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Some Results on Zeros of a Polynomial

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ABSTRACT: For a given polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ with real or complex coefficients, Gulzar (2015) proved for any real $p > 1, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, all the zeros of $p(z)$ lie in the closed circular region

$$|z| \le \max\left\{L_p, L_p^{\frac{1}{n+1}}\right\},$$

where

$$L_p = (n+1)^{\frac{1}{q}} \left(\sum_{j=1}^{n} \left| \frac{a_{n-1}a_{n-j} - a_n a_{n-j-1}}{a_n^2} \right|^p \right)^{\frac{1}{p}},\ a_{-1} = 0.$$

Recently, Gulzar also generalized the above result by using a lacunary type polynomial. Unfortunately, the proof is not correct. In this present paper an attempt has been taken to investigate and extend the above results were made. Moreover, we have produce some important corollaries which give the generalization of Mezerji and Bidkham (2014) results.

Key Words: Zeros, Cauchy bound, circular region, Lacunary type polynomials.

Contents

1. Introduction and statement of results

Let

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n$$

be a polynomial of degree n with real or complex coefficients. Concerning the bound for the moduli of zeros of $p(z)$, a famous result was obtained by Cauchy [3, Ch. VII, Sect. 27, Thm. 27.2] (see also [1], [8, Ch. VIII, Cor. 8.1.8]) is well-known.

Using Holder's inequality, Mezerji and Bidkham [2] proved the following Theorems.

Theorem A. *For any real* $p > 1, q > 1$ with

$$\frac{1}{p} + \frac{1}{q} = 1,$$

all the zeros of $p(z)$ *lie in the closed circular region*

$$|z| \le \left(1 + A_p^q\right)^{\frac{1}{q}},$$

where

$$A_p = \min_{-1 \le i \le n} \{A_{p,i}\},\ A_{p,i} = \left(\sum_{j=0}^{n} \left| \frac{a_i a_{n-j} - a_n a_{n-j-1}}{a_n^2} \right|^p \right)^{\frac{1}{p}},\ a_{-1} = 0.$$

Theorem B. *For any real* $p > 1, q > 1$ with

$$\frac{1}{p} + \frac{1}{q} = 1,$$

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all the zeros of $p(z)$ *never lie in the closed circular region*

$$|z| \leq \left(1+B_p^q\right)^{-\frac{1}{q}},$$

where

$$B_p = \min_{-1\leq i\leq n}\{B_{p,i}\},\ B_{p,i} = \left(\sum_{j=0}^{n}\left|\frac{a_i a_j - a_0 a_{j+1}}{a_0^2}\right|^p\right)^{\frac{1}{p}},\ a_{-1}=a_{n+1}=0.$$

Applying the similar arguments, in 2015, Gulzar [4] established the following result.

Theorem C. *For any real* $p>1, q>1$ with

$$\frac{1}{p}+\frac{1}{q}=1,$$

all the zeros of $p(z)$ *with* $a_{n-1}\neq 0$ *lie in the closed circular region*

$$|z| \leq \max\left\{L_p, L_p^{\frac{1}{n+1}}\right\},$$

where

$$L_p = (n+1)^{\frac{1}{q}}\left(\sum_{j=1}^{n}\left|\frac{a_{n-1}a_{n-j} - a_n a_{n-j-1}}{a_n^2}\right|^p\right)^{\frac{1}{p}},\ a_{-1}=0.$$

Recently, Gulzar [5] also generalized the above result by using lacunary type polynomial [3, Ch. VIII, Sect. 34, pp. 156]

$$P(z) = \sum_{j=0}^{\lambda} a_j z^j + a_n z^n,\ 1\leq\lambda\leq n-1,\ a_\lambda\neq 0$$

and obtained the following result.

Theorem D. *For any real* $p>1, q>1$ with

$$\frac{1}{p}+\frac{1}{q}=1,$$

all the zeros of $P(z)$ *lie in the closed circular region*

$$|z| \leq \max\left\{L, L^{\frac{1}{n+1}}\right\},$$

where

$$L = (n+1)^{\frac{1}{q}}\left(\sum_{j=0}^{\lambda+1}\left|\frac{a_\lambda a_{\lambda+1-j} - a_n a_{\lambda-j}}{a_n^2}\right|^p\right)^{\frac{1}{p}},\ a_{\lambda+1}=a_{-1}=0.$$

Unfortunately, the above theorem is not correct due to some Author's mistakes in the beginning of the proof as written by

$$\begin{aligned} F(z) &= (a_\lambda - a_n z)P(z) \\ &= (a_\lambda - a_n z)\left(a_n z^n + a_\lambda z^\lambda + a_{\lambda-1}z^{\lambda-1} + \cdots + a_1 z + a_0\right) \\ &= -a_n^2 z^{n+1} - a_\lambda a_n z^{\lambda+1} + (a_\lambda a_\lambda - a_n a_{\lambda-1})z^\lambda + \cdots + a_\lambda a_0 \\ &= -a_n^2 z^{n+1} + \sum_{j=0}^{\lambda+1}(a_\lambda a_{\lambda+1-j} - a_n a_{\lambda-j})z^{\lambda+1-j} \end{aligned}$$

and obtained the inequality

$$|F(z)| \geq |a_n|^2 |z|^{n+1} - \sum_{j=0}^{\lambda+1} |a_\lambda a_{\lambda+1-j} - a_n a_{\lambda-j}| \, |z|^{\lambda+1-j}.$$

