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Blow up, exponential decay of solutions for a G-heat equation with source term: Analytical and numerical results

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ABSTRACT: We consider an initial value problem related to the equation

$$u_t - \frac{1}{2}\left(\overline{\sigma}^2 Max\left(\Delta u,\ 0\right) - \underline{\sigma}^2 Max\left(0,\ -\Delta u\right)\right) = |u|^{p-2}\, u,$$

with homogeneous Diriclet boundary condition in a bounded domain.We prove under suitable conditions on initial energy a blow up and exponential decay of solutions, and also give the numerical examples to illustrate the blow up and exponential decay results.

Key Words: G-heat equation, blow up, exponential decay.

Contents

1. Introduction

In this paper, we consider the following boundary value problem

$$u_t - G\left(\Delta u\right) = |u|^{p-2}\, u, \qquad x \in \Omega, t > 0, \tag{1.1}$$

$$u\left(x,t\right) = 0, \qquad x \in \partial\Omega, t > 0, \tag{1.2}$$

with the initial conditions

$$u\left(x,0\right) = u_0\left(x\right), \qquad x \in \Omega, \tag{1.3}$$

where $G\left(\alpha\right) = \frac{1}{2}\left(\overline{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-\right)$, with $\alpha^+ = Max\left(\alpha,\ 0\right)$ and $\alpha^- = Max\left(0,\ -\alpha\right)$, and Ω is a bounded domain of $\mathbb{R}$ with a smooth boundary $\partial\Omega$. The G-heat equation (1.4)

$$u_t - G\left(\Delta u\right) = 0, \tag{1.4}$$

defined by Peng [12], is a nonlinear equation related to the G-normal distribution and generalises the classical normal distribution. This equation, which is a special kind of Hamilton-Jacobi-Bellman equation (Crandall et al., [1]), and has a unique viscosity solution (Peng, [10]). In the case when $\overline{\sigma} = \underline{\sigma}$ the nonlinear equation (1.4) becames the classical heat equation

$$u_t - \alpha\partial^2_{xx} = 0. \tag{1.5}$$

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In multidimensional cases we have the following typical nonlinear heat equation:

$$u_t - \sum_{i=1}^{d} G_i\left(\partial^2_{x_i x_i}\right) = 0, \tag{1.6}$$

where $G_i(\alpha) = \frac{1}{2}\left(\overline{\sigma}_i^2\alpha^+ - \underline{\sigma}_i^2\alpha^-\right)$ and $0 \leq \underline{\sigma}_i \leq \overline{\sigma}_i$ are given constants.

The existence and the uniqueness of the solution of the G-heat equation in the sense of viscosity solution can be found for example in Peng in [8] and Yong and Zhou in [17]. Peng in [11] described a new framework of a sublinear expectation space and the related notions and results of distributions, independence. A new notion of G-distributions is introduced which generalizes the G-normal-distribution in the sense that mean-uncertainty can be also described. Rcently problems of model uncetainties, measures of risk and suprheding in finance introduced by Peng in [9].

Many authors in [4], [5], [6], [7], [13], studied the blow up, exponential decay of solutions of heat equation with source terms in different suitable conditions.

This paper is organized as follows. In section 2, we present some lemmas, and theorem are introduced. In section 3, we show the blow up of solutions. In section 4, we prove exponential decay of solutions. In section 5, we illustrate numerically in one dimensional case the blowup and exponential decay results.

2. Assumptions and preliminaries

In this section, we present some material needed in the proof in our results.

Lemma 2.1 *(Poincare inequality). Let q be a number with $2 \leq q < \infty$ $(n = 1, 2)$ or $2 \leq q \leq \frac{2n}{n-2}(n \geq 3)$, then, there is a constant $c_* = c_*(\Omega, q)$ such that*

$$\|u\|_q \leq c_* \|\nabla u\|_2 \quad \textit{for} \;\; u \in H_0^1(\Omega).$$

Lemma 2.2 *(Young's inequality) Let $a,\ b \geq 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ for $1 < p,\ q < +\infty$, then one has the inequality*

$$ab \leq \delta a^p + C(\delta) b^q,$$

where $\delta > 0$ is an arbitrary constant, and $C(\delta)$ is a positive constant depending on δ.

