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Picard sequence in complete G_b-metric spaces and fixed points results

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ABSTRACT: Our idea in this article to introduce Picard sequence in complete G_b-metric spaces and explain some useful outcomes in the structure of G_b- metric spaces. We are extending the conclusions of Khojasteh et al. [7] and Yildirim et al. [30]. An example and some corollaries are also proved to show the usefulness of our results.

Key Words: Picard sequence, fixed point, G_b-metric space.

Contents

1. Introduction

Fixed point theory is one of the primary mechanisms that pre-owned to crack many numeric problems in applied analysis and has extensive applications in various disciplines. There turn out vast work on this topic. Many new spaces originated by many researchers, such as partial metric space [11], b-metric space [3,4], G-metric space [12], cone metric space [5], and many more. Sintunavarat, Kumam, and many other writers had their research work on various contractive mappings [2,10,18,19,20,21]. By affiliating the notion of G-metric spaces and b-metric spaces, Aghajani et al. [1] initiated the approach of G_b-metric spaces. A large number of research papers published by many researchers and they discussed the topological structures of G_b-metric spaces [6,8,9,13,14,15,16,17,22,23,24,25,26,27,28,29]. We establish a variety of fixed point results for single-valued mappings for a set of Picard sequences in G_b-metric space.

2. Preliminaries

In the beginning, we require essential definitions and consequences of G_b-metric spaces.

Definition 2.1 [1] *In a non-empty set W, a mapping $G_b : W \times W \times W \to \mathbb{R}^+$ fulfilling the subsequent conditions:*

(G1) $G_b(\omega,\kappa,\chi) = 0$ if and only if $\omega = \kappa = \chi$,

(G2) $0 < G_b(\omega,\omega,\kappa)$ for all $\omega,\kappa \in W$ with $\omega \neq \kappa$,

(G3) $G_b(\omega,\omega,\kappa) \leq G_b(\omega,\kappa,\chi)$ for all $\omega,\kappa,\chi \in W$ with $\omega \neq \chi$,

(G4) $G_b(\omega,\kappa,\chi) = P\{G_b(\omega,\chi,\kappa)\}$, where P is a permutation of ω,κ,χ,

(G5) $G_b(\omega,\kappa,\chi) \leq s\{G_b(\omega,r,r) + G_b(r,\kappa,\chi)\}$, for all $\omega,\kappa,\chi,r \in W$ (rectangle inequality).

The pair (W, G_b) is named as a G_b- metric space.

Definition 2.2 [1] A G_b-metric space is said to be symmetric if $G_b(\omega,\kappa,\kappa) = G_b(\kappa,\omega,\omega)$, for every $\omega,\kappa \in W$.

Definition 2.3 [1] A sequence $\{\omega_\lambda\}$ in W is called G_b-convergent to a point $\omega \in W$ if for every $\epsilon > 0$, there always occur a positive integer λ_p such that for every $m,\lambda \geq \lambda_p$, $G_b(\omega_m,\omega_\lambda,\omega) < \epsilon$.

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Definition 2.4 [1] A sequence $\{\omega_\lambda\}$ in W is said to be G_b-Cauchy if for every $\epsilon > 0$, there always occur a positive integer λ_p such that for all $m, \lambda, l \geq \lambda_p$, $G_b(\omega_m, \omega_\lambda, \omega_l) < \epsilon$.

Definition 2.5 [1] A G_b-metric space is said to be complete if every G_b-Cauchy sequence is G_b-convergent in W.

Definition 2.6 [1] In two G_b-metric spaces (W, G_b) and (W', G'_b), a mapping $\phi : W \to W'$ is called G_b-continuous at $\omega \in W$ if and only if it is G_b-sequentially continuous at it.

3. Main Results

Definition 3.1 Consider (W, G_b) be a G_b-metric space, $\omega_0 \in W$, and $\phi : W \to W'$ is a mapping. A sequence is called a Picard sequence with preliminary point ω_0, if $\omega_\lambda = \phi^\lambda \omega_0 = \phi\omega_{\lambda-1}$ for every $\lambda \in \Lambda$.