In fact,

$$F(z) = -a_n^2 z^{n+1} + a_n a_\lambda z^n + \sum_{j=0}^{\lambda+1} (a_\lambda a_{\lambda+1-j} - a_n a_{\lambda-j}) \, z^{\lambda+1-j}$$

and therefore,

$$\begin{aligned} |F(z)| &\geq |a_n|^2 |z|^{n+1} - |a_n a_\lambda| \, |z|^n - \sum_{j=0}^{\lambda+1} |a_\lambda a_{\lambda+1-j} - a_n a_{\lambda-j}| \, |z|^{\lambda+1-j} \\ &\ngeq |a_n|^2 |z|^{n+1} - \sum_{j=0}^{\lambda+1} |a_\lambda a_{\lambda+1-j} - a_n a_{\lambda-j}| \, |z|^{\lambda+1-j}, \end{aligned}$$

which shows that the inequality

$$|F(z)| \geq |a_n|^2 |z|^{n+1} - \sum_{j=0}^{\lambda+1} |a_\lambda a_{\lambda+1-j} - a_n a_{\lambda-j}| \, |z|^{\lambda+1-j}$$

is not true.

In the literature, there exist some results ([2,3,4,5,6,7]) on polynomial zeros by using Holder's inequality.

In this paper, we have obtained some results concerning the bounds for the zeros of a lacunary type polynomial $P(z)$ by using Holder's inequality. Our results produce an important corollaries which give the generalization of Theorem A, Theorem B and Theorem C, respectively. Moreover, some of the results give a zero free region. More precisely, we prove

Theorem 1.1 *For any real or complex t with $p > 1, q > 1$,*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

all the zeros of $P(z)$ lie in the closed circular region

$$|z| \leq \max\left\{ L_{p,t}, L_{p,t}^{\frac{1}{n+1}} \right\},$$

where

$$L_{p,t} = \left\{ \begin{array}{ll} (\lambda+3)^{\frac{1}{q}} \left(\sum_{j\in\Omega} \left| \frac{t a_j - a_n a_{j-1}}{a_n^2} \right|^p \right)^{\frac{1}{p}}; & \textit{for } 0 < \lambda < n-1 \textit{ with } a_{-1} = a_{\lambda+1} = 0 \\ (n+1)^{\frac{1}{q}} \left(\sum_{j=0}^{n} \left| \frac{t a_j - a_n a_{j-1}}{a_n^2} \right|^p \right)^{\frac{1}{p}}; & \textit{for } \lambda = n-1 \textit{ with } a_{-1} = 0 \end{array} \right\} \tag{1.1}$$

and

$$\Omega = \{0, 1, \ldots, \lambda, \lambda+1, n\}.$$

Clearly

$$\sum_{j=0}^{n} \left| \frac{t a_j - a_n a_{j-1}}{a_n^2} \right|^p = \sum_{j=0}^{n} \left| \frac{t a_{n-j} - a_n a_{n-j-1}}{a_n^2} \right|^p. \tag{1.2}$$

Putting $t = a_{n-1}$ in the above expression we have

$$\begin{aligned}\sum_{j=0}^{n}\left|\frac{a_{n-1}a_j - a_n a_{j-1}}{a_n^2}\right|^p &= \sum_{j=0}^{n}\left|\frac{a_{n-1}a_{n-j} - a_n a_{n-j-1}}{a_n^2}\right|^p \\ &= \sum_{j=1}^{n}\left|\frac{a_{n-1}a_{n-j} - a_n a_{n-j-1}}{a_n^2}\right|^p.\end{aligned}$$

So for the case of $\lambda = n-1$, we get from (1.1),

$$L_{p,t} = (n+1)^{\frac{1}{q}}\left(\sum_{j=1}^{n}\left|\frac{a_{n-1}a_{n-j} - a_n a_{n-j-1}}{a_n^2}\right|^p\right)^{\frac{1}{p}} \quad \text{when } t = a_{n-1}.$$

Also, for $0 < \lambda < n-1$, we get from (1.1),

$$L_{p,t} = (\lambda+3)^{\frac{1}{q}}\left(\sum_{j\in\Omega}\left|\frac{a_\lambda a_j - a_n a_{j-1}}{a_n^2}\right|^p\right)^{\frac{1}{p}} \quad \text{when } t = a_\lambda.$$

Hence, with the help of Theorem 1.1 we can easily obtained the following Corollary which is a generalization of Theorem C.