Lemma 2.3 *Let $G : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a non-increasing function and assume that there are tow constant $c > 0$ such that*

$$\int_s^\infty G(t)\, dt \leq c\ G(s), \quad \forall s > 0.$$

Then we have

$$G(t) \leq c\ G(0)\, e^{-\zeta t},$$

where $\xi > 0$ is a positive constant.

Theorem 2.1 *Assume that u be a strong solution of (1.1)-(1.3), and satisfying*

$$E(0) < 0.$$

Then the solution blow up in finite time.

3. Blow up of solution

In order to state and prove our result, we introduce the energy functional

$$E(t) = \begin{cases} E_1(t) = \frac{1}{4}\overline{\sigma}^2\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p, & \text{if } \Delta u \geq 0, \\ E_2(t) = \frac{1}{4}\underline{\sigma}^2\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p, & \text{if } \Delta u < 0. \end{cases} \tag{3.1}$$

We difine the following spaces

$$\Omega_1 = \{x \in \Omega,\ \text{such that } \ \Delta u \geq 0\},\ \Omega_2 = \{x \in \Omega,\ \text{such that } \Delta u < 0\}.$$

Lemma 3.1 *Let u be a solution of (1.1)-(1.3). Then, $E(t)$ is nonincreasing, that is,*

$$\frac{d}{dt}E\left(t\right)\leq 0.$$

Proof: By multiplying equation (1.1) by u_t and integrating over Ω, we obtain

$$\int_{\Omega}u_t^2dx-\frac{1}{2}\int_{\Omega}\left(\overline{\sigma}^2 Max\left(\Delta u,\ 0\right)-\underline{\sigma}^2 Max\left(0,\ -\Delta u\right)\right)u_t dx$$
$$=\int_{\Omega}\left|u\right|^{p-2}uu_t dx.$$

So, we get

$$\left.\begin{array}{l}\int_{\Omega}u_t^2dx+\frac{\overline{\sigma}^2}{4}\frac{d}{dt}\int_{\Omega}\left|\nabla u\right|^2dx,\quad x\in\Omega_1\\ \int_{\Omega}u_t^2dx+\frac{\underline{\sigma}^2}{4}\frac{d}{dt}\int_{\Omega}\left|\nabla u\right|^2dx,\quad x\in\Omega_2\end{array}\right\}=\frac{1}{p}\frac{d}{dt}\int_{\Omega}\left|u\right|^p dx.$$

Then, we have

$$\frac{d}{dt}E\left(t\right)=-\left\|u_t\right\|_2^2\leq 0. \tag{3.2}$$

□

Similar to [14], we give a definition for a strong solution of (1.1).

Definition 3.1 A strong solution of (1.1) is a function $u\in C\left(\left[0,T\right);H_0^1\left(\Omega\right)\right)\cap C^1\left(\left[0,T\right);L^2\left(\Omega\right)\right),$ satisfying (3.2) and

$$\int_0^t\int_{\Omega}\left(u_t\varphi-\frac{1}{2}\left(\overline{\sigma}^2 Max\left(\Delta u,\ 0\right)-\underline{\sigma}^2 Max\left(0,\ -\Delta u\right)\right)\varphi-\left|u\right|^{p-2}u\varphi\right)dxds=0,$$

for all t in $[0,T)$ and all test function φ in $C\left(\left[0,T\right),H_0^1\left(\Omega\right)\right).$

Now, we let

$$H\left(t\right)=-E\left(t\right),\quad t\geq 0, \tag{3.3}$$

Lemma 3.2 *Suppose that $E\left(0\right)<0$, then for all $t\geq 0$,*

$$0<H\left(0\right)\leq H\left(t\right)\leq\frac{1}{p}\left\|u\right\|_p^p. \tag{3.4}$$