Theorem 3.1 *Let $\phi : W \to W$ be a function in a complete G_b-metric space. Assume that there is a Picard sequence $\{\omega_\lambda\}$ with preliminary point ω_0 fulfills the condition:*

$$G_b(\omega_\lambda, \omega_{\lambda+1}, \omega_{\lambda+1}) \left[\leq \frac{G_b(\omega_{\lambda-1}, \omega_\lambda, \omega_\lambda) + G_b(\omega_{\lambda-1}, \omega_{\lambda+1}, \omega_{\lambda+1}) + G_b(\omega_\lambda, \omega_{\lambda+1}, \omega_{\lambda+1}) + G_b(\omega_\lambda, \omega_{\lambda+2}, \omega_{\lambda+2}) + \frac{c}{s}}{G_b(\omega_{\lambda-1}, \omega_\lambda, \omega_\lambda) + G_b(\omega_{\lambda-1}, \omega_{\lambda+1}, \omega_{\lambda+1}) + G_b(\omega_\lambda, \omega_{\lambda+1}, \omega_{\lambda+1}) + G_b(\omega_\lambda, \omega_{\lambda+2}, \omega_{\lambda+2}) + a}\right] \times G_b(\omega_{\lambda-1}, \omega_\lambda, \omega_\lambda), \quad (3.1)$$

where c, a are positive real numbers such that $c < a$. Then $\{\omega_\lambda\}$ is a Cauchy sequence.

Proof: *Consider an arbitrary point $\omega_0 \in W$ and let $\{\omega_\lambda\}$ is a Picard sequence with preliminary point ω_0. If $\phi(\omega_{\lambda_0}) = \omega_{\lambda_0+1}$ for some $\lambda_0 \in \Lambda$, then mapping ϕ has ω_{λ_0} as its fixed point and $\{\omega_\lambda\}$ is a Cauchy sequence.*
If $\omega_{\lambda+1} \neq \omega_\lambda$ for every $\lambda \in \Lambda$, then, from (3.1), we can conclude that $\{G_b(\omega_{\lambda-1}, \omega_\lambda, \omega_\lambda)\}$ is a decreasing sequence and there always be a non-negative real number such $\{G_b(\omega_{\lambda-1}, \omega_\lambda, \omega_\lambda)\} \to \gamma$.
Claim that $\gamma = 0$. If possible $\gamma > 0$, on taking the limit on both sides of (3.1), we get $\gamma \leq \frac{\gamma+\gamma+\gamma+\gamma+\frac{c}{s}}{\gamma+\gamma+\gamma+\gamma+a}\gamma < \gamma$, which is not feasible. So, $\gamma = 0$.
We can write the equation (3.1) as $G_b(\omega_\lambda, \omega_{\lambda+1}, \omega_{\lambda+1}) \leq JG_b(\omega_{\lambda-1}, \omega_\lambda, \omega_\lambda)$ where

$$J = \left[\frac{G_b(\omega_{\lambda-1}, \omega_\lambda, \omega_\lambda) + G_b(\omega_{\lambda-1}, \omega_{\lambda+1}, \omega_{\lambda+1}) + G_b(\omega_\lambda, \omega_{\lambda+1}, \omega_{\lambda+1}) + G_b(\omega_\lambda, \omega_{\lambda+2}, \omega_{\lambda+2}) + \frac{c}{s}}{G_b(\omega_{\lambda-1}, \omega_\lambda, \omega_\lambda) + G_b(\omega_{\lambda-1}, \omega_{\lambda+1}, \omega_{\lambda+1}) + G_b(\omega_\lambda, \omega_{\lambda+1}, \omega_{\lambda+1}) + G_b(\omega_\lambda, \omega_{\lambda+2}, \omega_{\lambda+2}) + a}\right],$$

by continuing the same process, we have

$$G_b(\omega_\lambda, \omega_{\lambda+1}, \omega_{\lambda+1}) \leq J^\lambda G_b(\omega_0, \omega_1, \omega_1).$$

Now consider

$$\begin{aligned}
&G_b(\omega_\lambda, \omega_m, \omega_m) \leq s\{G_b(\omega_\lambda, \omega_{\lambda+1}, \omega_{\lambda+1}) + G_b(\omega_{\lambda+1}, \omega_m, \omega_m)\}\\
&\leq sG_b(\omega_\lambda, \omega_{\lambda+1}, \omega_{\lambda+1}) + s^2G_b(\omega_{\lambda+1}, \omega_{\lambda+2}, \omega_{\lambda+2}) + s^3G_b(\omega_{\lambda+2}, \omega_{\lambda+3}, \omega_{\lambda+3})... + s^{m-\lambda}G_b(\omega_{m-1}, \omega_m, \omega_m)\\
&\leq sJ^1G_b(\omega_0, \omega_1, \omega_1) + s^2J^{\lambda+1}G_b(\omega_0, \omega_1, \omega_1) + s^3J^{\lambda+2}G_b(\omega_0, \omega_1, \omega_1) + ...s^{m-\lambda}J^{m-1}G_b(\omega_0, \omega_1, \omega_1)\\
&\leq \frac{sJ^\lambda}{1 - sJ}G_b(\omega_0, \omega_1, \omega_1) \to 0 \text{ as } \lambda \to \infty.
\end{aligned}$$