Corollary 1.1 *For any real $p > 1, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, all the zeros of $P(z)$ lie in the closed circular region*

$$|z| \le \max\left\{L_p, L_p^{\frac{1}{n+1}}\right\},$$

where

$$L_p = \left\{\begin{array}{ll} (\lambda+3)^{\frac{1}{q}}\left(\sum_{j\in\Omega}\left|\frac{a_\lambda a_j - a_n a_{j-1}}{a_n^2}\right|^p\right)^{\frac{1}{p}}; & \text{for } 0<\lambda<n-1 \text{ with } a_{-1} = a_{\lambda+1} = 0 \\ (n+1)^{\frac{1}{q}}\left(\sum_{j=1}^{n}\left|\frac{a_{n-1}a_{n-j} - a_n a_{n-j-1}}{a_n^2}\right|^p\right)^{\frac{1}{p}}; & \text{for } \lambda = n-1 \text{ with } a_{-1} = 0 \end{array}\right\}$$

and Ω is same as in Theorem 1.1.

Theorem 1.2 *For any real or complex t with $p > 1, q > 1$,*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

all the zeros of $P(z)$ never lie in an open circular region

$$|z| < \max\left\{\frac{1}{L'_{p,t}}, \frac{1}{(L'_{p,t})^{\frac{1}{n+1}}}\right\},$$

where

$$L'_{p,t} = \left\{\begin{array}{ll} (\lambda+3)^{\frac{1}{q}}\left(\sum_{j\in\Delta}\left|\frac{t a_j - a_0 a_{j+1}}{a_0^2}\right|^p\right)^{\frac{1}{p}}; & \text{for } 0<\lambda<n-1 \text{ with } a_{n+1} = a_{\lambda+1} = 0 \\ (n+1)^{\frac{1}{q}}\left(\sum_{j=0}^{n}\left|\frac{t a_j - a_0 a_{j+1}}{a_0^2}\right|^p\right)^{\frac{1}{p}}; & \text{for } \lambda = n-1 \text{ with } a_{n+1} = 0 \end{array}\right\} \tag{1.3}$$

and

$$\Delta = \{0, 1, \ldots, \lambda, n-1, n\}.$$

Note that the estimated bounds in Theorem 1.1 and Theorem 1.2 depend not only the coefficients of $P(z)$ but also the value of t. Here we estimate the expected values of t for which the bounds in Theorem 1.1, and Theorem 1.2 give the better approximation for the zeros of $P(z)$ by considering t being real. At first we determine the value of t for which the Theorem 1.1 gives better bound that can be obtain by reducing the modulus value of the coefficients. It is clear from the equation (2.1), the expected value of t may be obtain when the value of

$$S(t)=\left\{\begin{array}{c}\sum\limits_{j\in\Omega-(\lambda+1)}|ta_j-a_na_{j-1}|^2;\ \text{for }\lambda<n-1\\ \sum\limits_{j=0}^{n}|ta_j-a_na_{j-1}|^2;\ \text{for }\lambda=n-1\end{array}\right\}$$

is minimum (for $j=\lambda+1$, the corresponding term in the equation (2.1) does not involve the variable t and so we omitted that particular term from the above expression $S(t)$ for the case of $\lambda<n-1$).
For $\lambda<n-1$, we have

$$\begin{aligned}S(t)&=\sum_{j\in\Omega-\{\lambda+1\}}(ta_j-a_na_{j-1})\overline{(ta_j-a_na_{j-1})}\\&=\sum_{j\in\Omega-\{\lambda+1\}}(ta_j-a_na_{j-1})(t\overline{a_j}-\overline{a_na_{j-1}})\\&=\sum_{j\in\Omega-\{\lambda+1\}}\left(|a_j|^2t^2-(a_n\overline{a_j}a_{j-1}+\overline{a_n}a_j\overline{a_{j-1}})t+|a_n|^2|a_{j-1}|^2\right)\\&=t^2\sum_{j\in\Omega-\{\lambda+1\}}|a_j|^2-2t\sum_{j\in\Omega-\{\lambda+1\}}\mathrm{Re}(a_n\overline{a_j}a_{j-1})+|a_n|^2\sum_{j\in\Omega-\{\lambda+1\}}|a_{j-1}|^2.\end{aligned}$$

Clearly,

$$S'(t)=2t\sum_{j\in\Omega-\{\lambda+1\}}|a_j|^2-2\sum_{j\in\Omega-\{\lambda+1\}}\mathrm{Re}(a_n\overline{a_j}a_{j-1})$$

and

$$S''(t)=2\sum_{j\in\Omega-\{\lambda+1\}}|a_j|^2>0,$$

which shows that $S(t)$ attains its minimum value at

$$\frac{\sum\limits_{j\in\Omega-\{\lambda+1\}}\mathrm{Re}(a_n\overline{a_j}a_{j-1})}{\sum\limits_{j\in\Omega-\{\lambda+1\}}|a_j|^2}.$$

Similarly, for $\lambda=n-1$, the expected value of t is

$$\frac{\sum\limits_{j=0}^{n}\mathrm{Re}(a_n\overline{a_j}a_{j-1})}{\sum\limits_{j=0}^{n}|a_j|^2}.$$