Proof: From Lemma 3.1, we have

$$\frac{d}{dt}E\left(t\right)\leq 0,$$

and thus

$$H\left(t\right)\geq H\left(0\right)=-E\left(0\right)>0,\quad t\geq 0$$

From (3.1) and (3.3), we obtain

$$H\left(t\right)=\left\{\begin{array}{ll}-\frac{1}{4}\overline{\sigma}^2\left\|\nabla u\right\|_2^2+\frac{1}{p}\left\|u\right\|_p^p, & \text{if}\quad x\in\Omega_1\\ -\frac{1}{4}\underline{\sigma}^2\left\|\nabla u\right\|_2^2+\frac{1}{p}\left\|u\right\|_p^p, & \text{if}\quad x\in\Omega_2\end{array}\right.$$
$$\leq\frac{1}{p}\left\|u\right\|_p^p. \tag{3.5}$$

Proof: [Prof of theorem 1]We define the following fonctional

$$L\left(t\right)=\frac{1}{2}\int_{\Omega}u^2\left(x,t\right)dx,$$

and differentiate L, to get

$$\begin{aligned} L'(t) &= \int_\Omega u u_t(x,t)\,dx \\ &= \int_\Omega u\left(\frac{1}{2}\left(\overline{\sigma}^2 Max(\Delta u,\ 0) + \underline{\sigma}^2 Max(0,\ -\Delta u)\right) + |u|^{p-2}u\right)dx \\ &\geq \begin{cases} -\frac{1}{2}\overline{\sigma}^2\|\nabla u\|_2^2 + \frac{1}{p}\|u\|_p^p, & \text{if} \quad x\in\Omega_1 \\ -\frac{1}{2}\underline{\sigma}^2\|\nabla u\|_2^2 + \frac{1}{p}\|u\|_p^p, & \text{if} \quad x\in\Omega_2. \end{cases} \end{aligned} \tag{3.6}$$

Since, $H(t) > 0$, then we have

$$\begin{cases} -\frac{1}{2}\overline{\sigma}^2\|\nabla u\|_2^2 \geq -\frac{2}{p}\|u\|_p^p, & \text{if} \quad x\in\Omega_1 \\ -\frac{1}{2}\underline{\sigma}^2\|\nabla u\|_2^2 \geq -\frac{2}{p}\|u\|_p^p, & \text{if} \quad x\in\Omega_2. \end{cases} \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$L'(t) \geq -\frac{2}{p}\|u\|_p^p + \frac{1}{p}\|u\|_p^p = \left(1-\frac{2}{p}\right)\|u\|_p^p. \tag{3.8}$$

Next, we have, by the embedding of $L^p(\Omega) \hookrightarrow L^2(\Omega)$ spaces,

$$L^{\frac{p}{2}}(t) \leq \left(\frac{1}{2}\right)^{\frac{p}{2}}\|u\|_p^p. \tag{3.9}$$

By combining of (3.8) and (3.9), we get

$$L'(t) \geq \zeta L^{\frac{p}{2}}(t). \tag{3.10}$$

where $\zeta = \frac{p-2}{(1/2)^{\frac{p}{2}}p}$.

A direct integration of (3.10) then yields

$$L^{\frac{p}{2}-1}(t) \geq \frac{1}{L^{1-\frac{p}{2}}(0) - \zeta\frac{p-2}{p}t}. \tag{3.11}$$

Therefore L blows up in a time $t^* \leq \frac{p}{\zeta(p-2)L^{\frac{p}{2}-1}(0)}$. □

□

4. Exponential decay of solution

In order to state and prove the second main result, we define the following Nehari functionalmain

$$I(t) = \begin{cases} \frac{\overline{\sigma}^2}{2}\|\nabla u\|_2^2 - \|u\|_p^p, & \text{if} \quad x\in\Omega_1, \\ \frac{\underline{\sigma}^2}{2}\|\nabla u\|_2^2 - \|u\|_p^p, & \text{if} \quad x\in\Omega_2. \end{cases} \tag{4.1}$$

Lemma 4.1 *Assume that u be a strong solution, $E(0) > 0$, $I(0) > 0$, and*

$$Max(\beta_1,\ \beta_2) = \rho < 1, \tag{4.2}$$

where $\beta_1 = \frac{2c_*^p}{\overline{\sigma}^2}\left(\frac{2p\overline{\sigma}^2}{p-2}E(0)\right)^{\frac{p-2}{2}}$, $\beta_2 = \frac{2c_*^p}{\underline{\sigma}^2}\left(\frac{2p\underline{\sigma}^2}{p-2}E(0)\right)^{\frac{p-2}{2}}$, and c_* is the best embedding constant of $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, then $I(t) > 0$, for all $t \in [0,\ T]$.