Hence, sequence $\{\omega_\lambda\}$ is a Cauchy sequence. □

Theorem 3.2 *In a complete G_b-metric space $\phi : W \to W$ is a mapping that fulfills the condition:*

$$sG_b(\phi\omega, \phi\kappa, \phi\chi) \leq \left[\frac{G_b(\omega, \phi\kappa, \phi\chi) + G_b(\omega, \phi^2\kappa, \phi^2\chi) + G_b(\phi\omega, \kappa, \chi) + G_b(\phi\omega, \phi\kappa, \phi\chi) + c}{G_b(\omega, \phi\omega, \phi\omega) + G_b(\omega, \phi^2\omega, \phi^2\omega) + G_b(\kappa, \phi\kappa, \phi\chi) + G_b(\kappa, \phi^2\omega, \phi^2\chi) + a}\right] G_b(\omega, \kappa, \chi) \quad (3.2)$$

for all $\omega, \kappa, \chi \in W$ and c, a are positive real numbers such that $c < a$. Then

1. *there exists at least one fixed point for mapping ϕ, at which every Picard sequence $\{\omega_\lambda\}$ converges.*
2. *If ω, τ, σ are three different fixed points for ϕ, then $G_b(\omega, \tau, \sigma) > \frac{[as-c]}{4}$.*

Proof: *Let $\{\omega_\lambda\}$ be a Picard sequence with preliminary point ω_0. If $\omega_{\lambda_0+1} = \phi\omega_{\lambda_0}$ for some $\lambda_0 \in \Lambda$, then mapping ϕ has ω_{λ_0} as its fixed point and $\{\omega_\lambda\}$ is a Cauchy sequence.*
If $\omega_\lambda \neq \omega_{\lambda+1}$ for every $\lambda \in \Lambda$, then

$$\begin{aligned} sG_b(\omega_\lambda, \omega_{\lambda+1}, \omega_{\lambda+1}) &= sG_b(\phi\omega_{\lambda-1}, \phi\omega_\lambda, \phi\omega_\lambda)\{\omega_\lambda\} \\ &\leq \left[\frac{G_b(\omega_{\lambda-1}, \omega_{\lambda+1}, \omega_{\lambda+1}) + G_b(\omega_{\lambda-1}, \omega_{\lambda+2}, \omega_{\lambda+2}) + G_b(\omega_\lambda, \omega_\lambda, \omega_\lambda) + G_b(\omega_\lambda, \omega_{\lambda+1}, \omega_{\lambda+1}) + c}{G_b(\omega_{\lambda-1}, \omega_\lambda, \omega_\lambda) + G_b(\omega_{\lambda-1}, \omega_{\lambda+1}, \omega_{\lambda+1}) + G_b(\omega_\lambda, \omega_{\lambda+1}, \omega_{\lambda+1}) + G_b(\omega_\lambda, \omega_{\lambda+2}, \omega_{\lambda+2}) + a}\right] \\ &\qquad \times G_b(\omega_{\lambda-1}, \omega_\lambda, \omega_\lambda) \end{aligned}$$

$$\begin{aligned} sG_b(\omega_\lambda, \omega_{\lambda+1}, \omega_{\lambda+1}) &\leq \left[\frac{s\{G_b(\omega_{\lambda-1}, \omega_\lambda, \omega_\lambda) + G_b(\omega_\lambda, \omega_{\lambda+2}, \omega_{\lambda+2})\} + G_b(\omega_{\lambda-1}, \omega_{\lambda+1}, \omega_{\lambda+1}) + G_b(\omega_\lambda, \omega_{\lambda+1}, \omega_{\lambda+1}) + c}{G_b(\omega_{\lambda-1}, \omega_\lambda, \omega_\lambda) + G_b(\omega_{\lambda-1}, \omega_{\lambda+1}, \omega_{\lambda+1}) + G_b(\omega_\lambda, \omega_{\lambda+1}, \omega_{\lambda+1}) + G_b(\omega_\lambda, \omega_{\lambda+2}, \omega_{\lambda+2}) + a}\right] \\ &\qquad \times G_b(\omega_{\lambda-1}, \omega_\lambda, \omega_\lambda) \end{aligned}$$

Therefore, by Theorem 3.1, sequence $\{\omega_\lambda\}$ is a Cauchy sequence. As G_b is given to be a complete metric space, there is a point $\omega \in W$ such that sequence $\{\omega_\lambda\}$ is convergent to ω.