Hence, the expected value of t for the bound in Theorem 1.1 is given by

$$t_0=\left\{\begin{array}{l}\frac{\sum\limits_{j\in\Omega-\{\lambda+1\}}\mathrm{Re}(a_n\overline{a_j}a_{j-1})}{\sum\limits_{j\in\Omega-\{\lambda+1\}}|a_j|^2};\ \text{for }\lambda<n-1\\ \frac{\sum\limits_{j=0}^{n}\mathrm{Re}(a_n\overline{a_j}a_{j-1})}{\sum\limits_{j=0}^{n}|a_j|^2};\ \text{for }\lambda=n-1\end{array}\right\}. \tag{1.4}$$

Again by observing the equation (2.2), the expected value of t can be obtain when

$$M(t)=\left\{\begin{array}{c}\sum\limits_{j\in\Delta-\{n-1\}}\left|ta_j-a_0a_{j+1}\right|^2;\ \text{for } \lambda<n-1\\ \sum\limits_{j=0}^{n}\left|ta_j-a_0a_{j+1}\right|^2;\ \text{for } \lambda=n-1\end{array}\right\}$$

is minimum (for $j=n-1$, the corresponding term in the equation (2.2) does not involve the variable t and so we remove that particular term from the above expression $M(t)$ for the case of $\lambda<n-1$) for the bounds in Theorem 1.2 and is given by

$$t_0'=\left\{\begin{array}{c}\dfrac{\sum\limits_{j\in\Delta-\{n-1\}}\mathrm{Re}(a_0a_{j+1}\overline{a_j})}{\sum\limits_{j\in\Delta-\{n-1\}}|a_j|^2};\ \text{for } \lambda<n-1\\ \dfrac{\sum\limits_{j=0}^{n}\mathrm{Re}(a_0a_{j+1}\overline{a_j})}{\sum\limits_{j=0}^{n}|a_j|^2};\ \text{for } \lambda=n-1\end{array}\right\}, \tag{1.5}$$

Putting $t=t_0$ in Theorem 1.1 and $t=t_0'$ in Theorem 1.2 and combine them, we can easily obtain the following Corollary.

Corollary 1.2 *For any real $p>1,q>1$ with*

$$\frac{1}{p}+\frac{1}{q}=1,$$

all the zeros of $P(z)$ lie in the closed annular region

$$\max\left\{\frac{1}{L'_{p,t_0'}},\frac{1}{\left(L'_{p,t_0'}\right)^{\frac{1}{n+1}}}\right\}\leq|z|\leq\max\left\{L_{p,t_0},L_{p,t_0}^{\frac{1}{n+1}}\right\},$$

where $L_{p,t_0},L'_{p,t_0'}$ are the values of $L_{p,t},L'_{p,t}$ (see equations (1.1) and (1.3)) at t_0,t_0' (see equations (1.4) and (1.5)) respectively.

Theorem 1.3 *For any real or complex t with $p>1,q>1$,*

$$\frac{1}{p}+\frac{1}{q}=1,$$

all the zeros of $P(z)$ lie in an open circular region

$$|z|<\left(1+A_{p,t}^q\right)^{\frac{1}{q}},$$

where

$$A_{p,t}=\left\{\begin{array}{l}B_{p,t};\ \textit{for } 0<\lambda<n-1 \textit{ with } a_{-1}=a_{\lambda+1}=0\\ C_{p,t};\ \textit{for } \lambda=n-1 \textit{ with } a_{-1}=0\end{array}\right\}, \tag{1.6}$$

$$B_{p,t}=\left(\sum_{j\in\Omega}\left|\frac{ta_j-a_na_{j-1}}{a_n^2}\right|^p\right)^{\frac{1}{p}},C_{p,t}=\left(\sum_{j=0}^{n}\left|\frac{ta_j-a_na_{j-1}}{a_n^2}\right|^p\right)^{\frac{1}{p}}, \tag{1.7}$$

and Ω is same as in Theorem 1.1.

With the help of Theorem 1.3 and using the equation (1.2), we obtain the following Corollary as follows.

Corollary 1.3 *For any* $p>1, q>1$ *with*

$$\frac{1}{p}+\frac{1}{q}=1,$$

all the zeros of $P(z)$ *lie in an open circular region*

$$|z|<\left(1+A_p^q\right)^{\frac{1}{q}},$$

where

$$A_p=\left\{\begin{array}{c}\min\limits_{i\in\Omega}\left\{B_{p,a_i}\right\};\ \text{for } 0<\lambda<n-1 \text{ with } a_{-1}=a_{\lambda+1}=0\\ \min\limits_{-1\leq i\leq n}\left\{C_{p,a_i}\right\};\ \text{for } \lambda=n-1 \text{ with } a_{-1}=0\end{array}\right\},$$

$$B_{p,a_i}=\left(\sum_{j\in\Omega}\left|\frac{a_ia_j-a_na_{j-1}}{a_n^2}\right|^p\right)^{\frac{1}{p}}, C_{p,a_i}=\left(\sum_{j=0}^{n}\left|\frac{a_ia_{n-j}-a_na_{n-j-1}}{a_n^2}\right|^p\right)^{\frac{1}{p}},$$

and Ω *is same as in Theorem 1.1.*

Clearly, corollary 1.3 is a generalization of Theorem A. Also, with the help of Theorem 1.3 we obtain the next Corollary by putting $t=t_0$ (see equation (1.4)) as follows.