Proof: By continuity, there exists T_*, such that

$$I(t) \geq 0, \quad \text{for all } t \in [0,\ T_*]. \tag{4.3}$$

Now, we have for all $t \in [0,\ T_*]$:

$$\begin{aligned} E(u(t)) = E(t) &= \begin{cases} \frac{1}{4}\overline{\sigma}^2 \|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p, & \text{if } x \in \Omega_1 \\ \frac{1}{4}\underline{\sigma}^2 \|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p, & \text{if } x \in \Omega_2 \end{cases} \\ &= \begin{cases} \frac{1}{4}\overline{\sigma}^2 \|\nabla u\|_2^2 - \frac{1}{p}\left(\frac{\overline{\sigma}^2}{2}\|\nabla u\|_2^2 - I(t)\right), & \text{if } x \in \Omega_1 \\ \frac{1}{4}\underline{\sigma}^2 \|\nabla u\|_2^2 - \frac{1}{p}\left(\frac{\underline{\sigma}^2}{2}\|\nabla u\|_2^2 - I(t)\right), & \text{if } x \in \Omega_2 \end{cases} \\ &= \begin{cases} \frac{1}{4}\overline{\sigma}^2 \|\nabla u\|_2^2 - \frac{1}{2p}\overline{\sigma}^2\|\nabla u\|_2^2 + \frac{1}{p}I(t), & \text{if } x \in \Omega_1 \\ \frac{1}{4}\underline{\sigma}^2 \|\nabla u\|_2^2 - \frac{1}{2p}\underline{\sigma}^2\|\nabla u\|_2^2 + \frac{1}{p}I(t), & \text{if } x \in \Omega_2 \end{cases} \end{aligned} \tag{4.4}$$

Using (4.3), we obtain

$$E(u(t)) \geq \begin{cases} \frac{p-2}{p}\frac{\overline{\sigma}^2}{4}\|\nabla u\|_2^2, & \text{if } x \in \Omega_1 \\ \frac{p-2}{p}\frac{\underline{\sigma}^2}{4}\|\nabla u\|_2^2, & \text{if } x \in \Omega_2 \end{cases} \quad \text{for all } t \in [0,\ T_*]. \tag{4.5}$$

By (4.5) and Lemma 3.1, we get

$$\|\nabla u(t)\|_2^2 \leq \begin{cases} \frac{4p}{(p-2)\overline{\sigma}^2}E(t) \leq \frac{4p}{(p-2)\overline{\sigma}^2}E(0), & \text{if } x \in \Omega_1 \\ \frac{4p}{(p-2)\underline{\sigma}^2}E(t) \leq \frac{4p}{(p-2)\underline{\sigma}^2}E(0), & \text{if } x \in \Omega_2. \end{cases} \tag{4.6}$$

By the embedding of $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, we obtain

$$\begin{aligned} \|u(t)\|_p^p &\leq c_*^p \|\nabla u(t)\|_2^p \\ &\leq c_*^p \|\nabla u(t)\|_2^{p-2} \times \|\nabla u(t)\|_2^2. \end{aligned} \tag{4.7}$$

Combining (4.6) and (4.7) yields

$$\|u(t)\|_p^p \leq \begin{cases} \frac{2c_*^p}{\overline{\sigma}^2}\left(\frac{2p\overline{\sigma}^2}{p-2}E(0)\right)^{\frac{p-2}{2}} \frac{\overline{\sigma}^2}{2}\|\nabla u\|_2^2, & \text{if } x \in \Omega_1, \\ \frac{2c_*^p}{\underline{\sigma}^2}\left(\frac{2p\underline{\sigma}^2}{p-2}E(0)\right)^{\frac{p-2}{2}} \frac{\underline{\sigma}^2}{2}\|\nabla u\|_2^2, & \text{if } x \in \Omega_2. \end{cases} \tag{4.8}$$