Next, we show that ω is a fixed point.

$$\begin{aligned} sG_b(\omega_{\lambda+1}, \phi\omega, \phi\omega) &= sG_b(\phi\omega_\lambda, \phi\omega, \phi\omega) \\ &\leq \left[\frac{G_b(\omega_\lambda, \phi\omega, \phi\omega) + G_b(\omega_\lambda, \phi^2\omega, \phi^2\omega) + G_b(\omega_{\lambda+1}, \omega, \omega) + G_b(\omega_{\lambda+1}, \phi\omega, \phi\omega) + c}{G_b(\omega_\lambda, \omega_{\lambda+1}, \omega_{\lambda+1}) + G_b(\omega_\lambda, \omega_{\lambda+2}, \omega_{\lambda+2}) + G_b(\omega, \phi\omega, \phi\omega) + G_b(\omega, \omega_{\lambda+2}, \phi^2\omega)}\right] \\ &\qquad \times G_b(\omega_\lambda, \omega, \omega). \quad (3.3) \end{aligned}$$

Moreover, by the rectangle inequality, we have

$$\begin{aligned} G_b(\omega, \phi\omega, \phi\omega) &\leq sG_b(\omega, \omega_\lambda, \omega_\lambda) + sG_b(\omega_\lambda, \phi\omega, \phi\omega) \\ G_b(\omega, \phi\omega, \phi\omega) - sG_b(\omega, \omega_\lambda, \omega_\lambda) &\leq sG_b(\omega_\lambda, \phi\omega, \phi\omega) + s^2\{G_b(\omega_\lambda, \omega, \omega) + G_b(\omega, \phi\omega, \phi\omega)\}. \end{aligned}$$

as $\lambda \to \infty$, we deduce that

$$G_b(\omega, \phi\omega, \phi\omega) \leq lim \inf sG_b(\omega_\lambda, \phi\omega, \phi\omega) \leq lim \sup sG_b(\omega_\lambda, \phi\omega, \phi\omega) \leq s^2 G_b(\omega, \phi\omega, \phi\omega). \quad (3.4)$$

On taking $lim \inf$, as $\lambda \to \infty$, on both sides of (3.3) and (3.4), we obtain

$$\begin{aligned} G_b(\omega, \phi\omega, \phi\omega) &\leq lim \inf sG_b(\omega_{\lambda+1}, \phi\omega, \phi\omega) \\ &\leq \frac{G_b(\omega, \phi\omega, \phi\omega) + G_b(\omega, \phi^2\omega, \phi^2\omega) + G_b(\omega, \phi\omega, \phi\omega) + c}{G_b(\omega, \phi\omega, \phi\omega) + G_b(\omega, \omega, \phi^2\omega) + a} lim \sup sG_b(\omega_\lambda, \omega, \omega) = 0. \end{aligned}$$

This implies that $G_b(\omega,\phi\omega,\phi\omega)=0$. *And* $\phi(\omega)=\omega$.

Suppose that there exist two more fixed points τ, σ *such that* $\phi(\tau)=\tau$ *and* $\phi(\sigma)=\sigma$.

$$\begin{aligned}
sG_b(\omega,\tau,\sigma) &= sG_b(\phi\omega,\phi\tau,\phi\sigma),\\
&\le \left[\frac{G_b(\omega,\phi\tau,\phi\sigma)+G_b(\omega,\phi^2\tau,\phi^2\sigma)+G_b(\phi\omega,\tau,\sigma)+G_b(\phi\omega,\phi\tau,\phi\sigma)+c}{G_b(\omega,\phi\omega,\phi\omega)+G_b(\omega,\phi^2\omega,\phi^2\omega)+G_b(\tau,\phi\tau,\phi\sigma)+G_b(\tau,\phi^2\omega,\phi^2\sigma)+a}\right]G_b(\omega,\tau,\sigma)\\
&\le \left[\frac{G_b(\omega,\tau,\sigma)+G_b(\omega,\tau,\sigma)+G_b(\omega,\tau,\sigma)+G_b(\omega,\tau,\sigma)+c}{a}\right]G_b(\omega,\tau,\sigma)\\
&\le \left[\frac{4G_b(\omega,\tau,\sigma)+c}{a}\right]G_b(\omega,\tau,\sigma)\\
&= \left[\frac{4G_b(\omega,\tau,\sigma)^2+cG_b(\omega,\tau,\sigma)}{a}\right]\\
&[as-c]G_b(\omega,\tau,\sigma) < 4G_b(\omega,\tau,\sigma)^2
\end{aligned}$$

Hence, $G_b(\omega,\tau,\sigma) > \frac{[as-c]}{4}$.

