$(ii, iii)$ we get $e_3(H_2) = 0$. The forth claim (i.e. $e_2(H_2) = 0$) can be affirmed using (4.14)$(iv)$ in a straightforward manner. □

**Theorem 4.6** Every $L_1$-biconservative connected orientable Lorentzian hypersurface with shape operator of type $IV$ in $\mathbb{E}^5_1$, having at most two distinct principal curvatures, has constant 2nd mean curvature.

**Proof:** Suppose that, $H_2$ be non-constant. Considering the open subset $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$, we try to show $\mathcal{U} = \emptyset$. By assumption, the shape operator $A$ of $M_1^4$ is of type $IV$ with at most two distinct nonzero eigenvalue functions, then, with respect to a suitable (local) orthonormal tangent frame $\{e_1, \cdots, e_4\}$ on $M$, the shape operator $A$ has the matrix form $B_4$, such that $Ae_1 = -\lambda e_2$, $Ae_2 = \lambda e_1$, $Ae_3 = Ae_4 = 0$ and then, we have $P_2 e_1 = P_2 e_2 = 0$, $P_2 e_3 = \lambda^2 e_3$ and $P_2 e_4 = \lambda^2 e_4$. Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{4} \epsilon_i e_i(H_2) e_i$, from condition (2.10)$(ii)$ we get

$$\begin{aligned}
&(i) \quad 9H_2)\epsilon_1 e_1(H_2) = 0, \qquad (ii) \quad 9H_2)\epsilon_2 e_2(H_2) = 0, \\
&(iii) \quad (\lambda^2 - 9H_2)\epsilon_3 e_3(H_2) = 0, \quad (iv) \ (\lambda^2 - 9H_2)\epsilon_4 e_4(H_2) = 0,
\end{aligned} \tag{4.15}$$

which gives $e_1(H_2) = e_2(H_2) = e_3(H_2) = e_4(H_2) = 0$.
Then, it's 2nd mean curvatures is constant. □

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