Since $\rho < 1$, then

$$\|u(t)\|_p^p \leq c_*^p \times \begin{cases} \frac{\overline{\sigma}^2}{2}\|\nabla u\|_2^2, & \text{if } x \in \Omega_1, \\ \frac{\underline{\sigma}^2}{2}\|\nabla u\|_2^2, & \text{if } x \in \Omega_2. \end{cases} \tag{4.9}$$

This implies that

$$I(t) > 0, \qquad \text{for all } t \in [0,\ T_*]. \tag{4.10}$$

By repeating the above procedure, we can extend T_* to T. □

Theorem 4.1 *The local strong solution of* (1.1) *is global.*

Proof: For all $t \in [0,\ T]$, we have

$$E(u(t)) = E(t) = \begin{cases} \frac{1}{4}\overline{\sigma}^2 \|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p, & \text{if } x \in \Omega_1 \\ \frac{1}{4}\underline{\sigma}^2 \|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p, & \text{if } x \in \Omega_2 \end{cases}$$

Then

$$E(u(t)) \geq \begin{cases} \frac{p-2}{p}\frac{\overline{\sigma}^2}{4}\|\nabla u\|_2^2, & \text{if } x \in \Omega_1 \\ \frac{p-2}{p}\frac{\underline{\sigma}^2}{4}\|\nabla u\|_2^2, & \text{if } x \in \Omega_2. \end{cases}, \ t \in [0,\ T]$$

So that

$$\|\nabla u(t)\|_2^2 \leq \begin{cases} \frac{4p}{(p-2)\overline{\sigma}^2}E(t) \leq \frac{4p}{(p-2)\overline{\sigma}^2}E(0), & \text{if } x \in \Omega_1 \\ \frac{4p}{(p-2)\overline{\sigma}^2}E(t) \leq \frac{4p}{(p-2)\overline{\sigma}^2}E(0), & \text{if } x \in \Omega_2. \end{cases}, \ t \in [0,\ T] \tag{4.11}$$

Thanks the Lemma 2.1, we obtain

$$\|u(t)\|_2^2 \leq c\|\nabla u(t)\|_2^2 \leq CE(t) \leq C\ E(0). \tag{4.12}$$

This implies that the local solution is global in time. □

Theorem 4.2 *Let u be a solution of (1.1)-(1.3), then, there exists tow positive constants C, $\theta > 0$, such that*

$$E(t) \leq Ce^{-\theta t}. \tag{4.13}$$

Proof: *Multiplying first equation of (1.1) by $u(t)$ and integrating over $\Omega \times (S,\ T)$, we obtain*

$$\int_S^T\int_\Omega u(t)\left[u_t - \frac{1}{2}\left(\overline{\sigma}^2 Max(\Delta u,\ 0) - \underline{\sigma}^2 Max(0,\ -\Delta u)\right)\right]dxdt = \int_S^T\int_\Omega |u(t)|^p\,dxdt. \tag{4.14}$$

So that

$$\int_S^T\int_\Omega \left[u(t)\,u_t(t) - \frac{1}{2}\left(\overline{\sigma}^2 Max(\Delta u,\ 0) - \underline{\sigma}^2 Max(0,\ -\Delta u)\right)u\right]dxdt = \int_S^T\int_\Omega |u(t)|^p\,dxdt. \tag{4.15}$$

Then,we obtain

$$\begin{cases} \int_S^T\int_\Omega \left(uu_t - \frac{1}{2}\overline{\sigma}^2|\nabla u|^2\right)dxdt = \int_S^T\int_\Omega |u(t)|^p\,dxdt, & \text{if } x \in \Omega_1, \\ \int_S^T\int_\Omega u\left(u_t - \frac{1}{2}\underline{\sigma}^2|\nabla u|^2\right)dxdt = \int_S^T\int_\Omega |u(t)|^p\,dxdt, & \text{if } x \in \Omega_2. \end{cases} \tag{4.16}$$