□

Theorem 3.3 *In a complete* G_b*-metric space,* $\phi : W \to W$ *is a mapping that fulfills the next conditions:*

$$\begin{aligned}
&sG_b(\phi\omega,\phi\kappa,\phi\chi)\\
&\quad\le \left[\frac{G_b(\omega,\phi\kappa,\phi\chi)+G_b(\omega,\phi^2\kappa,\phi^2\chi)+G_b(\phi\omega,\kappa,\chi)+G_b(\phi\omega,\phi\kappa,\phi\chi)+c}{G_b(\omega,\phi\omega,\phi\omega)+G_b(\omega,\phi^2\omega,\phi^2\omega)+G_b(\kappa,\phi\kappa,\phi\chi)+G_b(\kappa,\phi^2\omega,\phi^2\chi)+a}\right]G_b(\omega,\kappa,\chi)+\\
&\qquad PG_b(\kappa,\chi,\phi\omega). \quad (3.5)
\end{aligned}$$

for all $\omega,\kappa,\chi \in W$ *and* c*, a positive real numbers such that* $c<a$*. Then*

1. *there exists at least one fixed point for the mapping* ϕ*, at which every Picard sequence* $\{\omega_\lambda\}$ *converges.*
2. *If* ω*,* τ*, and* σ *are three different fixed points for* ϕ*, then* $G_b(\omega,\tau,\sigma) > \frac{\{a[s-P]-c\}}{4}$.

Proof: *Let* $\{\omega_\lambda\}$ *be a Picard sequence with preliminary point* ω_0*. If* $\omega_{\lambda_0+1}=\phi\omega_{\lambda_0}$ *for some* $\lambda_0\in\Lambda$*, then the mapping* ϕ *has* ω_{λ_0} *as its fixed point and* $\{\omega_\lambda\}$ *is a Cauchy sequence.*

If $\omega_\lambda \neq \omega_{\lambda+1}$ *for every* $\lambda\in\Delta$*, then*

$$\begin{aligned}
&sG_b(\omega_\lambda,\omega_{\lambda+1},\omega_{\lambda+1}) = sG_b(\phi\omega_{\lambda-1},\phi\omega_\lambda,\phi\omega_\lambda)\\
&\le \left[\frac{G_b(\omega_{\lambda-1},\phi\omega_\lambda,\phi\omega_\lambda)+G_b(\omega_{\lambda-1},\phi^2\omega_\lambda,\phi^2\omega_\lambda)+G_b(\omega_\lambda,\phi\omega_{\lambda-1},\phi\omega_{\lambda-1})+G_b(\omega_\lambda,\phi\omega_\lambda,\phi\omega_\lambda)+c}{G_b(\omega_{\lambda-1},\phi\omega_{\lambda-1},\phi\omega_{\lambda-1})+G_b(\phi\omega_{\lambda-1},\phi^2\omega_{\lambda-1},\phi^2\omega_{\lambda-1})+G_b(\omega_\lambda,\phi\omega_\lambda,\phi\omega_\lambda)+G_b(\omega_\lambda,\phi^2\omega_{\lambda-1},\phi^2\omega_\lambda)+a}\right]\\
&\qquad\times G_b(\omega_{\lambda-1},\omega_\lambda,\omega_\lambda)+PG_b(\omega_\lambda,\omega_\lambda,\phi\omega_{\lambda-1})
\end{aligned}$$

$$\begin{aligned}
&= \left[\frac{G_b(\omega_{\lambda-1},\omega_{\lambda+1},\omega_{\lambda+1})+G_b(\omega_{\lambda-1},\omega_{\lambda+2},\omega_{\lambda+2})+G_b(\omega_\lambda,\omega_\lambda,\omega_\lambda)+G_b(\omega_\lambda,\omega_{\lambda+1},\omega_{\lambda+1})+c}{G_b(\omega_{\lambda-1},\omega_\lambda,\omega_\lambda)+G_b(\omega_{\lambda-1},\omega_{\lambda+1},\omega_{\lambda+1})+G_b(\omega_\lambda,\omega_{\lambda+1},\omega_{\lambda+1})+G_b(\omega_\lambda,\omega_{\lambda+1},\omega_{\lambda+2})+a}\right]\\
&\qquad\times G_b(\omega_{\lambda-1},\omega_\lambda,\omega_\lambda)+PG_b(\omega_\lambda,\omega_\lambda,\phi\omega_\lambda)
\end{aligned}$$

$$\begin{aligned}
&\le \left[\frac{s\{G_b(\omega_{\lambda-1},\omega_\lambda,\omega_\lambda)+G_b(\omega_\lambda,\omega_{\lambda+2},\omega_{\lambda+2})\}+G_b(\omega_{\lambda-1},\omega_{\lambda+1},\omega_{\lambda+1})+G_b(\omega_\lambda,\omega_{\lambda+1},\omega_{\lambda+1})+c}{G_b(\omega_{\lambda-1},\omega_\lambda,\omega_\lambda)+G_b(\omega_{\lambda-1},\omega_{\lambda+1},\omega_{\lambda+1})+G_b(\omega_\lambda,\omega_{\lambda+1},\omega_{\lambda+1})+G_b(\omega_\lambda,\omega_{\lambda+1},\omega_{\lambda+2})+a}\right]\\
&\qquad\times G_b(\omega_{\lambda-1},\omega_\lambda,\omega_\lambda)
\end{aligned}$$

By Theorem 3.1, sequence $\{\omega_\lambda\}$ *is a Cauchy sequence as* G_b *is given to be a complete metric space, there is a point* $\omega \in W$ *such that* $\{\omega_\lambda\}$ *is convergent to* ω.