Corollary 1.4 *For any* $p>1, q>1$ *with*

$$\frac{1}{p}+\frac{1}{q}=1,$$

all the zeros of $P(z)$ *lie in an open circular region*

$$|z|<\left(1+A_{p,t_0}^q\right)^{\frac{1}{q}},$$

where A_{p,t_0} *represent the value of* $A_{p,t}$ *(see equation (1.6)) at* $t=t_0$.

Theorem 1.4 *For any real or complex* t *with* $p>1, q>1$,

$$\frac{1}{p}+\frac{1}{q}=1,$$

all the zeros of $P(z)$ *never lie in the closed circular region*

$$|z|\leq\left(1+B_{p,t}^q\right)^{-\frac{1}{q}},$$

where

$$B_{p,t}=\left\{\begin{array}{c}D_{p,t};\ \text{for } 0<\lambda<n-1 \text{ with } a_{n+1}=a_{\lambda+1}=0\\ E_{p,t};\ \text{for } \lambda=n-1 \text{ with } a_{-1}=0\end{array}\right\}, \tag{1.8}$$

$$D_{p,t}=\left(\sum_{j\in\Delta}\left|\frac{ta_j-a_0a_{j+1}}{a_0^2}\right|^p\right)^{\frac{1}{p}} \text{ and } E_{p,t}=\left(\sum_{j=0}^{n}\left|\frac{ta_j-a_0a_{j+1}}{a_0^2}\right|^p\right)^{\frac{1}{p}}, \tag{1.9}$$

and Δ *is same as in Theorem 1.2.*

With the help of Theorem 1.4, we obtain the following Corollary which is a generalization of Theorem B as follows.

Corollary 1.5 *For any* $p>1, q>1$ *with*

$$\frac{1}{p}+\frac{1}{q}=1,$$

all the zeros of $P(z)$ *never lie in the closed circular region*

$$|z|<\left(1+B_p^q\right)^{-\frac{1}{q}},$$

where Ω *is same as in Theorem 1.1 and*

$$B_p=\left\{\begin{array}{c}\min\limits_{i\in\Omega}\{D_{p,a_i}\};\ \textit{for}\ 0<\lambda<n-1\ \textit{with}\ a_{-1}=a_{\lambda+1}=0\\ \min\limits_{-1\leq i\leq n}\{E_{p,a_i}\};\ \textit{for}\ \lambda=n-1\ \textit{with}\ a_{-1}=0\end{array}\right\},$$

$$D_{p,a_i}=\left(\sum_{j\in\Delta}\left|\frac{a_ia_j-a_0a_{j+1}}{a_0^2}\right|^p\right)^{\frac{1}{p}}\ \textit{and}\ E_{p,a_i}=\left(\sum_{j=0}^{n}\left|\frac{a_ia_j-a_0a_{j+1}}{a_0^2}\right|^p\right)^{\frac{1}{p}}.$$

Put $t=t_0'$ (see equation (1.5)) in Theorem 1.4 we have the following Corollary.

Corollary 1.6 *For any* $p>1,q>1$ *with*

$$\frac{1}{p}+\frac{1}{q}=1,$$

all the zeros of $P(z)$ *never lie in the closed circular region*

$$|z|\leq\left(1+B_{p,t_0'}^q\right)^{-\frac{1}{q}},$$

where

$$B_{p,t_0'}=\left\{\begin{array}{c}D_{p,t_0'};\ \textit{for}\ 0<\lambda<n-1\ \textit{with}\ a_{n+1}=a_{\lambda+1}=0\\ E_{p,t_0'};\ \textit{for}\ \lambda=n-1\ \textit{with}\ a_{-1}=0\end{array}\right\},$$

$$D_{p,t_0'}=\left(\sum_{j\in\Delta}\left|\frac{t_0'a_j-a_0a_{j+1}}{a_0^2}\right|^p\right)^{\frac{1}{p}},E_{p,t_0'}=\left(\sum_{j=0}^{n}\left|\frac{t_0'a_j-a_0a_{j+1}}{a_0^2}\right|^p\right)^{\frac{1}{p}},$$

$\Delta=\{0,1,\ldots,\lambda,n-1,n\}$ *and* t_0' *is given by the equation (1.5).*

From Corollary 1.4 and Corollary 1.6, we can immediately obtained the following Corollary.