We add and substract the terms in equations (4.16) respectively

$$\frac{\overline{\sigma}^2}{2}\int_S^T\int_{\Omega_1}\beta_1|\nabla u|^2\,dxdt,\ \frac{\underline{\sigma}^2}{2}\int_S^T\int_{\Omega_2}\beta_2|\nabla u|^2\,dxdt,$$

and use (4.8), to get

$$\begin{cases} (1-\beta_1)\int_S^T\int_\Omega \frac{1}{2}\overline{\sigma}^2|\nabla u|^2\,dxdt + \int_S^T\int_\Omega uu_t dxdt \\ = -\left(\frac{\overline{\sigma}^2}{2}\int_S^T\int_\Omega \beta_1|\nabla u|^2\,dxdt - \int_S^T\int_\Omega |u(t)|^p\,dxdt\right) \leq 0, \quad \text{if } x \in \Omega_1, \\ (1-\beta_2)\int_S^T\int_\Omega \frac{1}{2}\underline{\sigma}^2|\nabla u|^2\,dxdt + \int_S^T\int_\Omega uu_t dxdt \\ = -\left(\frac{\underline{\sigma}^2}{2}\int_S^T\int_\Omega \beta_2|\nabla u|^2\,dxdt - \int_S^T\int_\Omega |u(t)|^p\,dxdt\right) \leq 0, \quad \text{if } x \in \Omega_2. \end{cases} \tag{4.17}$$

So, we have

$$\begin{cases} (1-\beta_1)\int\limits_S^T\int\limits_\Omega \frac{1}{2}\overline{\sigma}^2 |\nabla u|^2 \, dxdt \leq \left| -\int\limits_S^T\int\limits_\Omega uu_t dxdt \right|, & if \;\; x\in\Omega_1, \\ (1-\beta_2)\int\limits_S^T\int\limits_\Omega \frac{1}{2}\underline{\sigma}^2 |\nabla u|^2 \, dxdt \leq \left| -\int\limits_S^T\int\limits_\Omega uu_t dxdt \right|, & if \;\; x\in\Omega_2. \end{cases} \tag{4.18}$$

For the right-hand side term, we use the Young inequality, we get

$$\begin{cases} (1-\beta_1)\int\limits_S^T\int\limits_\Omega \frac{1}{2}\overline{\sigma}^2 |\nabla u|^2 \, dxdt \leq \epsilon\int\limits_S^T\int\limits_\Omega |u|^2 \, dxdt + c_\epsilon \int\limits_S^T\int\limits_\Omega |u_t|^2 \, dxdt, & if \;\; x\in\Omega_1, \\ (1-\beta_2)\int\limits_S^T\int\limits_\Omega \frac{1}{2}\underline{\sigma}^2 |\nabla u|^2 \, dxdt \leq \epsilon\int\limits_S^T\int\limits_\Omega |u|^2 \, dxdt + c_\epsilon \int\limits_S^T\int\limits_\Omega |u_t|^2 \, dxdt, & if \;\; x\in\Omega_2. \end{cases} \tag{4.19}$$

Using (4.19), (3.2) *and definition of* $E(t)$*, we obtain*

$$\begin{cases} (1-\beta_1)\int\limits_S^T E(t)\, dxdt \leq \epsilon\int\limits_S^T\int\limits_\Omega |u|^2 \, dxdt + c_\epsilon \int\limits_S^T (-E(t))\, dt, & if \;\; x\in\Omega_1, \\ (1-\beta_2)\int\limits_S^T E(t)\, dxdt \leq \epsilon\int\limits_S^T\int\limits_\Omega |u|^2 \, dxdt + c_\epsilon \int\limits_S^T (-E(t))\, dt, & if \;\; x\in\Omega_2. \end{cases} \tag{4.20}$$

By Poincaré inequality, we have

$$\begin{cases} (1-\beta_1)\int\limits_S^T E(t)\, dxdt \leq \epsilon c_1\int\limits_S^T\int\limits_\Omega |\nabla u|^2 \, dxdt + c_\epsilon (E(S)-E(T)), & if \;\; x\in\Omega_1, \\ (1-\beta_2)\int\limits_S^T E(t)\, dxdt \leq \epsilon c_2\int\limits_S^T\int\limits_\Omega |\nabla u|^2 \, dxdt + c_\epsilon (E(S)-E(T)), & if \;\; x\in\Omega_2. \end{cases} \tag{4.21}$$