Next is to show that ω *is a fixed point.*

$$\begin{aligned}
sG_b(\omega_{\lambda+1}, \phi\omega, \omega) &= sG_b(\phi\omega_\lambda, \phi\omega, \phi\omega) \\
&= \left[\frac{G_b(\omega_\lambda, \phi\omega, \phi\omega) + G_b(\omega_\lambda, \phi^2\omega, \phi^2\omega) + G_b(\omega_{\lambda+1}, \omega, \omega) + G_b(\omega_{\lambda+1}, \phi\omega, \phi\omega) + c}{G_b(\omega_\lambda, \omega_{\lambda+11}, \omega_{\lambda+1}) + G_b(\omega_\lambda, \omega_{\lambda+2}, \omega_{\lambda+2}) + G_b(\omega, \phi\omega, \phi\omega) + G_b(\omega, \omega_{\lambda+2}, \phi^2\omega) + a}\right] \\
&\qquad \times G_b(\omega_\lambda, \omega, \omega) + PG_b(\omega, \omega, \omega_{\lambda+1}) \qquad (3.6)
\end{aligned}$$

taking lim inf *as* $\lambda \to \infty$, *on both sides of (3.6), by (3.4),we get*

$$\begin{aligned}
G_b(\omega, \phi\omega, \phi\omega) &\leq \liminf sG_b(\omega_{\lambda+1}, \phi\omega, \phi\omega) \\
&\leq \frac{G_b(\omega, \phi\omega, \phi\omega) + G_b(\omega, \phi^2\omega, \phi^2\omega) + G_b(\omega, \phi\omega, \phi\omega) + c}{G_b(\omega, \phi\omega, \phi\omega) + G_b(\omega, \omega, \phi^2\omega) + a} \limsup sG_b(\omega_\lambda, \omega, \omega) \\
&\qquad + \liminf PG_b(\omega, \omega, \omega_{\lambda+1}) = 0.
\end{aligned}$$

And $G_b(\omega, \phi\omega, \phi\omega) = 0$. *From this, we conclude that* $\phi(\omega) = \omega$.

Suppose there exist two more fixed points τ *and* σ *such that* $\phi(\tau) = \tau$ *and* $\phi(\sigma) = \sigma$.

$$\begin{aligned}
sG_b(\omega, \tau, \sigma) &= sG_b(\phi\omega, \phi\tau, \phi\sigma), \\
&\leq \left[\frac{G_b(\omega, \phi\tau, \phi\sigma) + G_b(\omega, \phi^2\tau, \phi^2\sigma) + G_b(\phi\omega, \tau, \sigma) + G_b(\phi\omega, \phi\tau, \phi\sigma) + c}{G_b(\omega, \phi\omega, \phi\omega) + G_b(\omega, \phi^2\omega, \phi^2\omega) + G_b(\tau, \phi\tau, \phi\sigma) + G_b(\tau, \phi^2\omega, \phi^2\sigma) + a}\right] \\
&\quad \times G_b(\omega, \tau, \sigma) + PG_b(\tau, \sigma, \phi\omega) \\
&\leq \left[\frac{G_b(\omega, \tau, \sigma) + G_b(\omega, \tau, \sigma) + G_b(\omega, \tau, \sigma) + G_b(\omega, \tau, \sigma) + c}{a}\right] G_b(\omega, \tau, \sigma) + PG_b(\omega, \tau, \sigma) \\
&= \left[\frac{4G_b(\omega, \tau, \sigma) + c}{a}\right] G_b(\omega, \tau, \sigma) + PG_b(\omega, \tau, \sigma) \\
[s - P]G_b(\omega, \tau, \sigma) &\leq \left[\frac{4G_b(\omega, \tau, \sigma) + c}{a}\right] G_b(\omega, \tau, \sigma) \\
[s - P]G_b(\omega, \tau, \sigma) &= \left[\frac{4G_b(\omega, \tau, \sigma)^2 + cG_b(\omega, \tau, \sigma)}{a}\right] \\
\{a[s - P] - c\}G_b(\omega, \tau, \sigma) &< 4G_b(\omega, \tau, \sigma)^2
\end{aligned}$$

Hence, $G_b(\omega, \tau, \sigma) > \frac{\{a[s-P]-c\}}{4}$.