Corollary 1.7 *For any* $p>1,q>1$ *with*

$$\frac{1}{p}+\frac{1}{q}=1,$$

all the zeros of $P(z)$ *lie in an open annular region*

$$\left(1+B_{p,t_0'}^q\right)^{-\frac{1}{q}}<|z|<\left(1+A_{p,t_0}^q\right)^{\frac{1}{q}},$$

where A_{p,t_0} *and* $B_{p,t_0'}^q$ *are same as in Corollary 1.4 and Corollary 1.6 respectively.*

2. Proof of Theorems

Proof: [Proof of Theorem 1.1] Clearly, for some real or complex t,

$$\begin{aligned}F(z)&=(t-a_nz)P(z)\\&=\left\{\begin{array}{c}-a_n^2z^{n+1}+\sum\limits_{j\in\Omega}(ta_j-a_na_{j-1})z^j;\ \text{for}\ \lambda<n-1\ \text{with}\ a_{-1}=a_{\lambda+1}=0\\ -a_n^2z^{n+1}+\sum\limits_{j=0}^{n}(ta_j-a_na_{j-1})z^j;\ \text{for}\ \lambda=n-1\ \text{with}\ a_{-1}=0\end{array}\right\},\end{aligned}$$

where $\Omega=\{0,1,\ldots,\lambda,\lambda+1,n\}$.
Now, for some $|z|>0$,

$$|F(z)|\geq\left\{\begin{array}{c}|a_n|^2|z|^{n+1}-\sum\limits_{j\in\Omega}|ta_j-a_na_{j-1}||z|^j;\ \text{for}\ \lambda<n-1\\ |a_n|^2|z|^{n+1}-\sum\limits_{j=0}^{n}|ta_j-a_na_{j-1}||z|^j;\ \text{for}\ \lambda=n-1\end{array}\right\}. \tag{2.1}$$

There are two possibilities:
Case I: For $\lambda < n-1$,

$$|F(z)| \geq |a_n|^2 |z|^{n+1} \left[1 - \sum_{j\in\Omega} \left| \frac{ta_j - a_n a_{j-1}}{a_n^2} \right| \frac{1}{|z|^{n+1-j}} \right].$$

Applying Holder's inequality, we have

$$|F(z)| \geq |a_n|^2 |z|^{n+1} \left[1 - \left(\sum_{j\in\Omega} \left| \frac{ta_j - a_n a_{j-1}}{a_n^2} \right|^p \right)^{\frac{1}{p}} \left(\sum_{j\in\Omega} \frac{1}{|z|^{(n+1-j)q}} \right)^{\frac{1}{q}} \right].$$

On $|z| \geq 1$,

$$\begin{aligned} |z|^{(n+1-j)q} &\geq |z|^q \text{ for each } j \in \Omega, \\ i.e., \ \frac{1}{|z|^{(n+1-j)q}} &\leq \frac{1}{|z|^q}. \end{aligned}$$

So,

$$\begin{aligned} \sum_{j\in\Omega} \frac{1}{|z|^{(n+1-j)q}} &\leq (\lambda+3) \frac{1}{|z|^q} \\ i.e., \ \left(\sum_{j\in\Omega} \frac{1}{|z|^{(n+1-j)q}} \right)^{\frac{1}{q}} &\leq (\lambda+3)^{\frac{1}{q}} \frac{1}{|z|}. \end{aligned}$$

Which implies

$$|F(z)| \geq |a_n|^2 |z|^{n+1} \left[1 - (\lambda+3)^{\frac{1}{q}} \left(\sum_{j\in\Omega} \left| \frac{ta_j - a_n a_{j-1}}{a_n^2} \right|^p \right)^{\frac{1}{p}} \frac{1}{|z|} \right] > 0$$

if

$$|z| > (\lambda+3)^{\frac{1}{q}} \left(\sum_{j\in\Omega} \left| \frac{ta_j - a_n a_{j-1}}{a_n^2} \right|^p \right)^{\frac{1}{p}} = L_{p,t}.$$

Again, for $|z| \leq 1$,

$$\begin{aligned} |z|^{(n+1-j)q} &\geq |z|^{(n+1)q} \text{ for each } j \in \Omega, \\ i.e., \ \frac{1}{|z|^{(n+1-j)q}} &\leq \frac{1}{|z|^{(n+1)q}}, \end{aligned}$$

which implies

$$\begin{aligned} \sum_{j\in\Omega} \frac{1}{|z|^{(n+1-j)q}} &\leq (\lambda+3) \frac{1}{|z|^{(n+1)q}} \\ i.e., \ \left(\sum_{j\in\Omega} \frac{1}{|z|^{(n+1-j)q}} \right)^{\frac{1}{q}} &\leq (\lambda+3)^{\frac{1}{q}} \frac{1}{|z|^{(n+1)}}. \end{aligned}$$

Hence,

$$|F(z)| \geq |a_n|^2 |z|^{n+1} \left[1 - (\lambda+3)^{\frac{1}{q}} \left(\sum_{j\in\Omega} \left| \frac{ta_j - a_n a_{j-1}}{a_n^2} \right|^p \right)^{\frac{1}{p}} \frac{1}{|z|^{(n+1)}} \right] > 0$$

if

$$|z|>\left[(\lambda+3)^{\frac{1}{q}}\left(\sum_{j\in\Omega}\left|\frac{ta_j-a_na_{j-1}}{a_n^2}\right|^p\right)^{\frac{1}{p}}\right]^{\frac{1}{n+1}}=L_{p,t}^{\frac{1}{n+1}}.$$

Now, there are two possibilities of $L_{p,t}$:
(1) If $L_{p,t}\geq 1$, then

$$|F(z)|>0 \text{ for } |z|>L_{p.t}=\max\left\{L_{p,t},L_{p,t}^{\frac{1}{n+1}}\right\}.$$