Using again (4.5)*, we obtain*

$$\begin{cases} (1-\beta_1)\int\limits_S^T E(t)\, dxdt \leq \epsilon c_1\int\limits_S^T \frac{4p}{(p-2)\overline{\sigma}^2} E(t)\, dt + c_\epsilon E(S), & if \;\; x\in\Omega_1, \\ (1-\beta_2)\int\limits_S^T E(t)\, dxdt \leq \epsilon c_2\int\limits_S^T \frac{4p}{(p-2)\underline{\sigma}^2} E(t)\, dt + c_\epsilon E(S), & if \;\; x\in\Omega_2. \end{cases} \tag{4.22}$$

It is clear that

$$\begin{cases} \zeta\int\limits_S^T E(t)\, dxdt \leq \epsilon c_1\int\limits_S^T \frac{4p}{(p-2)\overline{\sigma}^2} E(t)\, dt + c_\epsilon E(S), & if \;\; x\in\Omega_1, \\ \zeta\int\limits_S^T E(t)\, dxdt \leq \epsilon c_2\int\limits_S^T \frac{4p}{(p-2)\underline{\sigma}^2} E(t)\, dt + c_\epsilon E(S), & if \;\; x\in\Omega_2. \end{cases} \tag{4.23}$$

where $\zeta = Min\left((1-\beta_1),\ (1-\beta_2)\right)$. *Choosing* ε *small enough for that* $\left(\zeta - \epsilon c_1 \frac{4p}{(p-2)\overline{\sigma}^2}\right) > 0$ *and* $\left(\zeta - \epsilon c_2 \frac{4p}{(p-2)\underline{\sigma}^2}\right) > 0$*, then, we get*

$$\int\limits_S^T E(t)\, dt \leq cE(S). \tag{4.24}$$

Taking T *goes to* ∞*, we get*

$$\int\limits_S^\infty E(t)\, dt \leq cE(S). \tag{4.25}$$

By Lemma 2.3 (Theorem 8.1 of [2], page 103), we have

$$E(t) \leq cE(0)\, e^{-\theta t}. \tag{4.26}$$

□

5. Numerical example

In this section, we present an application to illustrate numerically the blowup and exponential decay results of Theorem 1 and theorem 4. For this purpose, we numerically solve problem (1.1)-(1.3), where the domain is taken to be $\Omega = [0,\ 1]$.

5.1. Numerical method

We first introduce a suitable numerical scheme to discretize (1.1)-(1.3) using finite differences for the time variable $t \in [0, T]$ and the space variable $x \in \Omega$. Comprehensive details about the finite difference methods are available in [3], [15], [16].

We subdivide the time interval $[0,\ T]$ into N equal subintervals $[t_{n-1},\ t_n]$, $t_n = n\ \delta t$, $n = 1, 2, ..., N+1$, where δt is the time step.

Let $U^n(x) = u(x,\ t_n)$, and use the finite-difference formulas: the first-order forward difference for

$$\partial_t U^n(x) = \frac{U^{n+1}(x) - U^n(x)}{\delta t}.$$

Then the time discrete problem of (1.1)-(1.3) reads: Given u_0, find $\{U^1,\ U^2, ..., U^{n+1}\}$ such that

$$\begin{cases} -\frac{1}{2}\overline{\sigma}^2 \Delta U^{n+1} + \frac{1}{\delta t} U^{n+1} = \frac{1}{\delta t} U^n + |U^n|^{p-2} U^n, & \text{if } \Delta U^n \geq 0 \\ -\frac{1}{2}\underline{\sigma}^2 \Delta U^{n+1} + \frac{1}{\delta t} U^{n+1} = \frac{1}{\delta t} U^n + |U^n|^{p-2} U^n, & \text{if } \Delta U^n < 0 \\ U^{n+1} = 0, & \text{on } \partial\Omega \\ U^0 = u_0(x), & \text{in } \Omega \end{cases} \tag{5.1}$$

Note that the above problem is linear in U^{n+1}, which is achieved by using the history data U^n in the second side of the equation.