□

Corollary 3.1 *In a complete* G_b*-metric space,* $\phi : W \to W$ *is a mapping that fulfills next conditions:*

$$sG_b(\phi\omega, \phi\kappa, \phi\chi) \leq \left[\frac{G_b(\omega, \phi\kappa, \phi\chi) + G_b(\omega, \phi^2\kappa, \phi^2\chi) + G_b(\phi\omega, \kappa, \chi) + G_b(\phi\omega, \phi\kappa, \phi\chi)}{G_b(\omega, \phi\omega, \phi\omega) + G_b(\omega, \phi^2\omega, \phi^2\omega) + G_b(\kappa, \phi\kappa, \phi\chi) + G_b(\kappa, \phi^2\omega, \phi^2\chi) + 1}\right] G_b(\omega, \kappa, \chi). \qquad (3.7)$$

for all $\omega, \kappa, \chi \in W$ *and such that*

1. *there exists at least one fixed point for the mapping* ϕ *at which every Picard sequence* $\{\omega_\lambda\}$ *converges.*
2. *If* ω, τ, *and* σ *are three different fixed points for* ϕ, *then* $sG(\omega, \tau, \sigma) > \frac{s}{4}$.

Proof: *Substituting c=0 and a=1 in Theorem 3.2* □

Corollary 3.2 *In a complete G_b-metric space, $\phi : W \to W$ is a mapping that fulfills next conditions:*

$$sG_b(\phi\omega, \phi\kappa, \phi\chi) \leq \left[\frac{G_b(\omega, \phi\kappa, \phi\chi) + G_b(\omega, \phi^2\kappa, \phi^2\chi) + G_b(\phi\omega, \kappa, \chi) + G_b(\phi\omega, \phi\kappa, \phi\chi)}{G_b(\omega, \phi\omega, \phi\omega) + G_b(\omega, \phi^2\omega, \phi^2\omega) + G_b(\kappa, \phi\kappa, \phi\chi) + G_b(\kappa, \phi^2\omega, \phi^2\chi) + 1}\right] G_b(\omega, \kappa, \chi).$$

for all $\omega, \kappa, \chi \in W$.
Then,

1. *there exists at least one fixed point for the mapping ϕ, at which every Picard sequence $\{\omega_\lambda\}$ converges.*

2. *If ω, τ, and σ are three different fixed points for ϕ, then $sG_b(\omega, \tau, \sigma) > \frac{[s-P]}{4}$.*

we are giving an example in support of corollary 3.3A.

Example 3.1 *Let $W = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$. Taking $s = 2$, and define $G_b : W \times W \times W \to \mathbb{R}^+$ as*

$$G_b\left(0, \frac{1}{3}, \frac{2}{3}\right) = \left(\frac{1}{3}, \frac{2}{3}, 0\right) = G_b\left(\frac{1}{3}, 0, \frac{2}{3}\right) = G_b\left(\frac{2}{3}, \frac{1}{3}, 0\right) = 2, G_b\left(0, \frac{1}{3}, \frac{1}{3}\right) = \frac{5}{3}$$
$$G_b\left(0, \frac{2}{3}, \frac{2}{3}\right) = G_b\left(\frac{2}{3}, \frac{2}{3}, 0\right) = \frac{1}{3}, G_b\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right) = \frac{2}{3}$$
$$G_b(0,0,0) = G_b\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = G_b\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) = G_b(1,1,1) = 0,$$

Let $\phi : W \to W$ be defined as $\phi(0) = 0, \phi^2(\frac{1}{3}) = \phi(\frac{1}{3}) = \frac{1}{3} = \phi^2(\frac{2}{3}), \phi(\frac{2}{3}) = \frac{2}{3}, \phi^2(1) = \phi(1) = 0$.

$$G_b\left(\phi 0, \phi\frac{1}{3}, \phi\frac{2}{3}\right) = G_b\left(0, \frac{1}{3}, \frac{2}{3}\right) = 2, G_b\left(0, \phi^2\frac{1}{3}, \phi^2\frac{2}{3}\right) = G_b\left(0, \frac{1}{3}, \frac{2}{3}\right) = 2$$
$$G_b\left(\phi 0, \frac{1}{3}, \frac{2}{3}\right) = G_b\left(0, \frac{1}{3}, \frac{2}{3}\right) = 2, G_b\left(\phi 0, \phi\frac{1}{3}, \phi\frac{2}{3}\right) = G_b\left(0, \frac{1}{3}, \frac{2}{3}\right) = 2$$
$$G_b(\phi 0, \phi 0, \phi 0) = G_b(0,0,0) = 0, G_b\left(\phi 0, \phi^2 0, \phi^2 0\right) = G_b(0,0,0) = 0$$
$$G_b\left(\frac{1}{3}, \phi\frac{1}{3}, \phi\frac{2}{3}\right) = G_b\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right) = \frac{1}{3}, G_b\left(\frac{1}{3}, \phi^2 0, \phi^2\frac{2}{3}\right) = G_b\left(\frac{1}{3}, 0, \frac{2}{3}\right) = 2$$