(2) If $L_{p,t}<1$, then

$$|F(z)|>0 \text{ for } |z|>L_{p,t}^{\frac{1}{n+1}}=\max\left\{L_{p,t},L_{p,t}^{\frac{1}{n+1}}\right\}.$$

Combining both possibilities, we have

$$|F(z)|>0 \text{ if } |z|>\max\left\{L_{p,t},L_{p,t}^{\frac{1}{n+1}}\right\}$$

and it leads us to the desired result.
Case II: For $\lambda=n-1$,

$$|F(z)|\geq|a_n|^2|z|^{n+1}\left[1-\sum_{j=0}^{n}\left|\frac{ta_j-a_na_{j-1}}{a_n^2}\right|\frac{1}{|z|^{n+1-j}}\right].$$

In this case, we take

$$L_{p,t}=(n+1)^{\frac{1}{q}}\left(\sum_{j=0}^{n}\left|\frac{ta_j-a_na_{j-1}}{a_n^2}\right|^p\right)^{\frac{1}{p}}.$$

Applying Holder's inequality and do the same procedure mentioned in case I to establish the desired result. This completes the proof of Theorem 1.1. □

Proof: [Proof of Theorem 1.2] For some real or complex t, we construct a polynomial define by

$$\begin{aligned}R(z) &= -a_0^2z^{n+1}+(ta_0-a_0a_1)z^n+(ta_1-a_0a_2)z^{n-1}+\cdots+(ta_j-a_0a_{j+1})z^{n-j}\\ &\quad+\cdots+(ta_{\lambda-1}-a_0a_\lambda)z^{n-(\lambda-1)}+ta_\lambda z^{n-\lambda}-a_0a_nz+ta_n,\end{aligned}$$

where

$$R(z)=(t-a_0z)z^nP\left(\frac{1}{z}\right).$$

Clearly, $R(z)$ can be written as

$$R(z)=\left\{\begin{array}{l}-a_0^2z^{n+1}+\sum\limits_{j\in\Delta}(ta_j-a_0a_{j+1})z^{n-j};\ \text{for } \lambda<n-1 \text{ with } a_{n+1}=a_{\lambda+1}=0\\ -a_0^2z^{n+1}+\sum\limits_{j=0}^{n}(ta_j-a_0a_{j+1})z^{n-j};\ \text{for } \lambda=n-1 \text{ with } a_{n+1}=0\end{array}\right\},$$

where $\Delta=\{0,1,\ldots,\lambda,n-1,n\}$. So,

$$|R(z)|\geq\left\{\begin{array}{l}|a_0|^2|z|^{n+1}-\sum\limits_{j\in\Delta}|ta_j-a_0a_{j+1}||z|^{n-j};\ \text{for } \lambda<n-1\\ |a_0|^2|z|^{n+1}+\sum\limits_{j=0}^{n}|ta_j-a_0a_{j+1}||z|^{n-j};\ \text{for } \lambda=n-1\end{array}\right\}. \tag{2.2}$$

There are two possibilities of λ:
Case I: For $\lambda < n-1$,

$$|R(z)| \geq |a_0|^2 \left(|z|^{n+1} - \sum_{j\in\Delta} \left| \frac{ta_j - a_0 a_{j+1}}{a_0^2} \right| \frac{1}{|z|^{j+1}} \right).$$

Applying Holder's inequality, we have

$$|R(z)| \geq |a_0|^2 \left[|z|^{n+1} - \left(\sum_{j\in\Delta} \left| \frac{ta_j - a_0 a_{j+1}}{a_0^2} \right|^p \right)^{\frac{1}{p}} \left(\sum_{j\in\Delta} \frac{1}{|z|^{(j+1)q}} \right)^{\frac{1}{q}} \right].$$

On $|z| \geq 1$ and for each $j \in \Delta$,

$$|z|^{(j+1)q} \geq |z|^q$$

which implies

$$\left(\sum_{j\in\Delta} \frac{1}{|z|^{(j+1)q}} \right)^{\frac{1}{q}} \leq (\lambda+3)^{\frac{1}{q}} \frac{1}{|z|}.$$

So,

$$|R(z)| \geq |a_0|^2 \left[|z|^{n+1} - (\lambda+3)^{\frac{1}{q}} \left(\sum_{j\in\Delta} \left| \frac{ta_j - a_0 a_{j+1}}{a_0^2} \right|^p \right)^{\frac{1}{p}} \frac{1}{|z|} \right] > 0$$

if

$$|z| > (\lambda+3)^{\frac{1}{q}} \left(\sum_{j\in\Delta} \left| \frac{ta_j - a_0 a_{j+1}}{a_0^2} \right|^p \right)^{\frac{1}{p}} = L'_{p,t}.$$

When $|z| \leq 1$ then, for each $j \in \Delta$,

$$|z|^{(j+1)q} \geq |z|^{(n+1)q},$$

which implies

$$\left(\sum_{j\in\Delta} \frac{1}{|z|^{(j+1)q}} \right)^{\frac{1}{q}} \leq (\lambda+3)^{\frac{1}{q}} \frac{1}{|z|^{n+1}}.$$