Problem(5.1) is solved iteratively as for given regular U^n, the solution U^{n+1} satisfies the boundary-value problem:

$$\begin{cases} -\frac{1}{2}\overline{\sigma}^2 \Delta U^{n+1} + \frac{1}{\delta t} U^{n+1} = F(U^n), & \text{in } \Omega_{1,h} \\ -\frac{1}{2}\underline{\sigma}^2 \Delta U^{n+1} + \frac{1}{\delta t} U^{n+1} = F(U^n), & \text{in } \Omega_{2,h} \\ U^{n+1} = 0, & \text{on } \partial\Omega_h \\ U^0 = u_0(x), & \text{in } \Omega_h \end{cases} \tag{5.2}$$

where $F(U^n) = \frac{1}{\delta t} U^n + |U^n|^{p-2} U^n$.

5.2. Numerical results

In this subsection, we present and discuss the blow up results of the numerical scheme(5.1). The numerical results are obtained using the Matlab codes.

1. Exponential decay

The parameters that have been set up for numerical expirements are:

- Number of discretisation points is: 20. Time step is: $\delta t = 0.005$.
- The diffusion coeficents are: $\overline{\sigma} = 3.5$, $\underline{\sigma} = 3.0$. The initial condition is: $u_0(x) = -10 \times \chi(\chi - 1)$.
- We chosen $p = 2.2$.

2. Blow up

The parameters that have been set up for numerical expirements are:

- Number of discretisation points is: 50.
- Time step is: $\delta t = 0.005$.
- The diffusion coeficents are: $\overline{\sigma} = 0.5$, $\underline{\sigma} = 0.4$.
- The initial condition is: $u_0(x) = -10 \times \chi(\chi - 1)$.

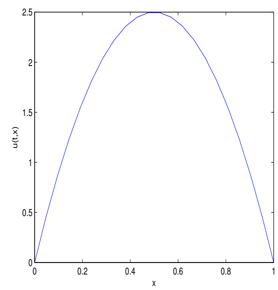

Figure 1: U^0

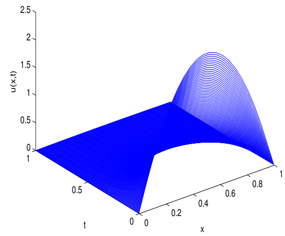

Figure 2: Numerical solutions of evolution equation to U^0 until U^{300}.

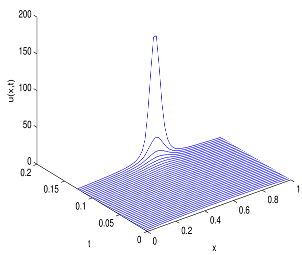

Figure 3: Numerical solutions of evolution equation to U^0 until U^{38}.

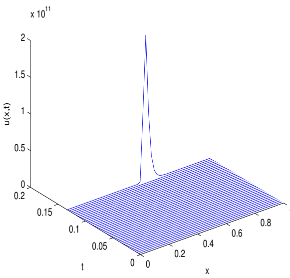

Figure 4: Numerical solutions of evolution equation to U^0 until U^{40}.

- We chosen $p = 4.2$.

n	t_n	$\|U^n\|_\infty$	$E_1(t_n) \simeq$	$E_2(t_n) \simeq$
0	0	2.50	−1.14E+02	−1.55E+02
1	0.005	2.52	−1.23E+02	−1.64E+02
7	0.035	2.77	−1.95E+02	−2.40E+02
13	0.075	3.09	−3.22E+02	−3.75E+02
20	0.100	3.68	−6.61E+02	−7.32E+02
23	0.115	4.07	−9.77E+02	−1.06E+03
27	0.135	4.86	−1.89E+03	−2.01E+03
37	0.185	83.89	−2.22E+06	−2.24E+06
39	0.195	2.53E+04	−1.16E+18	−1.16E+18
40	0.200	1.99E+11	−7.46E+46	−7.46E+46

Table 1.

Figures 1, shows the graphs of the initial data u_0.

Figure 2, present the evolution of solution U^n for iteration to $n = 0$ $(t = 0)$ at $n = 300$.

Table 1, lists numerical values of $\|U^n\|_\infty$ and the energy $E(t_n)$. It indicates the blowup of both the solution and the energy because their magnitude orders drastically jumped high (see Figures. 3 and 4).

Through Table 1, we note that the blow up of the solution and the energy to the problem (5.1) was clear when the time respectively $(t = 0.195)$, $(t = 0.200)$.

In conclusion, the above numerical application verifies and agrees with the blowup and exponential decay results of Theorem 1 and theorem 4.

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