Now

$$2.G_b\left(\phi 0, \phi\frac{1}{3}, \phi\frac{2}{3}\right) = 2.G_b\left(0, \frac{1}{3}, \frac{2}{3}\right) = 4$$
$$4 \leq \left[\frac{G_b\left(0, \phi\frac{1}{3}, \phi\frac{2}{3}\right) + G_b\left(0, \phi^2\frac{1}{3}, \phi^2\frac{2}{3}\right) + G_b\left(\phi 0, \frac{1}{3}, \frac{2}{3}\right) + G_b\left(\phi 0, \phi\frac{1}{3}, \phi\frac{2}{3}\right)}{G_b(0, \phi 0, \phi 0) + G_b(0, \phi^2 0, \phi^2 0) + G_b\left(\frac{1}{3}, \phi\frac{1}{3}, \phi\frac{2}{3}\right) + G_b\left(\frac{1}{3}, \phi^2 0, \phi^2\frac{2}{3}\right) + 1}\right] G\left(0, \frac{1}{3}, \frac{2}{3}\right)$$
$$4 \leq \left[\frac{G_b\left(0, \frac{1}{3}, \frac{2}{3}\right) + G_b\left(0, \frac{1}{3}, \frac{2}{3}\right) + G_b\left(0, \frac{1}{3}, \frac{2}{3}\right) + G_b\left(0, \frac{1}{3}, \frac{2}{3}\right)}{G_b(0,0,0) + G_b(0,0,0) + G_b\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right) + G_b\left(\frac{1}{3}, 0, \frac{2}{3}\right) + 1}\right] G\left(0, \frac{1}{3}, \frac{2}{3}\right)$$
$$= \frac{2+2+2+2}{0+0+\frac{1}{3}+2+1}.2 = \frac{48}{10}.$$

And

$$2.G_b\left(\phi\frac{1}{3},\phi\frac{2}{3},\phi 0\right)=2.G_b\left(\frac{1}{3},\frac{2}{3},0\right)=4$$

$$sG_b(\phi\omega,\phi\kappa,\phi\chi)\leq$$

$$\left[\frac{G_b(\omega,\phi\kappa,\phi\chi)+G_b(\omega,\phi^2\kappa,\phi^2\chi)+G_b(\phi\omega,\kappa,\chi)+G_b(\phi\omega,\phi\kappa,\phi\chi)}{G_b(\omega,\phi\omega,\phi\omega)+G_b(\omega,\phi^2\omega,\phi^2\omega)+G_b(\kappa,\phi\kappa,\phi\chi)+G_b(\kappa,\phi^2\omega,\phi^2\chi)+1}\right]G_b(\omega,\kappa,\chi)$$

$$4\leq\left[\frac{G_b\left(\frac{1}{3},\phi\frac{2}{3},\phi 0\right)+G_b\left(\frac{1}{3},\phi^2\frac{2}{3},\phi^2 0\right)+G_b\left(\phi\frac{1}{3},\frac{2}{3},0\right)+G_b\left(\phi\frac{1}{3},\phi\frac{2}{3},\phi 0\right)}{G_b\left(\frac{1}{3},\phi\frac{1}{3},\phi\frac{1}{3}\right)+G_b\left(\frac{1}{3},\phi^2\frac{1}{3},\phi^2\frac{1}{3}\right)+G_b\left(\frac{2}{3},\phi\frac{2}{3},\phi^2 0\right)+G_b\left(\frac{2}{3},\phi^2\frac{1}{3},\phi^2 0\right)+1}\right]G\left(\frac{1}{3},\frac{2}{3},0\right),$$

$$4\leq\left[\frac{G_b\left(\frac{1}{3},\frac{2}{3},0\right)+G_b\left(\frac{1}{3},\frac{2}{3},0\right)+G_b\left(\frac{1}{3},\frac{2}{3},0\right)+G_b\left(\frac{1}{3},\frac{2}{3},0\right)}{G_b\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)+G_b\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)+G_b\left(\frac{2}{3},\frac{2}{3},0\right)+G_b\left(\frac{2}{3},\frac{1}{3},0\right)+1}\right]G\left(\frac{1}{3},\frac{2}{3},0\right),$$

$$=\frac{2+2+2+2}{0+0+\frac{1}{3}+2+1}.2=\frac{48}{10}.$$

Hence, ϕ fulfills all the conditions of Corollary 3.3A. Moreover, there exists three distinct fixed points $\{0,\frac{1}{2},\frac{1}{3}\}$ for mapping ϕ and $G_b\left(0,\frac{1}{3},\frac{2}{3}\right)=2>\frac{2}{4}$.

Competing Interests

All the authors declare that they have no competing interests regarding this manuscript.

Author's Contributions

All authors contributed equally to the writing of this manuscript. All authors read and approved the final version.

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