So,

$$|R(z)| \geq |a_0|^2 \left[|z|^{n+1} - (\lambda+3)^{\frac{1}{q}} \left(\sum_{j\in\Delta} \left| \frac{ta_j - a_0 a_{j+1}}{a_0^2} \right|^p \right)^{\frac{1}{p}} \frac{1}{|z|^{n+1}} \right] > 0$$

if

$$|z| > \left[(\lambda+3)^{\frac{1}{q}} \left(\sum_{j\in\Delta} \left| \frac{ta_j - a_0 a_{j+1}}{a_0^2} \right|^p \right)^{\frac{1}{p}} \right]^{\frac{1}{n+1}} = \left(L'_{p,t}\right)^{\frac{1}{n+1}}.$$

There are two possibilities of $L'_{p,t}$:
(1) If $L'_{p,t} \geq 1$, then

$$|R(z)| > 0 \text{ for } |z| > L'_{p,t} = \max\left\{ L'_{p,t}, \left(L'_{p,t}\right)^{\frac{1}{n+1}} \right\}.$$

(2) If $L'_{p,t} < 1$, then

$$|R(z)| > 0 \text{ for } |z| > \left(L'_{p,t}\right)^{\frac{1}{n+1}} = \max\left\{ L'_{p,t}, \left(L'_{p,t}\right)^{\frac{1}{n+1}} \right\}.$$

Combining both possibilities, we have

$$|R(z)| > 0 \text{ if } |z| > \max\left\{L'_{p,t}, \left(L'_{p,t}\right)^{\frac{1}{n+1}}\right\}$$

and it leads us to the desired result.
Case II: For $\lambda = n-1$,

$$|R(z)| \geq |a_0|^2 |z|^{n+1}\left[1 - \sum_{j=0}^{n}\left|\frac{ta_j - a_n a_{j+1}}{a_0^2}\right| \frac{1}{|z|^{j+1}}\right].$$

Here we take

$$L'_{p,t} = \left(\sum_{j=0}^{n}\left|\frac{ta_j - a_n a_{j+1}}{a_0^2}\right|^p\right)^{\frac{1}{p}}.$$

Applying Holder's inequality and similar procedure as in case I, one can easily obtain the desired result. This completes the proof of Theorem 1.2. □

Proof: [Proof of Theorem 1.3] On $|z| > 1$, we have

$$|F(z)| \geq \left\{\begin{array}{ll} |a_n|^2 |z|^{n+1} - \sum\limits_{j\in\Omega} |ta_j - a_n a_{j-1}| \, |z|^j \, ; \text{ for } \lambda < n-1 \\ |a_n|^2 |z|^{n+1} - \sum\limits_{j=0}^{n} |ta_j - a_n a_{j-1}| \, |z|^j \, ; \text{ for } \lambda = n-1 \end{array}\right\}.$$

There are two possibilities of λ:
Case I: For $\lambda < n-1$,

$$|F(z)| \geq |a_n|^2 |z|^{n+1}\left[1 - \sum_{j\in\Omega}\left|\frac{ta_j - a_n a_{j-1}}{a_n^2}\right| \frac{1}{|z|^{n+1-j}}\right].$$

Using Holder's inequality, we have

$$\begin{aligned} |F(z)| &\geq |a_n|^2 |z|^{n+1}\left[1 - \left(\sum_{j\in\Omega}\left|\frac{ta_j - a_n a_{j-1}}{a_n^2}\right|^p\right)^{\frac{1}{p}} \left(\sum_{j\in\Omega}\frac{1}{|z|^{(n+1-j)q}}\right)^{\frac{1}{q}}\right] \\ &\geq |a_n|^2 |z|^{n+1}\left[1 - B_{p,t}\left(\sum_{j\in\Omega}\left(\frac{1}{|z|^q}\right)^{n+1-j}\right)^{\frac{1}{q}}\right] \\ &> |a_n|^2 |z|^{n+1}\left[1 - B_{p,t}\left(\sum_{j=0}^{\infty}\left(\frac{1}{|z|^q}\right)^{j}\right)^{\frac{1}{q}}\right] \\ &= |a_n|^2 |z|^{n+1}\left[1 - \frac{B_{p,t}}{\left(|z|^q - 1\right)^{\frac{1}{q}}}\right]. \end{aligned}$$

So, $|F(z)| > 0$ if

$$|z| \geq \left[1 + B_{p,t}^q\right]^{\frac{1}{q}}$$

and it leads to the desired result.
Case II: For $\lambda = n-1$, we have

$$|F(z)| \geq |a_n|^2 |z|^{n+1}\left[1 - \sum_{j=0}^{n}\left|\frac{ta_j - a_n a_{j-1}}{a_n^2}\right| \frac{1}{|z|^{n+1-j}}\right].$$

Using Holder's inequality and applying the same procedure as in case I, we can easily establish the desired result. This completes the proof of Theorem 1.3. □

Proof: [Proof of Theorem 1.4] The proof of Theorem 1.4 is omitted because it can be easily obtain from the line of the proof of Theorem 1.3 applying on $R(z)$. □

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